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THE DIOPHANTINE EQUATION $2x^2 + 1 = 3^n$

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Abstract. Let p be a rational prime and D a positive rational integer coprime with p. Denote by $N(D, 1, p)$ the number of solutions (x, n) of the equation $Dx^2 + 1 = p^n$ in rational integers $x \geq 1$ and $n \geq 1$. In a paper of Le, he claimed that $N(D, 1, p) \leq 2$ without giving a proof. Furthermore, the statement $N(D, 1, p) \leq 2$ has been used by Le, Bugeaud and Shorey in their papers to derive results on certain Diophantine equations. In this paper we point out that the statement $N(D, 1, p) \leq 2$ is incorrect by proving that $N(2, 1, 3) = 3$.

1. Introduction

Let D_1 and D_2 be coprime positive rational integers, and let p be a rational prime coprime with D_1D_2. Denote by $N(D_1, D_2, p)$ the number of solutions (x, n) of the following equation:

$$D_1x^2 + D_2 = p^n \quad \text{in rational integers} \quad x \geq 1, n \geq 1.$$

In paper [5], Le claimed that $N(D_1, 1, p) \leq 2$ and the proof could be found in [3] and [4]. Le used $N(D_1, 1, p) \leq 2$ and related results to deduce the main result of [5]. In the proof of Theorem 2 of [2], Bugeaud and Shorey used Le's result $N(D_1, 1, p) \leq 2$ to claim that $N(2, 1, 3) = 2$, by giving the solutions $(x, n) = (1, 1)$ and $(2, 2)$. (See also the remarks on page 59 of [2].)

By looking at papers [3] and [4], we cannot find a proof for the statement $N(D_1, 1, p) \leq 2$ which was claimed by Le [5]. Unfortunately, it is not difficult to verify that $N(2, 1, 3) \geq 3$ by considering $(x, n) = (1, 1)$, $(2, 2)$ and $(11, 5)$. In this paper we point out that the statement $N(D_1, 1, p) \leq 2$ is incorrect by proving that $N(2, 1, 3) = 3$.

2. $N(2, 1, 3) = 3$

To determine positive rational integral solutions (x, n) of $2x^2 + 1 = 3^n$ we apply unique factorization in the imaginary quadratic field $\mathbb{Q}(\sqrt{-2})$ to reduce the problem to a question about a Fibonacci-type integer sequence. Then, by Proposition 2.1, a result of Beukers [1], we prove that the Diophantine equation $2x^2 + 1 = 3^n$ has exactly three positive rational integral solutions, namely $(x, n) = (1, 1)$, $(2, 2)$ and $(11, 5)$.

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The following proposition is part of Lemma 7 of [1]:

Proposition 2.1. *Let $\theta = 1+\sqrt{-2}$ and $\alpha = \sqrt{-2}$. Then all rational integral solutions $n > 0$ of the equations $\alpha\theta^n - \bar{\alpha}\bar{\theta}^n = \alpha - \bar{\alpha}$ or $-(\alpha - \bar{\alpha})$ are $n = 1$, 2 and 5, where $\bar{\theta}$ and $\bar{\alpha}$ denote the algebraic conjugates of θ and α, respectively.*

Let $\mathbb{Z}$ be the set of rational integers and R denote the ring of algebraic integers in the quadratic field $\mathbb{Q}(\sqrt{-2})$. Then $R = \{a+b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$. It is known that R is a unique factorization domain. Let $\theta = 1+\sqrt{-2}$ and $\bar{\theta} = 1-\sqrt{-2}$. Then $\theta\bar{\theta} = 3$ and $\theta^2 = 2\theta - 3$. The equation $2x^2+1 = 3^n$ factors in R as

$$(1+x\sqrt{-2})(1-x\sqrt{-2}) = \theta^n\bar{\theta}^n, \quad \text{if} \quad x \in \mathbb{Z}.$$

Note that θ and $\bar{\theta}$ are irreducible in R. If $\theta\bar{\theta} \mid 1+x\sqrt{-2}$, then there exist rational integers i, j, $1 \le i \le n$, $1 \le j \le n$ such that $1+x\sqrt{-2} = u\theta^i\bar{\theta}^j$, where u is either 1 or -1. Suppose $i \le j$. Then

$$\begin{aligned}
1+x\sqrt{-2} &= u\theta^i\bar{\theta}^j \\
&= u(\theta\bar{\theta})^i\bar{\theta}^{j-i} \\
&= u3^i(2-\theta)^{j-i} \quad (\bar{\theta} = 2-\theta) \\
&= u3^i(2^{j-i} - \binom{j-i}{1}2^{j-i-1}\theta + \binom{j-i}{2}2^{j-i-2}\theta^2 + \cdots \\
&\qquad + (-1)^{j-i}\theta^{j-i}) \\
&= u3^i(A+B\sqrt{-2}),
\end{aligned}$$

where A and B are rational integers. Since $\{1, \sqrt{-2}\}$ is an integral basis of R, the equality $1+x\sqrt{-2} = u3^i(A+B\sqrt{-2})$ is impossible. Suppose $j < i$. Then by the same argument, we also reach a contradiction. We conclude that $3 = \theta\bar{\theta}$ does not divide $1+x\sqrt{-2}$. Similarly, we also know that $3 = \theta\bar{\theta}$ does not divide $1-x\sqrt{-2}$. Hence we have $\theta^n = u(1+x\sqrt{-2})$ or $\theta^n = u(1-x\sqrt{-2})$, where u is either 1 or -1. Equivalently, we have $\sqrt{-2}\theta^n = u(\sqrt{-2}-2x)$ or $\sqrt{-2}\theta^n = u(\sqrt{-2}+2x)$. From these equations we find that $\sqrt{-2}\theta^n = a+\theta$ or $a-\theta$ for some rational integer a. Conversely, for some rational integer $m > 0$, if $\sqrt{-2}\theta^m = a+\theta$ or $a-\theta$ for $a \in \mathbb{Z}$, then $(a\pm 1)^2+2 = 2\times 3^m$. This means that either $(\frac{a+1}{2}, m)$ or $(\frac{a-1}{2}, m)$ is a solution of $2x^2+1 = 3^n$. To summarize, we have proved that the equation $2x^2+1 = 3^n$ has a positive rational integral solution (x, n) for $n = m$ if and only if $\sqrt{-2}\theta^m = a+\theta$ or $a-\theta$ for $a \in \mathbb{Z}$.

The problem now is to determine exactly those powers n such that $\sqrt{-2}\theta^n$ can be expressed either in the form $a+\theta$ or $a-\theta$ for $a \in \mathbb{Z}$. Since $\{1, \theta\}$ is also an integral basis of R, $\sqrt{-2}\theta^n$ can be expressed as $\sqrt{-2}\theta^n = a_n + b_n\theta$, for a_n, $b_n \in \mathbb{Z}$. By $\theta^2 = 2\theta - 3$, we have

$$\begin{aligned}
a_{n+1} + b_{n+1}\theta &= \sqrt{-2}\theta^{n+1} \\
&= (\sqrt{-2}\theta^n)\theta \\
&= (a_n + b_n\theta)\theta \\
&= -3b_n + (a_n + 2b_n)\theta,
\end{aligned}$$

which implies that $b_{n+2} = 2b_{n+1} - 3b_n$. Thus the sequence of rational integers b_n is completely determined by this binary linear recurrence and the initial values $b_1 = 1$

and $b_2 = -1$. The sequence $\{b_n\}_{n=1}^{\infty}$ begins:

$$1,\ -1,\ -5,\ -7,\ 1,\ 23,\ 43,\ 17,\ -95,\ \cdots.$$

Since $b_1 = b_5 = 1$ and $b_2 = -1$, we are provided with the three solutions to the equation $2x^2 + 1 = 3^n$, namely, (x, n)=$(1, 1)$, $(2, 2)$, and $(11, 5)$. Now, the problem is to prove that there are no further occurrences of 1 or -1 in the sequence $\{b_n\}_{n=1}^{\infty}$.

Proposition 2.2. *Let the sequence of rational integers b_n be defined by the equations: $b_1 = 1$, $b_2 = -1$ and $b_{n+2} = 2b_{n+1} - 3b_n$. Then $b_n = 1$ or -1 only for $n = 1$, 2 and 5.*

Proof. To apply Proposition 2.1, we define $\alpha = \sqrt{-2}$. Then $\alpha\theta = \sqrt{-2}(1+\sqrt{-2}) = b_2 - b_1\bar{\theta}$. Suppose, for all rational integers k, $1 \le k \le n$, that $\alpha\theta^k = b_{k+1} - b_k\bar{\theta}$. Then we have

$$\begin{aligned}
\alpha\theta^{n+1} &= (\alpha\theta^n)\theta \\
&= (b_{n+1} - b_n\bar{\theta})(2 - \bar{\theta}) \\
&= 2b_{n+1} - 2b_n\bar{\theta} - b_{n+1}\bar{\theta} + b_n\bar{\theta}^2 \\
&= 2b_{n+1} - 2b_n\bar{\theta} - b_{n+1}\bar{\theta} + b_n(2\bar{\theta} - 3) \\
&= (2b_{n+1} - 3b_n) - b_{n+1}\bar{\theta} \\
&= b_{n+2} - b_{n+1}\bar{\theta}.
\end{aligned}$$

By induction, we prove that $\alpha\theta^n = b_{n+1} - b_n\bar{\theta}$ for $n > 0$.

From $\alpha\theta^n = b_{n+1} - b_n\bar{\theta}$, it follows that $\alpha\theta^n - \bar{\alpha}\bar{\theta}^n = b_n(\theta - \bar{\theta}) = b_n(\alpha - \bar{\alpha})$, which implies that

$$b_n = \frac{\alpha\theta^n - \bar{\alpha}\bar{\theta}^n}{\alpha - \bar{\alpha}}.$$

By Proposition 2.1, $b_n = 1$ or -1 only for $n = 1$, 2 and 5. □

To summarize, we have proved the following:

Theorem 2.3. *The Diophantine equation $2x^2 + 1 = 3^n$ has exactly three positive rational integral solutions, namely $(x, n) = (1, 1)$, $(2, 2)$ and $(11, 5)$.*

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