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THE SCENERY FACTOR OF THE $[T, T^{-1}]$ TRANSFORMATION IS NOT LOOSELY BERNOULLI

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Abstract. Kalikow (1982) proved that the $[T, T^{-1}]$ transformation is not isomorphic to a Bernoulli shift. We show that the scenery factor of the $[T, T^{-1}]$ transformation is not isomorphic to a Bernoulli shift. Moreover, we show that it is not Kakutani equivalent to a Bernoulli shift.

1. Introduction

The $[T, T^{-1}]$ transformation is a random walk on a random scenery. It is defined as follows. Let $X = \{1, -1\}^{\mathbb{Z}}$ and $Y = \{\text{red, blue}\}^{\mathbb{Z}}$. Let σ be the left shift on X ($\sigma(x)_i = x_{i+1}$) and let T be the left shift on Y. Let μ' be the (1/2,1/2) product measure on X and μ'' be the (1/2,1/2) product measure on Y.

We define the transformation $[T, T^{-1}] : X \times Y \to X \times Y$ by

$$[T, T^{-1}](x, y) = \begin{cases} (\sigma(x), T(y)) & \text{if } x_0 = 1, \\ (\sigma(x), T^{-1}(y)) & \text{if } x_0 = -1. \end{cases}$$

Let $\mathcal{F}$ be the Borel σ-algebra and $\mu = \mu' \times \mu''$. Then the $[T, T^{-1}]$ transformation is the four-tuple $(X \times Y, [T, T^{-1}], \mathcal{F}, \mu)$.

The $[T, T^{-1}]$ transformation was introduced for its ergodic-theoretic properties. It is easy to show that this transformation is a K transformation [7]. For many years it was an open question to determine whether the $[T, T^{-1}]$ transformation is isomorphic to a Bernoulli shift. Kalikow settled the question with the following theorem [3].

Theorem 1. *The $[T, T^{-1}]$ transformation is not isomorphic to a Bernoulli shift. Moreover, it is not loosely Bernoulli.*

The $[T, T^{-1}]$ transformation also has probabilistic interest. Given x let

$$S(i) = S_x(i) = \begin{cases} \sum_0^{i-1} x_j & \text{if } i > 0, \\ -\sum_i^{-1} x_j & \text{if } i < 0, \\ 0 & \text{if } i = 0. \end{cases}$$

Define $C(x, y)_i = y_{S(i)}$. We refer to this as the color observed at time i.

Probabilists have focused on two questions. The first question is of reconstruction. In this problem you are given the sequence $C(x, y)_i$, $i \geq 0$, and you are trying

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to reconstruct y. The best result for reconstruction is the following theorem by Matzinger [6].

Theorem 2. *There exists a function $F : X \times Y \to Y$ so that*

(1) *for all (x, y), if $C(x, y)_j = C(x', y')_j$ for all $j \geq 0$, then $F(x, y) = F(x', y')$ and*
(2) *there exists an even m such that $F(x, y)_j = y_{j+m}$ for all j or $F(x, y)_j = y_{-j+m}$ for all j a.s.*

In the course of the proof, Matzinger proves the following results. There is a function $H : X \times Y \to \mathbb{Z}^{\mathbb{N}}$ and sets D_i such that

(3) *for all (x, y) and i, if $C(x', y')_j = C(x, y)_j$ for all $j \leq e^{i^4}$, then $H(x, y)_i = H(x', y')_i$,*
(4) *$\lim \mu(D_i) = 1$, and*
(5) *if $C(x, y)_j = C(x', y')_j$ for all j and there exists an even m such that $y_j = y'_{j+m}$ with both $(x, y), (x', y') \in D_i$, then*

$$y_{j+S_x(H(x,y)_i)} = y'_{j+S_{x'}(H(x',y')_i)}.$$

Note: The last half of Theorem 2 does not appear in this form in [6]. To see how this follows, we choose D_i to be the set denoted by $\bigcap_{j \geq i} \left(E_0^j \cap E^j\right)$ in [6]. We choose $H(x, y)_i$ to be the value denoted by t_6^i in [6]. Then Statement 3 follows from Algorithm 7. Statement 4 follows from Lemmas 3 and 5. Statement 5 follows from Algorithms 3 and 7.

The second question of probabilistic interest is one of distinguishability. Each y and n determines a measure $m_{y,n}$ on $\{\text{red, blue}\}^{[n,\infty)}$ by

$$m_{y,n}(A) = \mu'(\{x \text{ such that } C(x, y) \in A\}).$$

Call y and y' **distinguishable** if $m_{y,n}$ and $m_{y',n}$ are mutually singular for all n. It is easy to see that if there exists an even m such that $y_i = y'_{i+m}$ for all i or $y_i = y'_{-i+m}$ for all i, then y and y' are not distinguishable. The following question was raised by den Hollander and Keane and independently by Benjamini and Kesten [1]. If y and y' are not distinguishable, does there necessarily exist an even m such that $y_i = y'_{i+m}$ for all i or $y_i = y'_{-i+m}$ for all i? This was recently answered in the negative by Lindenstrauss [5].

In this paper we use Theorem 2 to study the ergodic-theoretic properties of the $[T, T^{-1}]$ process. We call the factor that associates two points (x, y) and (x', y') if $C(x, y)_i = C(x', y')_i$ for all i the **scenery factor**, $(X \times Y, [T, T^{-1}], \mathcal{G}, \mu)$. The main result of this paper is the following.

Theorem 3. *The scenery factor is not isomorphic to a Bernoulli shift. Moreover, it is not loosely Bernoulli.*

Recently, Steif gave an elementary proof of a closely related theorem. He proved that the scenery factor is not a finitary factor of a Bernoulli shift [10].

2. Proof

The equivalence relation that associates (x, y) and (x', y') if

(1) $C(x, y)_i = C(x', y')_i$ for all i and
(2) $y = T^m y'$ for some even m

defines a factor, $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$. The factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is a two-point extension of the scenery factor a.s. Both of these statements follow from Theorem 2.

For any partition P and any $a, b \in P^{\mathbb{Z}}$ let

$$\bar{d}^P_{[0,N]}(a, b) = |\{i \ : \ i \in [0, N] \text{ and } a_i \neq b_i\}|/(N + 1).$$

For any two measures μ and ν on $P^{\mathbb{Z}}$ define

$$\bar{d}^P_{[0,N]}(\mu, \nu) = \inf_m \int \bar{d}_{[0,N]}(a, b)dm$$

where the infinum is taken over all joinings of μ and ν. We set P to be the time zero partition of $X \times Y$. A point (x, y) in $X \times Y$ defines a sequence in $P^{\mathbb{Z}}$ with ith component $P([T, T^{-1}]^i(x, y)) = (x_i, C_i(x, y))$.

Theorem 4. *$(X \times Y, [T, T^{-1}], \mathcal{F}, \mu)$ is isomorphic to $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu) \times (\Omega, \sigma, \Sigma, \nu)$, where $(\Omega, \sigma, \Sigma, \nu)$ is a Bernoulli shift.*

Proof. An atom of $\mathcal{H}$ is given by $z, a \in \{\text{red, blue}\}^{\mathbb{Z}}$ such that there exists an x so that $z_i = C(x, a)_i$ for all i. The atoms given by z, a and z', a' are equivalent if $z = z'$ and there exists an even m such that $a_i = a'_{i+m}$. Given an atom z, a of $\mathcal{H}$ define $\tilde{\mu}_{z,a}$ by

$$\tilde{\mu}_{z,a}(A) = \mu\{(x, y) \in A \mid C(x, y)_i = z_i \ \forall i \text{ and there exists an even } m \text{ such that } y = T^m a\}.$$

Given $x \in X$ define $\bar{x} = \{x' \ : \ x_i = x'_i \ \forall \ i \leq 0\}$. Also define $\mu_{(x,y)}$ by

$$\mu_{(x,y)}(A) = \mu\{(x', y') \in A \mid x' \in \bar{x} \text{ and } C(x, y)_i = C(x', y')_i \ \forall i\}.$$

By Thouvenot's relative isomorphism theory, the theorem is equivalent to checking that the $[T, T^{-1}]$ transformation is **relatively very weak Bernoulli** with respect to $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ (see [11], [8], [4]). This means that given almost every atom z, a of $\mathcal{H}$ and any $\epsilon > 0$ there exists an N and a set G such that

(1) $\tilde{\mu}_{z,a}(G) > 1 - \epsilon$ and
(2) for any $(x, y), (x', y') \in G$,

$$\bar{d}^P_{[0,N]}(\mu_{(x,y)}, \mu_{(x',y')}) < \epsilon.$$

Fix z and a. Let

$$S_M = \left\{(x, y) : \ \mu_{(x,y)} \left\{(\tilde{x}, \tilde{y}) : \ (\tilde{x}, \tilde{y}) \in D_M\right\} > 1 - \epsilon\right\}.$$

Let M be such that $\tilde{\mu}_{z,a}(S_M) > 1-\epsilon$. This exists for almost every atom by Theorem 2. Now let G be S_M restricted to the atom defined by z, a.

Let $(c, d), (e, f) \in D_M$ both be points in the atom determined by z and a. By item 5 of Theorem 2 we have that for all j,

$$d_{j+S(H(c,d)_M)} = f_{j+S(H(e,f)_M)}.$$

For any $(x, y), (x', y') \in G$, let

$$V = \{(c, d), (e, f) : \ d_{j+S(H(c,d)_M)} = f_{j+S(H(e,f)_M)} \text{ for all } j\}.$$

Thus for any $(x, y), (x', y') \in G$ and any joining γ of $\mu_{(x,y)}$ and $\mu_{(x',y')}$ by Theorem 2 we have

$$\gamma(V) > 1 - 2\epsilon.$$

We now alter γ to obtain a new joining Γ in the following way. Partition V into subsets such that (c, d, e, f) and (c', d', e', f') are in the same set if

(1) $c_i = c'_i$ for all $i = 1, \dots, H(c,d)_M$,
(2) $d = d'$,
(3) $e_i = e'_i$ for all $i = 1, \dots, H(c,d)_M = H(e,f)$ and
(4) $f = f'$.

This partitions V into at most countably many sets of positive γ measure. We define Γ so that on each of these sets Q we have $\Gamma(Q) = \gamma(Q)$. On each Q we define Γ such that

$$\Gamma\left(Q \cap \{(c,d),(e,f) : c_j = e_j \text{ for all } j > H(c,d)_M\}\right) = \gamma(Q).$$

Then we get that

$$\Gamma\{(c,d),(e,f) : c_j = e_j \text{ and } C(c,d)_j = C(e,f)_j \text{ for all } j > H(c,d)_M\} > 1 - 2\epsilon.$$

Let $N > e^{M^4}/\epsilon > H(c,d)_M/\epsilon$. Thus the joining Γ shows that

$$\bar{d}^P_{[0,N]}(\mu_{(x,y)}, \mu_{(x',y')}) < 3\epsilon.$$

□

Proof of Theorem 3. By Theorem 1 the $[T, T^{-1}]$ transformation is not isomorphic to a Bernoulli shift [3]. By Theorem 4 the $[T, T^{-1}]$ transformation is the direct product of the factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ with a Bernoulli shift. Thus the factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is not isomorphic to a Bernoulli shift. By [3] the $[T, T^{-1}]$ transformation is not loosely Bernoulli. Since the direct product of a loosely Bernoulli transformation and a Bernoulli shift is loosely Bernoulli, the factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is not loosely Bernoulli either.

The factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is a two-point extension of the scenery factor. It is weak mixing since it is the factor of the $[T, T^{-1}]$ transformation that is K (and thus weak mixing). The two-point extension of a Bernoulli shift that is weak mixing is isomorphic to a Bernoulli shift [9]. Thus the scenery factor is not isomorphic to a Bernoulli shift.

Similarly we can show that the scenery factor is not loosely Bernoulli. The factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is not loosely Bernoulli. The two-point extension of a loosely Bernoulli transformation is loosely Bernoulli [9]. Thus if the scenery factor were loosely Bernoulli, then the factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ would be as well. This can not be; so the scenery factor is not loosely Bernoulli and is not Kakutani equivalent to a Bernoulli shift [2]. □

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