

THE DIOPHANTINE EQUATION $2x^2 + 1 = 3^n$

MING-GUANG LEU AND GUAN-WEI LI

(Communicated by David E. Rohrlich)

ABSTRACT. Let p be a rational prime and D a positive rational integer coprime with p . Denote by $N(D, 1, p)$ the number of solutions (x, n) of the equation $Dx^2 + 1 = p^n$ in rational integers $x \geq 1$ and $n \geq 1$. In a paper of Le, he claimed that $N(D, 1, p) \leq 2$ without giving a proof. Furthermore, the statement $N(D, 1, p) \leq 2$ has been used by Le, Bugeaud and Shorey in their papers to derive results on certain Diophantine equations. In this paper we point out that the statement $N(D, 1, p) \leq 2$ is incorrect by proving that $N(2, 1, 3) = 3$.

1. INTRODUCTION

Let D_1 and D_2 be coprime positive rational integers, and let p be a rational prime coprime with $D_1 D_2$. Denote by $N(D_1, D_2, p)$ the number of solutions (x, n) of the following equation:

$$D_1 x^2 + D_2 = p^n \quad \text{in rational integers } x \geq 1, n \geq 1.$$

In paper [5], Le claimed that $N(D_1, 1, p) \leq 2$ and the proof could be found in [3] and [4]. Le used $N(D_1, 1, p) \leq 2$ and related results to deduce the main result of [5]. In the proof of Theorem 2 of [2], Bugeaud and Shorey used Le's result $N(D_1, 1, p) \leq 2$ to claim that $N(2, 1, 3) = 2$, by giving the solutions $(x, n) = (1, 1)$ and $(2, 2)$. (See also the remarks on page 59 of [2].)

By looking at papers [3] and [4], we cannot find a proof for the statement $N(D_1, 1, p) \leq 2$ which was claimed by Le [5]. Unfortunately, it is not difficult to verify that $N(2, 1, 3) \geq 3$ by considering $(x, n) = (1, 1)$, $(2, 2)$ and $(11, 5)$. In this paper we point out that the statement $N(D_1, 1, p) \leq 2$ is incorrect by proving that $N(2, 1, 3) = 3$.

2. $N(2, 1, 3) = 3$

To determine positive rational integral solutions (x, n) of $2x^2 + 1 = 3^n$ we apply unique factorization in the imaginary quadratic field $\mathbb{Q}(\sqrt{-2})$ to reduce the problem to a question about a Fibonacci-type integer sequence. Then, by Proposition 2.1, a result of Beukers [1], we prove that the Diophantine equation $2x^2 + 1 = 3^n$ has exactly three positive rational integral solutions, namely $(x, n) = (1, 1)$, $(2, 2)$ and $(11, 5)$.

Received by the editors July 2, 2002.

2000 *Mathematics Subject Classification*. Primary 11D61.

The authors research was supported in part by grant NSC 91-2115-M-008-006 of the National Science Council of the Republic of China.

The following proposition is part of Lemma 7 of [1]:

Proposition 2.1. *Let $\theta = 1 + \sqrt{-2}$ and $\alpha = \sqrt{-2}$. Then all rational integral solutions $n > 0$ of the equations $\alpha\theta^n - \bar{\alpha}\bar{\theta}^n = \alpha - \bar{\alpha}$ or $-(\alpha - \bar{\alpha})$ are $n = 1, 2$ and 5 , where $\bar{\theta}$ and $\bar{\alpha}$ denote the algebraic conjugates of θ and α , respectively.*

Let \mathbb{Z} be the set of rational integers and R denote the ring of algebraic integers in the quadratic field $\mathbb{Q}(\sqrt{-2})$. Then $R = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$. It is known that R is a unique factorization domain. Let $\theta = 1 + \sqrt{-2}$ and $\bar{\theta} = 1 - \sqrt{-2}$. Then $\theta\bar{\theta} = 3$ and $\theta^2 = 2\theta - 3$. The equation $2x^2 + 1 = 3^n$ factors in R as

$$(1 + x\sqrt{-2})(1 - x\sqrt{-2}) = \theta^n \bar{\theta}^n, \quad \text{if } x \in \mathbb{Z}.$$

Note that θ and $\bar{\theta}$ are irreducible in R . If $\theta\bar{\theta} \mid 1 + x\sqrt{-2}$, then there exist rational integers $i, j, 1 \leq i \leq n, 1 \leq j \leq n$ such that $1 + x\sqrt{-2} = u\theta^i \bar{\theta}^j$, where u is either 1 or -1 . Suppose $i \leq j$. Then

$$\begin{aligned} 1 + x\sqrt{-2} &= u\theta^i \bar{\theta}^j \\ &= u(\theta\bar{\theta})^i \bar{\theta}^{j-i} \\ &= u3^i (2 - \theta)^{j-i} \quad (\bar{\theta} = 2 - \theta) \\ &= u3^i (2^{j-i} - \binom{j-i}{1} 2^{j-i-1}\theta + \binom{j-i}{2} 2^{j-i-2}\theta^2 + \dots \\ &\quad + (-1)^{j-i} \theta^{j-i}) \\ &= u3^i (A + B\sqrt{-2}), \end{aligned}$$

where A and B are rational integers. Since $\{1, \sqrt{-2}\}$ is an integral basis of R , the equality $1 + x\sqrt{-2} = u3^i(A + B\sqrt{-2})$ is impossible. Suppose $j < i$. Then by the same argument, we also reach a contradiction. We conclude that $3 = \theta\bar{\theta}$ does not divide $1 + x\sqrt{-2}$. Similarly, we also know that $3 = \theta\bar{\theta}$ does not divide $1 - x\sqrt{-2}$. Hence we have $\theta^n = u(1 + x\sqrt{-2})$ or $\theta^n = u(1 - x\sqrt{-2})$, where u is either 1 or -1 . Equivalently, we have $\sqrt{-2}\theta^n = u(\sqrt{-2} - 2x)$ or $\sqrt{-2}\theta^n = u(\sqrt{-2} + 2x)$. From these equations we find that $\sqrt{-2}\theta^n = a + \theta$ or $a - \theta$ for some rational integer a . Conversely, for some rational integer $m > 0$, if $\sqrt{-2}\theta^m = a + \theta$ or $a - \theta$ for $a \in \mathbb{Z}$, then $(a \pm 1)^2 + 2 = 2 \times 3^m$. This means that either $(\frac{a+1}{2}, m)$ or $(\frac{a-1}{2}, m)$ is a solution of $2x^2 + 1 = 3^n$. To summarize, we have proved that the equation $2x^2 + 1 = 3^n$ has a positive rational integral solution (x, n) for $n = m$ if and only if $\sqrt{-2}\theta^m = a + \theta$ or $a - \theta$ for $a \in \mathbb{Z}$.

The problem now is to determine exactly those powers n such that $\sqrt{-2}\theta^n$ can be expressed either in the form $a + \theta$ or $a - \theta$ for $a \in \mathbb{Z}$. Since $\{1, \theta\}$ is also an integral basis of R , $\sqrt{-2}\theta^n$ can be expressed as $\sqrt{-2}\theta^n = a_n + b_n\theta$, for $a_n, b_n \in \mathbb{Z}$. By $\theta^2 = 2\theta - 3$, we have

$$\begin{aligned} a_{n+1} + b_{n+1}\theta &= \sqrt{-2}\theta^{n+1} \\ &= (\sqrt{-2}\theta^n)\theta \\ &= (a_n + b_n\theta)\theta \\ &= -3b_n + (a_n + 2b_n)\theta, \end{aligned}$$

which implies that $b_{n+2} = 2b_{n+1} - 3b_n$. Thus the sequence of rational integers b_n is completely determined by this binary linear recurrence and the initial values $b_1 = 1$

and $b_2 = -1$. The sequence $\{b_n\}_{n=1}^{\infty}$ begins:

$$1, -1, -5, -7, 1, 23, 43, 17, -95, \dots$$

Since $b_1 = b_5 = 1$ and $b_2 = -1$, we are provided with the three solutions to the equation $2x^2 + 1 = 3^n$, namely, $(x, n) = (1, 1)$, $(2, 2)$, and $(11, 5)$. Now, the problem is to prove that there are no further occurrences of 1 or -1 in the sequence $\{b_n\}_{n=1}^{\infty}$.

Proposition 2.2. *Let the sequence of rational integers b_n be defined by the equations: $b_1 = 1$, $b_2 = -1$ and $b_{n+2} = 2b_{n+1} - 3b_n$. Then $b_n = 1$ or -1 only for $n = 1, 2$ and 5 .*

Proof. To apply Proposition 2.1, we define $\alpha = \sqrt{-2}$. Then $\alpha\theta = \sqrt{-2}(1 + \sqrt{-2}) = b_2 - b_1\bar{\theta}$. Suppose, for all rational integers k , $1 \leq k \leq n$, that $\alpha\theta^k = b_{k+1} - b_k\bar{\theta}$. Then we have

$$\begin{aligned} \alpha\theta^{n+1} &= (\alpha\theta^n)\theta \\ &= (b_{n+1} - b_n\bar{\theta})(2 - \bar{\theta}) \\ &= 2b_{n+1} - 2b_n\bar{\theta} - b_{n+1}\bar{\theta} + b_n\bar{\theta}^2 \\ &= 2b_{n+1} - 2b_n\bar{\theta} - b_{n+1}\bar{\theta} + b_n(2\bar{\theta} - 3) \\ &= (2b_{n+1} - 3b_n) - b_{n+1}\bar{\theta} \\ &= b_{n+2} - b_{n+1}\bar{\theta}. \end{aligned}$$

By induction, we prove that $\alpha\theta^n = b_{n+1} - b_n\bar{\theta}$ for $n > 0$.

From $\alpha\theta^n = b_{n+1} - b_n\bar{\theta}$, it follows that $\alpha\theta^n - \bar{\alpha}\bar{\theta}^n = b_n(\theta - \bar{\theta}) = b_n(\alpha - \bar{\alpha})$, which implies that

$$b_n = \frac{\alpha\theta^n - \bar{\alpha}\bar{\theta}^n}{\alpha - \bar{\alpha}}.$$

By Proposition 2.1, $b_n = 1$ or -1 only for $n = 1, 2$ and 5 . □

To summarize, we have proved the following:

Theorem 2.3. *The Diophantine equation $2x^2 + 1 = 3^n$ has exactly three positive rational integral solutions, namely $(x, n) = (1, 1)$, $(2, 2)$ and $(11, 5)$.*

REFERENCES

- [1] F. Beukers, *The multiplicity of binary recurrences*, *Compositio Math.* **40** (1980), 251–267. MR **81g**:10019
- [2] Y. Bugeaud and T. N. Shorey, *On the number of solutions of the generalized Ramanujan-Nagell equation*, *J. reine angew. Math.* **539** (2001), 55–74. MR **2002k**:11041
- [3] M.-H. Le, *Divisibility of the class numbers of a class of imaginary quadratic fields*, *Kexue Tongbao* **32** (1987), 724–727. (in Chinese)
- [4] M.-H. Le, *On the Diophantine equation $D_1x^2 + D_2 = 2^{n+2}$* , *Acta Arith.* **64** (1993), 29–41. MR **94e**:11030
- [5] M.-H. Le, *On the Diophantine equation $(x^3 - 1)/(x - 1) = (y^n - 1)/(y - 1)$* , *Trans. Amer. Math. Soc.* **351** (1999), 1063–1074. MR **99e**:11033

DEPARTMENT OF MATHEMATICS, NATIONAL CENTRAL UNIVERSITY, CHUNG-LI, TAIWAN 32054,
REPUBLIC OF CHINA

E-mail address: mleu@math.ncu.edu.tw

DEPARTMENT OF MATHEMATICS, NATIONAL CENTRAL UNIVERSITY, CHUNG-LI, TAIWAN 32054,
REPUBLIC OF CHINA