A NEW CHARACTERIZATION OF THE UNIT BALL OF $H^2$

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Abstract. We derive a new expression for the norm of $H^2$ functions; we present some well-known results in a different setting.

Introduction

In 1915, Pick [3] proved the following result.

Theorem 1. Let $g$ be an analytic function on the unit disc $\Delta$ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in $\Delta$ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k \overline{z_l}} \lambda_k \overline{\lambda_l} \geq 0.$$  

Ahlfors [1], page 3, gives an elegant proof of this characterization of the unit ball of $H^\infty$.

In this note we shall present a characterization of the unit ball of $H^2$. Our main tool will be an explicit solution of the “minimal interpolation problem” for $H^2$ (see [2], page 141). As a byproduct we obtain a new proof of Pick’s theorem.

Description of the main result

Let $z_1, z_2, \ldots, z_n$ be a sequence in $\Delta$, and let $b$ be the Blaschke product generated by the sequence

$$b(z) = \prod_{j=1}^{n} \frac{z - \overline{z}_j}{1 - \overline{z}_j z}.$$  

We shall prove that the following conditions are equivalent for continuous functions $f$ on $\Delta$:

1) $f$ lies in the unit ball of $H^2$.

2) For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k) \overline{f(z_l)}}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.$$  

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Preliminaries

For mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ and for $w_1, w_2, \ldots, w_n$ in $\mathbb{C}$ we define

$$\Lambda = \{f \in H^2 : f(z_j) = w_j, \; j = 1, 2, \ldots, n\}.$$

$\Lambda$ is not empty; it contains the Lagrange interpolation polynomial

$$\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k)b'(z_k)},$$

where $l(z) = \prod_{j=1}^{n} (z - z_j)$.

In the context of $H^p$ spaces it is more natural to work with the Blaschke interpolation function

$$\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \overline{z}_k z}{z - z_k} \cdot \frac{w_k}{b'(z_k)(1 - |z_k|^2)},$$

with $b(z)$ defined as in (2). Of course $\beta \in \Lambda$. However, for our purposes we are better off with

$$\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k)b'(z_k)}.$$

$\varphi \in \Lambda$, and $\varphi$ is analytic on some neighbourhood of $\sum$. $\Lambda$ is a hyperplane in $H^2$.

With $\varphi$ and $b$ defined as in (4) and (2) we have

$$\Lambda = \{\varphi + bg; \; g \in H^2\}.$$

**Theorem 2.** $\varphi$ is the unique solution of the “minimal interpolation problem”, i.e., for every $f \in \Lambda \setminus \{\varphi\}$ we have $\|f\|_2 > \|\varphi\|_2$.

**Proof.** It suffices to show that $\varphi \perp (f - \varphi)$ for every $f \in \Lambda$ (since under those circumstances $\|f\|^2 = \|\varphi\|^2 + \|f - \varphi\|^2$).

From the decomposition $f = \varphi + bg$ we have

$$\langle f - \varphi, \varphi \rangle = \langle bg, \varphi \rangle = \frac{1}{2\pi} \int_0^{2\pi} b(e^{it})g(e^{it})\overline{\varphi(e^{it})} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} b(e^{it})g(e^{it})\overline{b(e^{it})} \sum_{k=1}^{n} \frac{w_k}{(e^{-it} - \overline{z}_k)b'(z_k)} dt.$$

Note that $|b(e^{it})|^2 = 1$. Thus,

$$\langle f - \varphi, \varphi \rangle = \sum_{k=1}^{n} \frac{\overline{w}_k}{2\pi b'(z_k)} \int_0^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it}\overline{z}_k} dt$$

$$= \sum_{k=1}^{n} \frac{\overline{w}_k}{b'(z_k)} \frac{1}{2\pi i} \int_{|z| = 1} \frac{g(z)}{1 - \overline{z}_k z} dz = 0,$$

because the integrand is analytic on $\Delta$. 


It will be convenient to have an explicit expression for \(\|\varphi\|_2\):

\[
\|\varphi\|_2^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |\varphi(e^{it})|^2 dt = \frac{1}{2\pi i} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k w_l}{b'(z_k)b'(z_l)} \int_{0}^{2\pi} dt \frac{e^{it} - z_k}{b'(z_k)(e^{-it} - z_l)}
\]

\[
= \frac{1}{2\pi i} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k w_l}{b'(z_k)b'(z_l)} \int_{0}^{2\pi} \frac{dz}{(z - z_k)(1 - \bar{z}_l z)}
\]

\[
= \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \frac{w_k w_l}{b'(z_k)b'(z_l)}.
\]

There are, of course, many other expressions for \(\|\varphi\|_2\).

**Theorem 3.**

\[
\|\varphi\|_2 = \max \left\{ \left| \sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} \right| : f \in H^2, \|f\|_2 \leq 1 \right\}.
\]

**Proof.**

\[
\sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} = \frac{1}{2\pi i} \int_{0}^{2\pi} f(z) \varphi(z) \frac{dz}{b(z)}
\]

hence, by Schwarz’s inequality we have

\[
\left| \sum_{k=1}^{n} \frac{w_k f(z_k)}{b'(z_k)} \right| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(e^{it})| \cdot |\varphi(e^{it})| dt \leq \|f\|_2 \cdot \|\varphi\|_2 \leq \|\varphi\|_2.
\]

Equality holds for the function \(f : z \rightarrow \frac{1}{\|\varphi\|_2} \sum_{k=1}^{n} \frac{w_k}{(1 - \bar{z}_k z)b'(z_k)}\).

An immediate result from Theorem 3 is

**Corollary.** For every sequence \(z_1, z_2, \ldots, z_n\) of mutually distinct points of \(\Delta\) we have

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \frac{1}{b'(z_k)b'(z_l)} \leq 1.
\]

**Proof.** Take \(w_1 = w_2 = \ldots = w_n = 1\). Then \(1 \in \Lambda\) and since

\[
\|1\|_2 = 1,
\]

we have

\[
1 = \|\varphi\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \frac{1}{b'(z_k)b'(z_l)}.
\]

The equality sign certainly occurs if \(0 \in \{z_1, z_2, \ldots, z_n\}:

\[
1 = \varphi(0)^2 \leq \frac{1}{2\pi} \int_{0}^{2\pi} |\varphi(e^{it})|^2 dt = \|\varphi\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{1 - z_k z_l} \frac{1}{b'(z_k)b'(z_l)}.
\]

If \(0 \notin \{z_1, z_2, \ldots, z_n\}\), there is strict inequality.

Because of the uniqueness of \(\varphi\) there can be equality only if

\[
b(z) \sum_{k=1}^{n} \frac{1}{(z - z_k)b'(z_k)} = 1.
\]
In this identity for rational functions we let $z \to \infty$. Since $z_j \neq 0$, $\lim_{z \to \infty} b(z)$ has a finite value. Therefore, the left-hand side has limit zero.

**Remark.** The corollary shows that a function satisfying (1) also satisfies (3).

The fact that $\varphi \in \Lambda$ has an interesting reformulation. We start with a lemma.

**Lemma 1.** The partial fraction decomposition of $\varphi$ is

$$\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \overline{z_l}z_l)(1 - \overline{z_l}z_k)b'(z_k)b'(z_l)}$$

**Proof.** An elegant way to prove this is to compute both sides of the following identity.

For $z \in \Delta$ we have

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1 - \zeta z} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1 - \zeta z} \frac{d\zeta}{\zeta}.$$

The left-hand side is equal to

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta = \varphi(z),$$

while the right-hand side is equal to the complex conjugate of

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{b(\zeta)}{1 - \zeta z} \sum_{k=1}^{n} \frac{\overline{w_k}}{(\zeta - z_k)b'(z_k)} \frac{d\zeta}{1 - \zeta z},$$

i.e., to the complex conjugate of

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{b(\zeta)}{1 - \zeta z} \sum_{k=1}^{n} \frac{\overline{w_k}}{(1 - \overline{z_l}z_k)b'(z_k)} \frac{d\zeta}{1 - \overline{z_l}z_l}.$$

Calculation of the residues at the points $z_1, z_2, \ldots, z_n$ lead to (5).

The condition $\varphi \in \Lambda$ implies that $\varphi(z_j) = w_j$, $j = 1, \ldots, n$, i.e.,

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \overline{z_l}z_j)(1 - \overline{z_l}z_k)b'(z_k)b'(z_l)} = w_j.$$

This is equivalent to the assertion that the matrices

$$B = (\beta_{l_k})$$

and its conjugate $\overline{B} = (\overline{\beta}_{l_k})$ where

$$\beta_{l_k} = \frac{1}{(1 - \overline{z_l}z_k)b'(z_k)}$$

are inverses of each other, i.e., $B$ and $\overline{B}$ are unitary.
Proof of the main result

**Theorem 4.** Let $f$ be a continuous function on the unit disc in the complex plane. Then the following conditions are equivalent:

1. $f$ is analytic and $f$ lies in the unit ball of $H^2$.
2. For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1. 
\]

**Proof.** We split up the proof into two lemmas.

**Lemma 2.** Let $f$ belong to the unit ball of $H^2$, and let a sequence of mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ be given. Then (3) holds.

**Proof.** Define $w_j = f(z_j)$. $f$ lies in the hyperplane $\Lambda$ and the element $\varphi$ of $\Lambda$ with minimal norm satisfies

\[
\|\varphi\|_2 \leq \|f\|_2 \leq 1. 
\]

Use of the explicit expression for $\|\varphi\|_2$ leads to (3).

**Lemma 3.** Let $f$ be continuous and assume that $f$ satisfies (3). Then $f$ is analytic and $f$ lies in the unit ball of $H^2$.

**Proof.** We apply (3) for the case $n = 1$; an easy computation shows that

\[
|f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}} 
\]

for every choice of $z \in \Delta$.

Let $0 < r < \rho < 1$, and let $z_1, z_2, z_3, \ldots$ be an enumeration of the rational points of $\Sigma_r$. For every $n$ there is a function $\varphi_n$ with

\[
\varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n, 
\]

and

\[
\|\varphi_n\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1. 
\]

Thus, $\varphi_n$ lies in the unit ball of $H^2$, and so by Lemma 2 we have for every sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$ in $\Delta$

\[
\sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\varphi_n(\zeta_k)\overline{\varphi_n(\zeta_l)}}{1 - \zeta_k \overline{\zeta_l}} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1. 
\]

It follows from (6) that

\[
|\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1 - |\zeta|^2}}, 
\]

hence the sequence $\varphi_1, \varphi_2, \ldots$ is uniformly bounded on $\Sigma_r$. Therefore, it contains a locally uniformly convergent subsequence $\varphi_{n_j}$. At the points $z_1, z_2, \ldots$ the subsequence converges to $f$. By the continuity of $f$ and the fact that $\{z_1, z_2, \ldots\}$ is dense in $\Delta_r$ we see that

\[
\lim_{n_j \to \infty} \varphi_{n_j} = f. 
\]
This shows that $f$ is analytic on $\Delta_\rho$ for all $\rho < 1$. Because of uniform convergence on $\Gamma_r$, we have
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \leq 1.
\]
Thus, $f \in H^2$ and $\|f\|_2 \leq 1$.

Lemma 2 and Lemma 3 together constitute a proof of the theorem.

**Corollary.** For $f \in H^2$ we define
\[
\nu(f) = \sup \left\{ \sum_{k=1}^n \sum_{l=1}^n \frac{f(z_k)f(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} : z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta \right\}.
\]
Then $\nu(f) = \|f\|_2^2$.

**Proof.** Assume that $\nu(f) = 1$. Then by Lemma 3 $\|f\|_2^2 \leq 1$. If $\|f\|_2^2 < \lambda^2 < 1$ for some $\lambda$, then we have $\|\frac{1}{\lambda} f\| < 1$ but $\nu(\frac{1}{\lambda} f) > 1$ which is impossible by Lemma 2.

In a similar way we can show that $\|f\|_2^2 = 1$ implies that $\nu(f) = 1$. By the homogeneity of $\nu$ and $\|\cdot\|_2$ it follows that for all $f \in H^2$: $\nu(f) = \|f\|_2^2$.

**Pick’s theorem**

As an application of our results we shall give a proof of Pick’s theorem.

Let $g$ belong to the unit ball of $H^\infty$, and let $z_1, z_2, \ldots, z_n$ be a sequence of mutually distinct points in $\Delta$. Let $w_1, w_2, \ldots, w_n$ be an arbitrary sequence of complex numbers. We consider the hyperplanes $\Lambda$ and $\Lambda_g$ where
\[
\Lambda_g = \{ f \in H^2 : f(z_j) = w_j g(z_j), j = 1, 2, \ldots, n \}.
\]

Of course, if $f \in \Delta$, then $g \cdot f \in \Delta_g$, and by Theorem 2 applied to $\Lambda_g$ we have
\[
\|g f\|_2^2 \geq \sum_{k=1}^n \sum_{l=1}^n \frac{w_k g(z_k)w_l g(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}.
\]
Let $\varphi$ be, as before, the element of $\Lambda$ with smallest norm. From $\|g\|_\infty \leq 1$ we obtain
\[
\|g \varphi\|_2 \leq \|\varphi\|_2.
\]
Combining these steps leads to
\[
\sum_{k=1}^n \sum_{l=1}^n \frac{w_k \overline{w_l}}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \geq \|\varphi\|_2^2 \geq \sum_{k=1}^n \sum_{l=1}^n \frac{w_k \overline{w_l} g(z_k)g(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)},
\]
i.e., to
\[
\sum_{k=1}^n \sum_{l=1}^n \frac{1 - g(z_k)g(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{w_k \overline{w_l}}{b'(z_k)b'(z_l)} \geq 0,
\]
and since the sequence \( w_1, w_2, \ldots, w_n \) is arbitrary, we have for all choices of \( \lambda_1, \lambda_2, \ldots, \lambda_n \),

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k) \overline{g(z_l)}}{1 - z_k \overline{z_l}} \cdot \lambda_k \overline{\lambda_l} \geq 0.
\]

By the choice \( n = 1, \lambda_1 = 1 \) we see that the converse is trivial.

References


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