PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 12, Pages 3693–3694
S 0002-9939(03)07200-9
Article electronically published on July 9, 2003

A "NONLINEAR" PROOF OF PITT'S COMPACTNESS THEOREM

M. FABIAN AND V. ZIZLER

(Communicated by Jonathan M. Borwein)

ABSTRACT. Using Stegall's variational principle, we present a simple proof of Pitt's theorem that bounded linear operators from ℓ_q into ℓ_p are compact for $1 \leq p < q < +\infty$.

Let φ be a bounded below lower semicontinuous function defined on a reflexive Banach space X such that $\liminf_{\|x\|\to\infty} \|x\|^{-1}\varphi(x) > 0$. Then Stegall's variational principle yields $x_0 \in X$ and $f \in X^*$ such that $\varphi + f$ attains its minimum at x_0 (see [4], or [5], or e.g. [1, Theorem 10.20] or [3, Corollary 5.22]). Recall that a bounded linear operator T from a Banach space X into a Banach space Y is called a compact operator if $\overline{T(B_X)}$ is a compact set in Y, where B_X denotes the closed unit ball of X.

Theorem (Pitt) (see, e.g., [2, p. 54] or [1, p. 175]). *Let $1 \leq p < q < +\infty$. Then every bounded linear operator from ℓ_q into ℓ_p is compact.*

Proof. Let T be a bounded linear operator from ℓ_q into ℓ_p. Then the function

$$\varphi(x) = \|x\|_q{}^q - \|Tx\|_p{}^p, \quad x \in \ell_q$$

is bounded below and $\varphi(x) > \|x\|_q$ if $\|x\|_q$ is large enough. By Stegall's variational principle there is a point $x \in \ell_q$ and there is $\xi \in \ell_q{}^*$ such that

$$\varphi(x+h) - \varphi(x) - \xi(h) \geq 0 \quad \text{for every} \ \ h \in \ell_q.$$

Then from the linearity of ξ,

$$\varphi(x+h) + \varphi(x-h) - 2\varphi(x) \geq 0 \quad \text{for every} \ \ h \in \ell_q.$$

Thus

$$\|x+h\|_q{}^q + \|x-th\|_q{}^q - 2\|x\|^q \geq \|T(x+h)\|_p{}^p + \|T(x-h)\|_p{}^p - 2\|Tx\|_p{}^p \text{ for all } h \in \ell_q.$$

Let (x_i) be a bounded sequence in ℓ_q. By passing to a subsequence, if necessary, we may assume that (x_i) converges weakly to some $y \in \ell_q$. We will show that $\|Tx_i - Ty\| \to 0$ as $i \to \infty$. Indeed, by substituting $h = t(x_i - y)$ in the last inequality, we get

$$\begin{aligned}\|x + t(x_i - y)\|_q{}^q + \|x - t(x_i - y)\|_q{}^q - 2\|x\|_q{}^q \\ \geq \|Tx + tT(x_i - y)\|_p{}^p + \|Tx - tT(x_i - y)\|_p{}^p - 2\|Tx\|_p{}^p\end{aligned}$$

Received by the editors April 6, 2001.
2000 *Mathematics Subject Classification.* Primary 46B25.
Key words and phrases. ℓ_p space, compact operator, variational principle.
Supported by grants GA ČR 201-98-1449, AV 1019003, and NSERC 7926.

for all $i = 1, 2, \dots$ and all $t > 0$.

Thus we get that for all $t > 0$,

$$\limsup_{i\to\infty} \|x \pm t(x_i - y)\|_q{}^q = \|x\|_q{}^q + t^q \limsup_{i\to\infty} \|x_i - y\|_q{}^q.$$

In order to see this, we use the fact that if $z \in \ell_q$ and $w_i \to 0$ weakly, then

$$\limsup \|z + w_i\|^q = \|z\|^q + \limsup \|w_i\|^q.$$

Indeed, first assume that $z_i = 0$ for all $i \geq i_0$ for some i_0. Choose $\|\tilde{w}_i - w_i\|_q \to 0$ where $\tilde{w}_i = 0$ for all $i \leq i_0$. Since the desired equality trivially holds for $\tilde{w}_i$ and the q-th power of the norm function is Lipschitz on bounded sets, we get that the equality holds for all finitely supported z. The conclusion then follows from the Lipschitz property of the function in question on bounded sets and from the density of finitely supported elements in ℓ_q. For the same reason, since $Tx_i \to Ty$ weakly, we get for all $t > 0$,

$$\limsup_{i\to\infty} \|Tx \pm tT(x_i - y)\|_p{}^p = \|Tx\|_p{}^p + t^p \limsup_{i\to\infty} \|T(x_i - y)\|_p{}^p.$$

Thus

$$2t^q \limsup_{i\to\infty} \|x_i - y\|_q{}^q \geq 2t^p \limsup_{i\to\infty} \|T(x_i - y)\|_p{}^p$$

for all $t > 0$, and therefore, $\|T(x_i - y)\|_p \to 0$ as $i \to \infty$. □

The usual proofs of Pitt's theorem involve the theory of Schauder bases in ℓ_p spaces (cf., e.g., [1, p. 175] or [2, p. 54]).

References

1. M. Fabian, P. Habala, P. Hájek, V. Montesinos Santaluciá, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry,* CMS Books in Mathematics, Springer-Verlag, New York, 2001. MR **2002f:**46001
2. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92, Springer-Verlag, Berlin, 1977. MR **58:**17766
3. R. R. Phelps, *Convex functions, monotone operators, and differentiability*, Lecture Notes in Math. No. **1364**, 2nd Edition, Springer-Verlag, Berlin, 1993. MR **94f:**46055
4. Ch. Stegall, *Optimization of functions on certain subsets of Banach spaces*, Math. Annalen **236** (1978), 171–176. MR **80a:**46022
5. Ch. Stegall, *Optimization and differentiation in Banach spaces*, Linear Algebra and Appl. **84** (1986), 191–211. MR **88a:**49005

Mathematical Institute, Czech Academy of Sciences, Žitná 25, 11567 Praha 1, Czech Republic

E-mail address: fabian@math.cas.cz

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

E-mail address: vzizler@math.ualberta.ca