CHARACTERIZATION OF CLIFFORD-VALUED HARDY SPACES
AND COMPENSATED COMPACTNESS

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Abstract. In this paper, the general Clifford $\mathbb{R}^{n;s}$-valued Hardy spaces and conjugate Hardy spaces are characterized. In particular, each function in $\mathbb{R}^{n}$-valued Hardy space can be determined by half of its function components through Riesz transform, and the explicit determining formulas are given. The products of two functions in the Hardy space give six kinds of compensated quantities, which correspond to six paracommutators, and their boundedness, compactness and Schatten-von Neumann properties are given.

1. Introduction

Clifford-valued ($\mathbb{R}^{n}$-valued) Hardy spaces have been the subject of many works, in particular, by F. Sommen, F. Brackx, R. Delanghe, J. Ryan, T. Qian, M. Mitrea, K. Gürlebeck, W. Sprößig, S. Bernstein, etc.; see [10], [11], [3], [9], [5], [4], [1], etc. In this paper, we give the general Clifford-valued ($\mathbb{R}^{n;s}$-valued) a new characterization.

To explain the Hardy space theory on the general Clifford algebras $\mathbb{R}^{n;s}$, let us go back to the classical Hardy space theory on the complex numbers $\mathbb{C}$. Denote the square-integrable complex-valued function space on the real line $\mathbb{R}$ by $L^{2}(\mathbb{R};\mathbb{C})$.

The classical Hardy space $H^{2}(\mathbb{R},\mathbb{C})$ is defined to be

$$H^{2}(\mathbb{R},\mathbb{C}) = \{ f \in L^{2}(\mathbb{R},\mathbb{C}) \mid F(x+iy) = P_{y} * f(x), \frac{\partial F}{\partial y} = 0 \},$$

where $P_{y} * f(x)$ is the Poisson integral of $f(x)$. By Fourier theory it turns out to be

$$H^{2}(\mathbb{R},\mathbb{C}) = \{ f + iHf \mid f \in L^{2}(\mathbb{R},\mathbb{R}) \} = \{ f \in L^{2}(\mathbb{R},\mathbb{C}) \mid \text{supp} \hat{f} \subset [0, +\infty) \}.$$

In this space, each complex-valued function $f + iHf$ is determined by its real component. There is a decomposition $L^{2}(\mathbb{R},\mathbb{C}) = H^{2}(\mathbb{R},\mathbb{C}) \oplus \overline{H}^{2}(\mathbb{R},\mathbb{C})$, where $\overline{H}^{2}(\mathbb{R},\mathbb{C})$ is the conjugate Hardy space.

In this paper, we generalize these to the general Clifford algebras $\mathbb{R}^{n;s}$ and give the characterizations of the general Clifford-valued Hardy spaces and conjugate Hardy spaces. In particular, $\mathbb{R}^{n}$-valued Hardy spaces are characterized, and each...
function in this kind of Hardy space is determined by half of its function components through Riesz transform. Moreover, for those representations, the explicit formulas are given. Comparing with the decomposition of square-integrable function space on the real line $R$ into the direct sum of Hardy space and conjugate Hardy space, the square-integrable Clifford algebra-valued function spaces are decomposed into the orthogonal sum of Clifford-valued Hardy and conjugate Hardy spaces.

In the classical case, taking two functions $f = f_0 + iHf_0$ and $g = g_0 + iHg_0 \in H^2(R, C)$, one has studied the product $fg = f_0g_0 - Hf_0Hg_0 + i(f_0Hg_0 + Hf_0g_0)$; both its real part and imaginary part have compensated compactness. They are called compensated quantities. A one-to-one correspondence between paracommutator $T_b(A)f$ and compensated quantity $Q_A(f, g)$ has been established in [8]. The paracommutator was studied systematically by Janson, Peetre and Peng ([6], [7]). Therefore the BMO-boundedness, VMO-compactness and Schatten-von Neumann properties of $H_b f$ or $fg$ can be read in theorems in [8]. In [13], [14], Zhijian Wu studied the product of a left monogenic function and a right monogenic function. In this paper, we shall consider the product of two left monogenic functions in Hardy spaces. This gives a nature method to obtain some examples of compensated quantities.

In §2 we give some preliminaries. In §3 we characterize the Hardy space $H^{(b)}(R^n, R_{n,s})$. By the characterization of Hardy space, we give some examples of the compensated quantities and paracommutators arising from the product of two functions in Hardy space.

2. Preliminaries

This section is an overview of some basic facts which are concerned with Clifford algebras. We set up the general formalism which will be used in the sequel (cf. [2]).

Let $V_{(n,s)} \ (0 \leq s \leq n)$ be an $n$-dimensional ($n \geq 1$) real linear space with basis $\{e_1, \cdots, e_n\}$, and provided with a bilinear form $(v, w), \ v, w \in V_{(n,s)}$, such that $(e_i, e_j) = 0, i \neq j$; $(e_i, e_i) = 1, i = 1, \cdots, s$; and $(e_i, e_i) = -1, i = s + 1, \cdots, n$.

If $v = \sum_{i=1}^{n} v_i e_i \in V_{(n,s)}$, then the associated quadratic form reads

$$(v, v) = \sum_{i=1}^{s} v_i^2 - \sum_{i=s+1}^{n} v_i^2.$$ Consider the $2^n$-dimensional real linear space $C(V_{(n,s)})$ with basis

$$\{e_A = e_{h_1} \cdots e_{h_r} : A = (h_1, \cdots, h_r) \in PN, 1 \leq h_1 < \cdots < h_r \leq n\}$$

where $N$ stands for the set $\{1, \cdots, n\}$ and $e_0 = e_0$.

Now a product on $C(V_{(n,s)})$ is defined by

$$e_A e_B = (-1)^{n((A \cap B) \setminus S)}(-1)^{p(A,B)}e_{A \Delta B}$$

where $S$ stands for the set $\{1, \cdots, n\}$, $n(A) = |A|,$

$$p(A, B) = \sum_{j \in B} p(A, j), \ p(A, j) = \{i \in A : i > j\}$$

and the sets $A$, $B$ and $A \Delta B$ are ordered in the prescribed way. It is easy to check that $C(V_{n,s})$ turned into a linear, associative, but non-commutative algebra over $R$; it is called the universal Clifford algebra over $V_{(n,s)}$. 

In this paper, we use the real Clifford algebra $R_{n,s}$, which means that in $\mathbb{R}^n$ the basis $\{e_1, \ldots, e_n\}$ satisfies $e_ie_i = 1, i = 1, \ldots, s$, and $e_ie_i = -1, i = s + 1, \ldots, n$. When $s = 0$, we simply set $R_{n,s} = R_n$.

If $f$ is Lipschitz continuous, then at any point of differentiability $x \in U$ ($U$ is an open subset of $\mathbb{R}^{n+1}$) of $f(x) = \sum f_I(x)e_I$, we have
\[
(Df)(x) = \sum_{j=0}^{n} \frac{\partial f_I}{\partial x_j}(x)e_je_I, \quad (fD)(x) = \sum_{j=0}^{n} \frac{\partial f_I}{\partial x_j}(x)e_Ie_j,
\]
\[
D^k f = D(D^{k-1}) f, \quad fD^k = (fD^{k-1}) D.
\]
We shall call $f$ left $k$-monogenic (right $k$-monogenic, or two-sided $k$-monogenic, respectively) if $D^k f = 0$ ($fD^k = 0$, or $D^k f = fD^k = 0$, respectively); $D$ is called the Cauchy-Riemann operator.

3. Hardy space $H^{(k)}(\mathbb{R}^n, R_{n,s})$

**Definition 1.** Let $R_{n,s}$ be a real Clifford algebra. A Clifford module $L^2(\mathbb{R}^n, R_{n,s})$ is defined to be
\[
L^2(\mathbb{R}^n, R_{n,s}) = \{ f : \mathbb{R}^n \to R_{n,s}, f(x) = \sum_I f_I(x)e_I \mid f_I \in L^2(\mathbb{R}^n, R), \forall I \}.
\]
The Hardy space $H^{(k)}(\mathbb{R}^n, R_{n,s})$ is defined to be
\[
H^{(k)}(\mathbb{R}^n, R_{n,s}) = \{ f \in L^2(\mathbb{R}^n, R_{n,s}) \mid F(x_0, x) = P_{x_0} * f(x), x_0 > 0, \quad D^{k-1} F(x, x_0) \neq 0, \quad D^k F(x_0, x) = 0 \},
\]
where $k \in \mathbb{N}, 0 \leq s \leq n$, and $P_{x_0}$ is the Poisson kernel.

In these Hilbert spaces, the inner product is denoted by
\[
(f, g) = \int_{\mathbb{R}^n} \sum_I f_I(x)g_I(x)dx,
\]
where $f(x), g(x) \in L^2(\mathbb{R}^n, R_{n,s})$.

In detail, for arbitrary $f(x) = \sum f_I(x)e_I \in H^{(1)}(\mathbb{R}^n, R_{n,s})$, let
\[
F(x_0, x) = P_{x_0} * f(x) = \sum_I (P_{x_0} * f_I(x))e_I
\]
and
\[
F_I(x_0, x) = P_{x_0} * f_I(x), \quad \partial_j F_I(x, x_0) = \frac{\partial F_I(x, x_0)}{\partial x_j}.
\]
Then we have
\[
F(x_0, x) = \sum_I F_I(x_0, x)e_I, \quad DF(x_0, x) = \sum_I \sum_{j=0}^{n} \partial_j F_Ie_je_I.
\]
The latter can be rewritten as
\[
DF(x_0, x) = (e_0, e_1, \ldots, e_n, \ldots, e_{(1, 2, \ldots, n)})A(F_0, F_1, \ldots, F_n, \ldots, F_{(1, 2, \ldots, n)})^T,
\]
where $e_0, e_1, \ldots, e_n, \ldots, e_{(1, 2, \ldots, n)}$ is an ordered basis of $R_{n,s}$. It is arranged according to $I$ in lexicographical order. $A$ is a $2^n \times 2^n$ matrix uniquely determined when the order of this basis is set.
Example 1. $H^{(1)}(R^2, R_{2,1}) = \{ f \in L^2(R^2, R_{2,1}) \mid F(x_0, x) = P_{x_0} \ast f(x), x_0 > 0, DF(x_0, x) = 0 \}$, $DF = (e_0, e_1, e_2, e_{12})A(F_0, F_1, F_2, F_{12})^T$, where

$$A = \begin{bmatrix} \partial_0 & 0_1 & -\partial_2 & 0 \\ \partial_1 & 0_0 & 0 & \partial_2 \\ 0 & 0 & 0 & \partial_1 \\ -\partial_2 & \partial_1 & \partial_0 & 0 \end{bmatrix}.$$  

3.1. Characterization of $H^{(k)}(R^n, R_{n,s})$.

Theorem 1. In the Hardy space $H^{(1)}(R^n, R_n) = \{ f \in L^2(R^n, R_n) \mid F(x_0, x) = P_{x_0} \ast f(x), x_0 > 0, DF(x_0, x) \neq 0 \}$, each function $f = \sum_{j} f_j e_I$ is determined by its $2^{n-1}$ linearly independent function components. Explicitly for $\Omega = \{ A = (h_1, \cdots, h_r) \in \mathbb{R}^n, 1 \leq h_1 < \cdots < h_r \leq n \}$, denote $\Lambda = \{ A = (h_1, h_2, \cdots, h_{2k+1}), 1 \leq h_1 < \cdots < h_{2k+1} \leq n \}$ when $n = 2m$, $k = 0, 1, \cdots, m - 1$, when $n = 2m + 1$, $k = 0, 1, \cdots, m$. Then each $f_j$ can be represented by the Riesz transforms of all $f_k$; $f_k$ can also be represented by the Riesz transforms of all $f_j$, where $J \in \Lambda, K \in \Omega \setminus \Lambda$.

Proof. For convenience, we consider $DF$ in its Fourier transform

$$\widehat{DF}(x_0, \xi) = \sum_{l} \sum_{j=0}^{n} \partial_{j} \widehat{F}_l(x_0, \xi)e_j e_I.$$  

Since $\partial_{0} \widehat{F}_l(x_0, \xi) = -|\xi| \widehat{F}_l(x_0, \xi)$, $\partial_{j} \widehat{F}_l(x_0, \xi) = i\xi_j \widehat{F}_l(x_0, \xi)$, $j = 1, 2, \cdots, n$, for all $I$, then

$$\widehat{DF}(x_0, \xi) = \sum_{l} (-|\xi| \widehat{F}_l e_0 + \sum_{j=1}^{n} i\xi_j \widehat{F}_l e_j) e_I \quad (1)$$

$$= (e_0, e_1, \cdots, e_I, \cdots, e_{(1,2,\cdots,n)})B(\widehat{F}_0, \widehat{F}_1, \cdots, \widehat{F}_I, \cdots, \widehat{F}_{(1,2,\cdots,n)})^T.$$  

1) Let us prove that $B$ is a Hermitian matrix. Denote

$$B = (u_0, u_1, \cdots, u_I, \cdots, u_{(1,2,\cdots,n)})^T = (v_0, v_1, \cdots, v_I, \cdots, v_{(1,2,\cdots,n)}),$$

where $u_0, u_1, \cdots, u_I, \cdots, u_{(1,2,\cdots,n)}$ are the rows of $B$, and $v_0, v_1, \cdots, v_I, \cdots, v_{(1,2,\cdots,n)}$ are the columns of $B$.

Let

$$\overrightarrow{u_0} = -|\xi|e_0 - \sum_{j=1}^{n} i\xi_j e_j, \quad \overrightarrow{u_I} = \overrightarrow{u_0} e_I, \quad \overrightarrow{u_0} = -|\xi|e_0 + \sum_{j=1}^{n} i\xi_j e_j, \quad \overrightarrow{u_I} = \overrightarrow{u_0} e_I,$$

for all $I$.

We will prove that $u_I$ are the coordinates of $\overrightarrow{u_I}$ with respect to the basis $\{ e_0, e_1, \cdots, e_{(1,2,\cdots,n)} \}$, and $v_I$ are the transpose of $u_I$. Here $\overrightarrow{u_I}$ are the coordinates of $\overrightarrow{u_I}$ with respect to the basis, $I = (0), (1), \cdots, (1,2,\cdots,n)$.

In fact, let $\lambda = (\widehat{F}_0, \widehat{F}_1, \cdots, \widehat{F}_I, \cdots, \widehat{F}_{(1,2,\cdots,n)})^T$. From (1), we have

$$\widehat{DF} = (e_0, e_1, \cdots, e_{(1,2,\cdots,n)}) (u_0 \lambda, u_1 \lambda, \cdots, u_I \lambda, \cdots, u_{(1,2,\cdots,n)} \lambda)^T.$$  

Obviously, $u_I \lambda = u_I (\widehat{F}_0, \widehat{F}_1, \cdots, \widehat{F}_I, \cdots, \widehat{F}_{(1,2,\cdots,n)})^T$ is the coefficient of $e_I$ in $\widehat{DF}$, for each $I$. 


On the other hand, for each \( j = 1, 2, \ldots, n \), there is a unique \( I_j \) satisfying \( e_j e_I = (-1)^j e_I \).

From (1), the \( e_I \) component in \( \mathcal{D}F \) is

\[
-|\xi|\hat{F}_I e_I + \sum_{j=1}^n i \xi_j \hat{F}_{I_j} e_{I_j} = (-|\xi|\hat{F}_I + \sum_{j=1}^n (-1)^j i \xi_j \hat{F}_{I_j}) e_I,
\]

so

\[
(2) \quad u_I \lambda = -|\xi|\hat{F}_I + \sum_{j=1}^n (-1)^j i \xi_j \hat{F}_{I_j}
\]

and

\[
(3) \quad \overline{u_I} = -|\xi|e_0 e_I - \sum_{j=1}^n i \xi_j e_I = -|\xi|e_I + \sum_{j=1}^n (-1)^j i \xi_j e_I.
\]

From (2) and (3), we obtain that \( u_I \) is the component of \( \overline{u_I} \) with respect to the basis \( \forall I \).

Now to prove that \( v_I \) is the transpose of \( w_I \), i.e. to prove

\[
(e_0, e_1, \ldots, e_{(1,2,\ldots,n)}) v_I = \overline{w_I}.
\]

Let \( \lambda = (0, 0, \cdots, \hat{f}, \cdots, 0)^T \) be the vector in \( R^{2^n} \) with all zeros except for an \( \hat{f} \) (\( \hat{f} \in L^2(R^n, R) \)) in the \( I \)th entry. From (1), we have

\[
(-|\xi|e_0 e_I + \sum_{j=1}^n i \xi_j e_I) \hat{f} = (e_0, e_1, \ldots, e_I, \cdots, e_{(1,2,\ldots,n)}) v_I \hat{f}.
\]

This is \( \overline{w_I} = (e_0, e_1, \cdots, e_I, \cdots, e_{(1,2,\ldots,n)}) v_I \), so the transpose of \( w_I \) is precisely \( v_I \).

These cases and

\[
(4) \quad \overline{w_I} = -|\xi|e_0 e_I + \sum_{j=1}^n (-1)^j i \xi_j e_{I_j},
\]

\[
\overline{w_I} = -|\xi|e_1 e_I - \sum_{j=1}^n (-1)^j i \xi_j e_{I_j}
\]

show that \( u_I \) is the conjugate transpose of \( v_I \) for each \( I \). So \( B^T = B \), i.e. \( B \) is a Hermitian matrix.

2) Let us prove \( BB = -2|\xi|B \). By the properties of Fourier transform, we know that the Cauchy-Riemann operator \( D \) corresponds to the operator \( \hat{D} = -|\xi| + \sum_{j=1}^n i \xi_j e_j \), and the latter corresponds to \( B \), so we get

\[
(5) \quad BB = -2|\xi|B.
\]

3) We are going to prove that \( r_B = 2^{n-1} \). From (4), we get

\[
(B + 2|\xi|E_{2^n}) B = -B^T B = 0,
\]
i.e. $BB^T = B^T B = 0$. Therefore, $-2|\xi|$ and 0 are eigenvalues of $B$. Denote by $V_0, V_{-2|\xi|}$ the eigenspace of $B$ associated with 0, $-2|\xi|$, respectively; $V_0'$ is the eigenspace of $B^T$ associated with 0. Then

$$\text{dim} V_{-2|\xi|} = \text{dim} V_0' \geq r_B.$$  

Applying the Plus Nullity Theorem, we get

$$r_B + \text{dim} V_0 = 2^n$$

and

$$r_B^T + \text{dim} V_0' = 2^n.$$

Therefore

$$\text{dim} V_0 = \text{dim} V_0' = \text{dim} V_{-2|\xi|}.$$  

Since $B$ is a hermitian matrix, $B$ can be diagonalized. If $B$ has other eigenvalues different from $-2|\xi|$ and 0, let $d$ be the sum of the dimensions of their eigenspaces. Then

$$\text{dim} V_{-2|\xi|} + \text{dim} V_0 + d = 2^n.$$  

Together with (6) and (7), we have

$$2^n = \text{dim} V_{-2|\xi|} + \text{dim} V_0 + d \geq 2^n + d > 2^n,$$

which is a contradiction. So the eigenvalues of $B$ are $-2|\xi|$ and 0, and we get

$$\text{dim} V_{-2|\xi|} = r_B.$$  

Thus from $\text{dim} V_{-2|\xi|} = \text{dim} V_0'$ and (7), we know $r_B = 2^{n-1}$. Therefore each $f$ can be determined by $2^{n-1}$ of its function components.

In fact, the rows $u_J$ are the maximum linearly independent in all rows of $B$, where

$$J \in \Lambda = \{ A = (h_1, h_2, \ldots, h_{2k+1}), 1 \leq h_1 < \cdots < h_{2k+1} \leq n \}.$$  

This can be deduced from $u_J = -|\xi| e_J - \sum_{j=1}^n i \xi_j e_j$.  

If

$$\sum_{J \in \Lambda} a_J u_J = 0,$$

there must be an $a_J = 0$. Hence the $u_J$ are linearly independent and, together with $r_B = 2^{n-1}$, we know all the rows of $B$ can be determined by these $u_J$.

On the other hand, for each $j = 1, \cdots, n$, there is a $J_j \in \Omega \setminus \Lambda$, which uniquely satisfies $e_J e_J = (-1)^{|J_j|} J_j$, so

$$u_J = -|\xi| e_J - \sum_{j=1}^n (-1)^{|J_j|} i \xi_j e_J.$$  

Solving the equations $u_J (\hat{F}_0, \hat{F}_1, \cdots, \hat{F}(1, \cdots, n))^T = 0$, we get

$$f_J(x) = -\sum_{j=1}^n (-1)^{|J_j|} R_j f_{J_j},$$  

where $J \in \Lambda$, i.e. each $f_J$ can be represented by the Riesz transforms of all $f_K$, where $J \in \Lambda, K \in \Omega \setminus \Lambda$. Similarly, $f_K$ can be represented by the Riesz transforms of all $f_J$. The proof is finished.  

\qed
Remark. From the proof of the above theorem, we can write each function \( f \in H^{(1)}(R^n, R_n) \) as follows:

1) When \( n \) is even,

\[
f = \sum_{l} f_l e_l = \left( f_0 + R_1 f_0 e_1 + R_2 f_0 e_2 + \cdots + R_n f_n e_n \right) + \left( f_{12} + R_1 f_{12} e_1 + R_2 f_{12} e_2 + \cdots + R_n f_{12n} e_{12} \right) + \cdots + \left( f_{n-1.n} + R_1 f_{n-1.n} e_1 + R_2 f_{n-1.n} e_2 + \cdots + R_n f_{n-1.n} e_{n-1.n} \right) + \cdots + \left( f_{12-n} + R_1 f_{12-n} e_1 + R_2 f_{12-n} e_2 + \cdots + R_n f_{12-n} e_{12-n} \right).
\]

2) When \( n \) is odd,

\[
f = \sum_{l} f_l e_l = \left( f_0 + R_1 f_0 e_1 + R_2 f_0 e_2 + \cdots + R_n f_n e_n \right) + \left( f_{12} + R_1 f_{12} e_1 + R_2 f_{12} e_2 + \cdots + R_n f_{12n} e_{12} \right) + \cdots + \left( f_{n-1.n} + R_1 f_{n-1.n} e_1 + R_2 f_{n-1.n} e_2 + \cdots + R_n f_{n-1.n} e_{n-1.n} \right) + \cdots + \left( f_{2-n} + R_1 f_{2-n} e_1 + R_2 f_{2-n} e_2 + \cdots + R_n f_{2-n} e_{2-n} \right).
\]

In fact, we can represent the function by using other function components.

Example 2. The Hardy space \( H^{(1)}(R^1, R_1) = \{ f \in L^2(R^1, R_1) \mid F(x_0, x) = P_{x_0} f(x), x_0 > 0, \text{ } DF(x_0, x) = 0 \} \) is exactly the classical Hardy space. For each \( f \in H^{(1)}(R^1, R_1) \), it can be represented by the Hilbert transform (when \( n = 1 \), there is only one Riesz transform, namely the Hilbert transform) of one of its function components.

Example 3. For each \( f \in H^{(1)}(R^3, R_3) \),

\[
\begin{align*}
\tilde{D} F(x_0, \xi) &= (\xi_0, \xi_1, \xi_2, \xi_3, \xi_{(1,2)}, \xi_{(1,3)}, \xi_{(2,3)}, \xi_{(1,2,3)}) \\
B(\tilde{F}_0, \tilde{F}_1, \tilde{F}_2, \tilde{F}_{(1,2)}, \tilde{F}_{(1,3)}, \tilde{F}_{(2,3)}, \tilde{F}_{(1,2,3)})^T
\end{align*}
\]

where

\[
B = \begin{bmatrix}
u_0 & u_1 & u_2 & u_3 & u_{(1,2)} & u_{(1,3)} & u_{(2,3)} & u_{(1,2,3)} \\
-\xi_1 & -\xi_2 & -i\xi_3 & 0 & 0 & 0 & 0 & 0 \\
i\xi_1 & -\xi_2 & 0 & -\xi_3 & 0 & 0 & 0 & 0 \\
i\xi_2 & 0 & -\xi_1 & 0 & 0 & 0 & 0 & 0 \\
i\xi_3 & 0 & 0 & -\xi_2 & 0 & 0 & 0 & 0 \\
0 & -i\xi_2 & i\xi_1 & 0 & 0 & 0 & 0 & 0 \\
0 & -i\xi_3 & 0 & i\xi_1 & 0 & 0 & 0 & 0 \\
0 & 0 & -i\xi_3 & i\xi_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i\xi_3 & -i\xi_2 & i\xi_1 & -\xi_1 \\
0 & 0 & 0 & 0 & i\xi_3 & -i\xi_2 & i\xi_1 & -\xi_1
\end{bmatrix}.
\]
By using Theorem 1, 
\[ f_1(x) = R_1f_0(x) + R_2f_{(1,2)}(x) + R_3f_{(1,3)}(x), \]
\[ f_2(x) = R_2f_0(x) - R_1f_{(1,2)}(x) + R_3f_{(2,3)}(x), \]
\[ f_3(x) = R_3f_0(x) - R_1f_{(1,3)}(x) - R_2f_{(2,3)}(x), \]
\[ f_{(1,2,3)}(x) = R_3f_{(1,2)}(x) - R_2f_{(1,3)}(x) + R_1f_{(2,3)}(x). \]

In fact, \( f_0, f_{(1,2)}, f_{(1,3)}, f_{(2,3)} \) can be represented by the Riesz transform of other function components.

**Corollary 1.** In the Hardy space \( H^{(k)}(R^n, R_m) = \{ f(x) \in L^2(R^n, R_m) | F(x_0, x) = P_{x_0} \ast f(x), x_0 > 0, D^{k-1}F(x_0, x) \neq 0, D^kF(x_0, x) = 0 \} \), each \( f \) is determined by \( 2^{n-1} \) linearly independent function components.

**Proof.** Let \( F(x_0, x) = \sum_j F_j(x_0, x)e_I \) and \( DF(x_0, x) = \sum_{j=0}^n \partial_jF_Ie_I \). For convenience, we consider \( DF \) in its Fourier transform. Using the formula
\[ \widehat{DF}(x_0, \xi) = (e_0, e_1, \cdots, e_I, \cdots, e_{(1,2,\cdots,n)})B(\widehat{F}_0, \widehat{F}_1, \cdots, \widehat{F}_n)^T \]
k times, we obtain
\[ \widehat{D^kF}(x_0, \xi) = (e_0, e_1, \cdots, e_I, \cdots, e_{(1,2,\cdots,n)})B^k(\widehat{F}_0, \widehat{F}_1, \cdots, \widehat{F}_n)^T. \]

Now we prove \( r_{B^k} = r_B = 2^{n-1} \). From Theorem 1, we know \( BB = -2|\xi|B \). Therefore
\[ B^k = -2|\xi|B^{k-1} = (-2|\xi|^2B^{k-2} = \cdots = (-2|\xi|)^{k-1}B. \]

Then \( r_{B^k} = 2^{n-1} \). This ends the proof. \( \square \)

**Theorem 2.** \( H^{(1)}(R^n, R_{m,n}) = \{ 0 \} \).

**Proof.** For convenience, we still consider \( DF \) in its Fourier transform. Denote \( B = (u_0, u_1, \cdots, u_I, \cdots, u_{(1,2,\cdots,n)})^T = (v_0, v_1, \cdots, v_I, \cdots, v_{(1,2,\cdots,n)}) \). Let \( \overline{u}_0 = -|\xi|v_0 + \sum_{j=1}^n i\xi_j e_j, \overline{u}_I = \overline{u}_0 e_I, \) and \( \overline{u}_0 = -|\xi|v_0 + \sum_{j=1}^n i\xi_j e_j, \overline{u}_I = \overline{u}_0 e_I, \) for all \( I \).

As in Theorem 1, we can prove that \( u_I = 0 \) with respect to the basis \( \{ e_0, e_1, \cdots, e_{(1,2,\cdots,n)} \} \), and \( v_I = 0 \). Then we get
\begin{enumerate}
\item \( B \) is a symmetric matrix.
\item \( B^2 = 2|\xi|^2E_{2^n} \).
\end{enumerate}

The method to prove this is similar to Theorem 1.

Therefore \( r_B = 2^n \), i.e. \( F(x_0, x) = 0 \). Thus \( H^{(1)}(R^n, R_{m,n}) = \{ 0 \} \). \( \square \)

**Corollary 2.**
\[ H^{(k)}(R^n, R_{m,n}) = \{ 0 \}, k \in \mathbb{N}. \]

More generally, we have the following theorem.

**Theorem 3.** For each \( f \in H^{(1)}(R^n, R_{m,n}) = \{ f \in L^2(R^n, R_{m,n}) | F = P_{x_0} \ast f, x_0 > 0, DF(x_0, x) = 0 \} \), if \( \sum_{j=1}^n x_j^2 \neq 0 \) or \( \sum_{j=1}^n x_j^2 = 0 \), then \( f = 0 \); otherwise, \( f \) can be determined by \( 2^{n-1} \) linearly independent function components.
Corollary 4. In the conjugate Hardy space \( B \), using a method similar to that in Theorem 1, we get \( \sum_{j=1}^{s} \xi_j^2 E_{2^n} \). If \( \sum_{j=1}^{s} \xi_j^2 \neq 0 \), then \( r_B = 2^n \). Therefore \( f = 0 \). If \( \sum_{j=1}^{s} \xi_j^2 = 0 \), then \( B \) is a hermitian matrix. Using Theorem 1, we have \( r_B = 2^{n-1} \). Therefore \( f \) is determined by \( 2^{n-1} \) linearly independent function coefficients.

\[ B = (u_0, u_1, \cdots, u_I, \cdots, u_{(12\cdots n)})^T = (v_0, v_1, \cdots, v_I, \cdots, v_{(12\cdots n)}) \]

Corollary 3. For each \( f \in H^{(k)}(R^n, R_{n,s}) = \{ f \in L^2(R^n, R_{n,s}) \mid F = P_{x_0} * f, x_0 > 0, D^{k-1}F(x_0, x) \neq 0, D^{k}F(x_0, x) = 0 \} \), if \( \sum_{j=1}^{s} x_j^2 \neq 0 \), then \( f = 0 \); otherwise, \( f \) can be determined by \( 2^{n-1} \) linearly independent function components.

Similarly, we denote the conjugate Hardy space \( \overline{H}^{(k)}(R^n, R_{n,s}) \) by

\[ \overline{H}^{(k)}(R^n, R_{n,s}) = \{ f \in L^2(R^n, R_{n,s}) \mid \overline{F}(x_0, x) = P_{x_0} * f(x), x_0 > 0, \overline{D}^{k-1}F(x_0, x) \neq 0, \overline{D}^{k}F = 0 \} \]

where \( k \in \mathbb{N}, 0 \leq s \leq n \).

From the above theorems, we have

Corollary 4. In the conjugate Hardy space \( \overline{H}^{(k)}(R^n, R_n) = \{ f(x) \in L^2(R^n, R_n) \mid F(x_0, x) = P_{x_0} * f(x), x_0 > 0, \overline{D}^{k-1}F(x_0, x) \neq 0, \overline{D}^{k}F(x_0, x) = 0 \} \), each \( f \) is determined by \( 2^{n-1} \) linearly independent function components.

Corollary 5. \( \overline{H}^{(k)}(R^n, R_{n,n}) = \{0\} \).

F. Sommen proved a decomposition of \( L^2(R^n, R_n) \) first in [12]. The inner product he used was Clifford algebra-valued. Here we can get the orthogonal decomposition of \( L^2(R^n, R_n) \) by using our characterization of Hardy space and conjugate Hardy space:

\[ L^2(R^n, R_n) = H^{(1)}(R^n, R_n) \oplus \overline{H}^{(1)}(R^n, R_n) \]

Now we give a natural method to obtain some compensated quantities. For each \( f, g \in H^{(1)}(R^n, R_n) \), for convenience, we assume \( n \) is an even number. By using [5], in the product \( fg \), we have

\[ Rfg \mid \mid e_1 \]

\[ = (f_0 + R_1f_1 + R_2f_2 + \cdots + R_nf_n) (g_1 + R_1g_1 + R_2g_2 + \cdots + R_ng_n) e_1 \]

\[ = ([f_0g_1 - \sum_{j=1}^{n} R_j f_j g_1] + \cdots + \sum_{j,k=1,j<k}^{n} (R_j f_j R_k g_1 - R_k f_j g_1) e_{jk}] e_1 \]
and

\[
Rf_1e_1Rg_1e_j
= (f_1 + R_1f_1e_1 + R_2f_1e_2 + \cdots + R_nf_1e_n)e_j
\quad \cdot (g_1 + R_1g_1e_1 + R_2g_1e_2 + \cdots + R_ng_1e_n)e_j
\]

\[
= (f_1 + R_1f_1e_1 + R_2f_1e_2 + \cdots + R_nf_1e_n)
\quad \cdot (g_1 - \sum_{j \in I} R_jg_je_j + \sum_{j \notin I} R_jg_je_j)e_1e_J
\]

\[
= [(f_1g_1 + \sum_{j \in I} R_jf_1R_jg_1 - \sum_{j \notin I} R_jf_1R_jg_1)
\quad + \sum_{j \in I} (g_1R_jf_1 - f_1R_jg_1)e_j
\quad + \sum_{k \notin I} (g_1R_kf_1 + f_1R_kg_1)e_k
\quad + \sum_{j, k \notin I, j < k} (R_jf_1R_kg_1 + R_kf_1R_jg_1)e_{jk}
\quad + \sum_{j, k \notin I, j < k} (R_jf_1R_kg_1 - R_kf_1R_jg_1)e_{jk}
\quad - \sum_{j, k \notin I, j < k} (R_jf_1R_kg_1 - R_kf_1R_jg_1)e_{jk}]e_1e_J,
\]

where \( I \in \Omega \setminus \Lambda, \ I \neq \{0\} \).

If we denote

\[
Q_0(u, v) = uv - \sum_{j=1}^n R_juR_jv, \quad Q_j(u, v) = uR_jv - R_juv,
\]

\[
Q_{j,k}(u, v) = R_juR_kv - R_kvR_jv,
\]

\[
P_0(u, v) = uv + \sum_{j \in I} R_juR_jv - \sum_{k \notin I} R_kvR_kv, \quad P_j(u, v) = R_juv - uR_jv,
\]

\[
P_{j,k}(u, v) = R_juR_kv + R_kvR_jv,
\]

where \( j, k = 1, \cdots, n, \ j < k \), for real-valued functions \( u, v \in L^2(R^n) \), then in the product \( fg \), there are six kinds of components as above.

For a real-valued symbol function \( b \), write

\[
\langle Q_j(u, v), b \rangle = \langle H_j^b(u, v), \rangle, \quad \langle P_j(u, v), b \rangle = \langle T_j^b(u, v), \rangle, \quad \text{where} \ j = 0, 1, \cdots, n,
\]

\[
\langle Q_{j,k}(u, v), b \rangle = \langle H_{j,k}^b(u, v), \rangle, \quad \langle P_{j,k}(u, v), b \rangle = \langle T_{j,k}^b(u, v), \rangle,
\]

where, \( j, k = 1, 2, \cdots, n, \ j < k \).

Denote the corresponding Fourier kernels of \( H_j^b \) by \( A_j(\xi, \eta) \), and the Fourier kernels of \( H_{j,k}^b \) by \( A_{j,k}(\xi, \eta) \). Then we obtain

\[
A_0(\xi, \eta) = 1 - \xi \cdot \eta |\xi| |\xi| = \frac{1}{2} (\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|})^2, \quad A_j(\xi, \eta) = -i (\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}),
\]
where \( j = 1, \ldots, n, \)
\[
A_{j,k}(\xi, \eta) = \frac{\eta_j \xi_k - \eta_k \xi_j}{|\xi||\eta|}, \quad j, k = 1, \ldots, n, \ j < k.
\]

Similarly, denote the corresponding Fourier kernels of \( T^j_b, T^{j,k}_b \) by \( B_j(\xi, \eta), \ j = 0, 1, \ldots, n, \) and \( B_{j,k}(\xi, \eta), \) respectively. Then
\[
B_0(\xi, \eta) = 1 + \sum_{j \in I} \frac{\xi_j \eta_j}{|\xi||\eta|} - \sum_{k \notin I} \frac{\xi_k \eta_k}{|\xi||\eta|}, \quad B_j(\xi, \eta) = i \left( \frac{\xi_j}{|\xi|} + \frac{\eta_j}{|\eta|} \right), \ j = 1, \ldots, n,
\]
\[
B_{j,k}(\xi, \eta) = \frac{\eta_j \xi_k + \eta_k \xi_j}{|\xi||\eta|}, \quad j, k = 1, \ldots, n, \ j < k.
\]

Therefore we have six kinds of kernels: \( A_0; A_j, j = 1, \ldots, n; A_{j,k}, j, k = 1, \ldots, n, \ j < k; B_0; B_j, j = 1, \ldots, n; \) and \( B_{j,k}, j, k = 1, \ldots, n, j < k.\)

Using theorems in \( [8], \) we get \( \langle Q_0(u, v), b \rangle \) is a compensated quantity which belongs to the real Hardy space \( H^1(R^2), \) and its corresponding paracommutator is bounded if and only if \( b \in BMO(R^2). \) It is compact if and only if \( b \in \text{VMO}(R^2); \) for \( \frac{3}{2} = 1 < p < \infty, \) it belongs to \( S_p \) if and only if \( b \in B_p(R^2); \) for \( 0 < p \leq \frac{3}{2} = 1, \) it belongs to \( S_p \) if and only if \( b \) is a polynomial.

\[
\langle Q_j(u, v), b \rangle, \ j = 1, \ldots, n, \ \text{are compensated quantities which belong to the real Hardy space } H^1(R^2), \ \text{and their corresponding paracommutators are bounded if and only if} \ b \in \text{BMO}(R^2). \ \text{They are compact if and only if} \ b \in \text{VMO}(R^2); \ \text{for} \ 2 < p < \infty, \ \text{they belong to} \ S_p \ \text{if and only if} \ b \in B_p(R^2); \ \text{for} \ 0 < p \leq 2, \ \text{they belong to} \ S_p \ \text{if and only if} \ b \text{ is a polynomial.}
\]

Similarly \( Q_{j,k}, j, k = 1, \ldots, n, \ j < k, \) have the same properties. The quantities \( P_j(u, v), j = 0, 1, \ldots, n, \ P_{j,k}(u, v), j, k = 1, \ldots, n, j < k, \) are not compensated, they belong to \( L^1(R^2) \) only. Their corresponding paracommutators are bounded if \( b \in L^\infty(R^2), \) and can never be compact unless \( b \equiv 0.\)

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