BLOW-UP FOR DEGENERATE PARABOLIC EQUATIONS WITH NONLOCAL SOURCE

YOUPENG CHEN, QILIN LIU, AND CHUNHONG XIE

(Communicated by David S. Tartakoff)

ABSTRACT. This paper deals with the blow-up properties of the solution to the degenerate nonlinear reaction diffusion equation with nonlocal source
\[ x^q u_t - (x^\gamma u_x)_x = \int_0^a u^p dx \quad \text{in} \quad (0, a) \times (0, T) \]
subject to the homogeneous Dirichlet boundary conditions. The existence of a unique classical nonnegative solution is established and the sufficient conditions for the solution exists globally or blows up in finite time are obtained. Furthermore, it is proved that under certain conditions the blow-up set of the solution is the whole domain.

1. Introduction

Let \( T \leq +\infty, a, q, \gamma \) and \( p \) be constants with \( T > 0, a > 0, \gamma \in [0, 1), |q| + \gamma \neq 0 \) and \( p > 1 \). Let \( D = (0, a) \) and \( \Omega = D \times (0, t) \), and let \( \overline{D} \) and \( \overline{\Omega} \) be their respective closures. We consider the following initial boundary value problem of the degenerate nonlinear reaction diffusion equation with nonlocal source:
\[
\begin{align*}
  x^q u_t - (x^\gamma u_x)_x &= \int_0^a u^p dx, \quad (x, t) \in (0, a) \times (0, T), \\
  u(0, t) &= u(a, t) = 0, \quad t \in (0, T), \\
  u(x, 0) &= u_0(x), \quad x \in [0, a],
\end{align*}
\]
where \( u_0(x) \in C^{2+\alpha}(\overline{D}) \) for some constant \( \alpha \in (0, 1), u_0(0) = u_0(a) = 0, u_0(x) \geq 0 \). Since \( |q| + \gamma \neq 0 \), the coefficients of \( u_t, u_x \) and \( u_{xx} \) may tend to 0 or \( +\infty \) as \( x \) tends to 0, we can regard the equation as degenerate and singular.

Floater \cite{Floater2002} and Chan & Liu \cite{ChanLiu2003} investigated the blow-up properties of the following degenerate parabolic problem:
\[
\begin{align*}
  x^q u_t - u_{xx} &= u^p, \quad (x, t) \in (0, a) \times (0, T), \\
  u(0, t) &= u(a, t) = 0, \quad t \in (0, T), \\
  u(x, 0) &= u_0(x), \quad x \in [0, a],
\end{align*}
\]
where \( q > 0 \) and \( p > 1 \). Under certain conditions on the initial datum \( u_0(x) \), Floater \cite{Floater2002} proved that for the case \( 1 < p \leq q + 1 \), if the solution \( u(x, t) \) of (1.2) blows up in finite time, then it blows up at the boundary \( x = 0 \). This contrasts with a result of Friedman and Mcleod \cite{FriedmanMcleod1975}, who showed that for the case \( q = 0 \), the
blow-up set of the solution \( u(x, t) \) of (1.2) is a proper compact subset of \( D \). The motivation for studying problem (1.2) comes from Ockendon’s model (see [13]) for the flow in a channel of a fluid whose viscosity is temperature dependent

\[
xu_t = u_{xx} + e^u,
\]

where \( u \) represents the temperature of the fluid. Floater in [7] approximated \( e^u \) by \( u^p \) and considered the results of [7] to the following degenerate quasilinear parabolic equation:

\[
x^q u_t = (u^m)_{xx} + u^p
\]

subject to homogeneous Dirichlet conditions in the critical exponent case \( q = (p - 1)/m \), where \( q > 0, m \geq 1 \) and \( p > 1 \). They pointed out that the general classification of blow-up solutions for the degenerate equation (1.4) stays the same for the quasilinear equation

\[
u_t = (u^m)_{xx} + u^p,
\]

see [2] and [15, Chapter 4]. Chan & Liu in [4] continued to study problem (1.2) for the case \( p > q + 1 \). They proved that under certain conditions \( x = 0 \) is not a blow-up point and the blow-up set is a proper compact subset of \( D \).

In this paper, we continue to consider (1.2) with the reaction term \( u^p \) replaced by \( \int_0^a u^p \, dx \) and investigate the effect of the singularity, degeneracy and nonlocal reaction on the behavior of the solution of (1.1). The difficulties are the establishment of the corresponding comparison principle and the construction of an upper solution of (1.1). It is different from [4] and [7]; we prove that under certain conditions the blow-up set of the solution of (1.1) is the whole domain. This is also consistent with the conclusion that in a nonlocal problem blow-up can be global (see [1, 16, 17]).

This paper is organized as follows: in section 2, we show the existence of a unique classical solution; in section 3, we give some criteria for the solution \( u(x, t) \) to exist globally or to blow up in finite time.

## 2. Local existence

In order to prove the existence of a unique positive solution to (1.1), we first show the following comparison result.

**Lemma 2.1.** Let \( b(x, t) \) be a continuous nonnegative function defined on \( [0, a] \times [0, r] \) for any \( r \in (0, T) \), and \( u(x, t) \in C(\Omega_r) \cap C^{2,1}(\Omega_r) \) satisfies

\[
x^q u_t - (x^q u_x)_x \geq \int_0^a b(x, t)u(x, t) \, dx, \quad (x, t) \in (0, a) \times (0, r],
\]

(2.1)

\[
u(0, t) \geq 0, \quad u(a, t) \geq 0, \quad t \in [0, r],
\]

\[
u(x, 0) \geq 0, \quad x \in [0, a].
\]

Then \( u(x, t) \geq 0 \) on \( [0, a] \times [0, T) \).

**Proof.** First, similar to the proof of Lemma 2.1 in [15], by using the positive Lemma 2.2.1 for uniformly parabolic equations in [14], we can easily obtain the following conclusion:
If \( w(x,t) \in C(\Omega_t) \cap C^{2,1}(\Omega_t) \) satisfies
\[
  x^q w_t - (x^q w_x)_x - \int_0^a b(x,t) w(x,t) dx \\
  \geq \eta e^{ct} [x^g + (\gamma - \gamma')(1 - \gamma')/x^{2-\gamma'} - a(1 + a^{\gamma'-\gamma}) \max_{(x,t) \in [0,a] \times [0,r]} b(x,t)],
\]
then \( w(x,t) > 0, (x,t) \in (0,a) \times (0,r], \)
\begin{align*}
  w(0,t) & > 0, \quad w(a,t) \geq 0, \quad t \in [0,r], \quad x \in [0,a], \quad \text{and} \\
  w(x,0) & \geq 0, \quad x \in [0,a].
\end{align*}

Next, let \( \gamma' \in (\gamma,1) \) be a positive constant and
\[
  w(x,t) = u(x,t) + \eta(1 + x^{\gamma'-\gamma}) e^{ct},
\]
where \( \eta > 0 \) is sufficiently small and \( c \) is a positive constant to be determined. Then \( w(x,t) > 0 \) on the parabolic boundary of \( \Omega_t \) and
\[
  x^q w_t - (x^q w_x)_x - \int_0^a b(x,t) w(x,t) dx \\
  \geq \eta e^{ct}[(\gamma' - \gamma)(1 - \gamma')/x^{2-\gamma'} - a(1 + a^{\gamma'-\gamma}) \max_{(x,t) \in [0,a] \times [0,r]} b(x,t)].
\]

If \( \max_{(x,t) \in [0,a] \times [0,r]} b(x,t) < (\gamma' - \gamma)(1 - \gamma')/(a^3 - \gamma'(1 + a^{\gamma'-\gamma})), \)
\[
  x^q w_t - (x^q w_x)_x - \int_0^a b(x,t) w(x,t) dx \\
  \geq \eta e^{ct}[(\gamma' - \gamma)(1 - \gamma')/x^{2-\gamma'} - a(1 + a^{\gamma'-\gamma}) \max_{(x,t) \in [0,a] \times [0,r]} b(x,t)] \geq 0.
\]

On the other hand, if \( \max_{(x,t) \in [0,a] \times [0,r]} b(x,t) > (\gamma' - \gamma)(1 - \gamma')/(a^3 - \gamma'(1 + a^{\gamma'-\gamma})), \)
let \( x_0 \) be the root of the algebraic equation
\[
  a(1 + a^{\gamma'-\gamma}) \max_{(x,t) \in [0,a] \times [0,r]} b(x,t) = (\gamma' - \gamma)(1 - \gamma')/x^{2-\gamma'},
\]
and let \( c > 0 \) be sufficiently large such that
\[
  (2.3) \quad c > \left\{ \begin{array}{ll}
    (\max_{(x,t) \in [0,a] \times [0,r]} b(x,t)) a(1 + a^{\gamma'-\gamma})/x_0^q & \text{for } q \geq 0, \\
    (\max_{(x,t) \in [0,a] \times [0,r]} b(x,t)) a(1 + a^{\gamma'-\gamma})/a^q & \text{for } q < 0.
  \end{array} \right.
\]

Then we have
\[
  x^q w_t - (x^q w_x)_x - \int_0^a b(x,t) w(x,t) dx \\
  \geq \left\{ \begin{array}{ll}
    \eta e^{ct}[(\gamma' - \gamma)(1 - \gamma')/x^{2-\gamma'} - a(1 + a^{\gamma'-\gamma}) \max_{(x,t) \in [0,a] \times [0,r]} b(x,t)] & \text{for } x \leq x_0, \\
    \eta e^{ct} [cx^q - a(1 + a^{\gamma'-\gamma}) \max_{(x,t) \in [0,a] \times [0,r]} b(x,t)] & \text{for } x > x_0
  \end{array} \right. \geq 0.
\]

From the above conclusion, we know that \( w(x,t) > 0 \) on \( [0,a] \times [0,r] \). Letting \( \eta \to 0^+ \), we then have \( u(x,t) \geq 0 \) on \( [0,a] \times [0,r] \). From the arbitrariness of \( r \in (0,T) \), we complete the proof.

Obviously, \( \tilde{u} = 0 \) is a lower solution of (1.1); we need to construct an upper solution.

**Lemma 2.2.** There exists a positive constant \( t_0 > 0 \) such that the problem (1.1) has an upper solution \( h(x,t) \in C(\Omega_{t_0}) \cap C^{2,1}(\Omega_{t_0}) \).
Proof. Let
\[ \psi(x) = \left( \frac{x}{a} \right)^{1-\gamma} (1 - \frac{x}{a}) + \frac{1}{2} \left( 1 - \frac{x}{a} \right)^{\frac{1}{2}}, \]
and let \( k_0 \) be a positive constant such that \( k_0 \psi(x) \geq u_0(x) \). Denote the positive constant \( \int_0^1 \left[ s^{1-\gamma} (1 - s) + s^{\frac{1}{2}\gamma} (1 - s)^{\frac{1}{2}} \right] ds \) by \( b_0 \). Let \( k_2 \in (0, \frac{1}{2\gamma}) \) be a positive constant such that
\[ k_2 < (2^a a^{1-\gamma} b_0 k_0^{p-1})^{-\frac{\gamma}{\gamma}}, \]
and let \( k(t) \) be the positive solution of the following initial value problem:
\[ k'(t) = \begin{cases} \frac{b_0 k^p(t)}{a^q k_2^{q+\frac{\gamma}{\gamma}} (1 - k_2)^{\frac{1}{q}\gamma} [1 + k_2^{\frac{q}{q}} (1 - k_2)^{\frac{1}{q}\gamma}]} & , \quad q \geq 0, \\ \frac{b_0 k^p(t)}{a^q k_2^{q+\frac{\gamma}{\gamma}} [1 + k_2^{\frac{q}{q}} (1 - k_2)^{\frac{1}{q}\gamma}]} & , \quad q < 0, \end{cases} \quad k(0) = k_0. \]
Since \( k(t) \) is increasing, we can choose \( t_0 > 0 \) such that \( k(t) \leq 2k_0 \) for all \( t \in [0, t_0] \).
Let \( h(x, t) = k(t) \psi(x) \); then \( h(x, t) \geq 0 \) on \( \Omega_{t_0} \). We would like to show that \( h(x, t) \) is an upper solution of (1.1) in \( \Omega_{t_0} \). To do this, let us construct a function \( J \) by
\[ J = x^q h_t - (x^{\gamma} h_x)x - \int_0^a k^p dx, \quad (x, t) \in \Omega_{t_0}. \]
Then,
\[ J = x^q k'(t) \psi(x) - (x^{\gamma} \psi'(x))'k(t) - \int_0^a k^p(t) \psi^p(x) dx \]
\[ = x^q k'(t) \psi(x) + [(2 - \gamma)/a^{2-\gamma} + (1-\gamma)^2/4 - x^{\gamma-\frac{1}{2}} (a - x)^{\frac{1}{2}} + 1/2x^{\gamma-\frac{1}{2}} (a - x)^{-\frac{1}{2}}] k(t) - ab_0 k^p(t) \]
\[ \geq x^q k'(t) \psi(x) + x^{\gamma-\frac{1}{2}} (a - x)^{-\frac{1}{2}} k(t)/(2a^{1-\frac{1}{2}}) - ab_0 k^p(t), \quad (x, t) \in \Omega_{t_0}. \]
For \( (x, t) \in (0, ak_2) \times (0, t_0] \cup (a(1 - k_2), a) \times (0, t_0] \), by (2.4), we have
\[ J \geq x^q k'(t) \psi(x) - ab_0 k^p(t) \]
\[ \geq [k_2^{q-\gamma}/(2a^{2-\gamma}) - ab_0 k^{p-1}(t_0)] k(t) \]
\[ \geq [k_2^{q-\gamma}/(2a^{2-\gamma}) - ab_0 (2k_0)^{p-1}] k(t) \]
\[ \geq 0. \]
For \( (x, t) \in [ak_2, a(1 - k_2)] \times (0, t_0], \) by (2.5), we have
\[ J \geq x^q k'(t) \psi(x) - ab_0 k^p(t) \]
\[ \geq \begin{cases} \frac{a^q k_2^q k'(t) [k_2 (1 - k_2)^{1-\gamma} + k_2^{\frac{q}{q}} (1 - k_2)^{\frac{1}{q}\gamma}]}{1 - ab_0 k^p(t),} & , \quad q \geq 0, \\ \frac{a^q (1 - k_2)^q k'(t) [k_2 (1 - k_2)^{1-\gamma} + k_2^{\frac{q}{q}} (1 - k_2)^{\frac{1}{q}\gamma}]}{1 - ab_0 k^p(t),} & , \quad q < 0 \end{cases} \]
\[ \geq 0. \]
Thus, \( J(x, t) \geq 0 \) in \( \Omega_{t_0} \). It follows from \( h(0, t) = h(a, t) = 0 \) and \( h(x, 0) = k_0 \psi(x) \geq u_0(x) \) that \( h(x, t) \) is an upper solution of (1.1) in \( \Omega_{t_0} \). \( \square \)
To show the existence of the classical solution $u(x, t)$ of (1.1), let us introduce a “cut-off function” $\rho(x)$. By Dunford and Schwartz [6] p. 1640], there exists a nondecreasing function $\rho(x) \in C^3(\mathbb{R})$ such that $\rho(x) = 0$ if $x \leq 0$, and $\rho(x) = 1$ if $x \geq 1$. Let $0 < \delta < \frac{1 - \gamma}{2 + \gamma}a$,
\[
\rho_\delta(x) = \begin{cases} 
0, & x \leq \delta, \\
\rho\left(\frac{x}{\delta} - 1\right), & \delta < x < 2\delta, \\
1, & x \geq 2\delta,
\end{cases}
\]
and let $u_{0\delta}(x) = \rho_\delta(x)u_0(x)$. We note that
\[
\frac{\partial}{\partial x}u_{0\delta}(x) = \begin{cases} 
0, & x \leq \delta, \\
-\frac{1}{\delta}\rho\left(\frac{x}{\delta} - 1\right)u_0(x), & \delta < x < 2\delta, \\
0, & x \geq 2\delta.
\end{cases}
\]
Since $\rho$ is nondecreasing, we have $\frac{\partial}{\partial x}u_{0\delta}(x) \leq 0$. From $0 \leq \rho(x) \leq 1$, we have $u_0(x) \geq u_{0\delta}(x)$ and $\lim_{\delta \to 0} u_{0\delta}(x) = u_0(x)$.

Let $D_\delta = (\delta, a), \omega_\delta = D_\delta \times (0, t_0]$, $\overline{D_\delta}$ and $\overline{\omega_\delta}$ be their respective closures, and let $S_\delta = \{\delta, a\} \times (0, t_0]$. We consider the following regularized problem:
\[
x'^p u_{0\delta} - (x'^q u_{0\delta})_x = \int_0^a u_\delta^p dx, \quad (x, t) \in \omega_\delta, \\
u_\delta(\delta, t) = u_\delta(a, t) = 0, \quad t \in (0, t_0], \\
u_\delta(x, 0) = u_{0\delta}(x), \quad x \in \overline{D_\delta}.
\]
(2.6)
By using Schauder’s fixed point theorem, we have

**Theorem 2.3.** Problem (2.6) admits a unique nonnegative solution $u_\delta \in C^{2+\alpha, 1+\frac{\gamma}{2}}(\overline{\omega_\delta})$. Moreover, $0 \leq u_\delta(x, t) \leq h(x, t), (x, t) \in \overline{\omega_\delta}$, where $h(x, t)$ is given by Lemma 2.2.

**Proof.** From the proof of Lemma 2.1, we know that there exists at most one nonnegative solution $u_\delta$. To prove existence, we use Schauder’s fixed point theorem. Let
\[
X = \{v \in C^{0, \frac{\gamma}{2}}(\overline{\omega_\delta}) : 0 \leq v(x, t) \leq h(x, t), (x, t) \in \overline{\omega_\delta}\}.
\]
Obviously, $X$ is a closed convex subset of the Banach space $C^{0, \frac{\gamma}{2}}(\overline{\omega_\delta})$. For any $v \in X$, let us consider the following linearized uniformly parabolic problem:
\[
x'^q w_\delta - (x'^r w_\delta)_x = \int_0^a v^p dx, \quad (x, t) \in \omega_\delta, \\
w_\delta(\delta, t) = w_\delta(a, t) = 0, \quad t \in (0, t_0], \\
w_\delta(x, 0) = u_{0\delta}(x), \quad x \in [\delta, a].
\]
(2.7)
It is easy to see that $\hat{w}(x, t) = 0$ and $\hat{w}(x, t) = h(x, t)$ are lower and upper solutions of the problem (2.7). We also note that $x^{-q+r}, x^{-q-1+r}, x^{-q} \in C^{0, \frac{(r-q)}{2}}(\overline{\omega_\delta})$, $x^{-q} \int_0^a v^p dx \in C^{0, \frac{(r-q)}{2}}(\overline{\omega_\delta}), u_{0\delta} \in C^{2+\alpha, 1+\frac{\gamma}{2}}(\overline{\omega_\delta})$. It follows from Theorem 4.2.2 of Lade et al. [10] p. 143] that the problem (2.7) has a unique solution $w_\delta(x, t; v) \in C^{2+\alpha, 1+\frac{\gamma}{2}}(\overline{\omega_\delta})$, which satisfies $0 \leq w_\delta(x, t; v) \leq h(x, t)$. Thus, we can define a mapping $Z$ from $X$ into $C^{2+\alpha, 1+\frac{\gamma}{2}}(\overline{\omega_\delta})$ such that $Z v(x, t) = w_\delta(x, t; v)$, where $w_\delta(x, t; v)$ denotes the unique solution of the problem (2.7) corresponding to $v(x, t) \in X$. To use Schauder’s fixed point theorem, we need to verify that $Z$ maps $X$ into itself and $Z$ is continuous and compact.
In fact, \( Z X \subset X \) and the embedding operator from the Banach space \( C^{2+\alpha,1+\frac{2}{q}}(\overline{\Omega}_0) \) to the Banach space \( C^{\alpha,\frac{2}{q}}(\overline{\Omega}_0) \) is compact. Therefore \( Z \) is compact. To show \( Z \) is continuous, let us consider in \( X \) a sequence \( \{v_n(x,t)\} \) which converges to \( v(x,t) \) uniformly in the norm \( \| \cdot \|_{\alpha,\frac{2}{q}} \). We know that \( v(x,t) \in X \). Let \( u_{bn}(x,t) \) and \( u_3(x,t) \) be the solutions of the problem (2.7) corresponding to \( v_n(x,t) \) and \( v(x,t) \), respectively. Without loss of generality, let us assume that

\[
\|v_n(x,t)\|_{\alpha,\frac{2}{q}} \leq \|v(x,t)\|_{\alpha,\frac{2}{q}} + 1 \text{ for any } n \geq 1.
\]

Let \( w(x,t) = u_{bn}(x,t) - u_3(x,t) \). Then we have

\[
x^q w_t - (x^7w_x)_x = \int_0^1 (v^n_p - v^p)dx, \quad (x,t) \in \omega_\delta,
\]
\[
w(0,t) = 0, w(a,t) = 0, \quad t \in (0,t_0],
\]
\[
w(x,0) = 0, \quad x \in \mathcal{D}_\delta.
\]

From theorem 4.5.2 of Ladyzenskaya et al. \[11\] p. 320, there exists a positive constant \( C \) (independent of \( v_n \) and \( v \)) such that

\[
\|w\|_{2+\alpha,1+\frac{2}{q}} \leq C\|\int_0^1 (v^n_p - v^p)dx\|_{\alpha,\frac{2}{q}} \leq C a p\|((v + \tau(v_n - v))^{p-1}}\|_{\alpha,\frac{2}{q}}\|v_n - v\|_{\alpha,\frac{2}{q}} \leq C a p[3(\|v\|_{\alpha,\frac{2}{q}} + 1)]^{p-1}\|v_n - v\|_{\alpha,\frac{2}{q}},
\]

where \( \tau \in (0,1) \). This shows that the mapping \( Z \) is continuous. By Schauder’s fixed point theorem, we complete the proof.

Now we can prove the following local existence result.

**Theorem 2.4.** There exists some \( t_0(< T) \) such that problem (1.1) has a unique nonnegative solution \( u(x,t) \in C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}) \).

**Proof.** From Theorem 2.3, the problem (2.6) has a unique nonnegative solution \( u_\delta \in C^{2+\alpha,1+\frac{2}{q}}(\overline{\Omega}_0) \). It follows from Lemma 2.1 that \( u_{\delta_1}(x,t) \leq u_{\delta_2}(x,t) \in \omega_{\delta_1} \) if \( \delta_1 > \delta_2 \). Therefore \( \lim_{\delta \to 0} u_\delta(x,t) \) exists for all \( (x,t) \in (0,a] \times [0,t_0] \). Let \( u(x,t) = \lim_{\delta \to 0} u_\delta(x,t) \), \( (x,t) \in (0,a] \times [0,t_0] \) and define \( u(0,t) = 0, t \in [0,t_0] \). We would like to show that \( u(x,t) \) is a classical solution of (1.1) in \( \Omega_{t_0} \). For any \( (x_1,t_1) \in \Omega_{t_0} \), there exist three domains \( Q' = (a_1', a_2') \times (t_1',t_2'] \), \( Q'' = (a_1'', a_2'') \times (t_1'',t_2'] \), and \( Q''' = (a_1''', a_2''') \times (t_1''',t_2'] \) such that \( (x_1,t_1) \in Q' \subset Q'' \subset Q''' \subset (0,a] \times (0,t_0] \) with \( 0 < a_1'' < a_1' < a_1 < x_1 < a_2 < a_2'' < a_2' < a \), \( 0 < t_2'' < t_2' < t_2 < t_2'' \). Since \( u_\delta(x,t) \leq h(x,t) \) in \( Q''' \) and \( h(x,t) \) is finite on \( \overline{Q''} \), we have for any constant \( \tilde{q} > 1 \),

(i) \( \|u_\delta\|_{L^{\tilde{q}}(Q''')} \leq \|h\|_{L^{\tilde{q}}(Q''')} \leq k_3 \) for some positive constant \( k_3 \).

(ii) \( \|x^{-q} \int_{0}^{x} u(x,t)dx\|_{L^{\tilde{q}}(Q''')} \leq (a^*)^{-q}\|\int_{0}^{x} h(x,t)dx\|_{L^{\tilde{q}}(Q''')} \leq k_4 \) for some positive constant \( k_4 \), where \( a^* = a'' \) if \( q > 0 \) and \( a^* = a'' \) if \( q < 0 \).

By the local \( L^q \) estimate of Ladyzenskaya et al. \[11\] pp. 341-342, 352], \( u_\delta \in W^{2,1}_{\tilde{q}}(Q''') \). By embedding theorems \[11\] pp. 61 and 80, \( W^{2,1}_{\tilde{q}}(Q''') \hookrightarrow H^{\alpha,\frac{2}{q}}(Q''') \) if
we choose \( \tilde{q} > \frac{2}{1-\alpha} \). Then, \( \|u_\delta\|_{H^{\alpha}(Q')} \leq k_5 \) for some positive constant \( k_5 \). Now
\[
\|x^{-q}\int_0^a u_\delta^p dx\|_{H^{\alpha}(Q')} \leq (\alpha^*)^{-q} \|\int_0^a h^p(x,t) dx\|_{\infty} + \sup_{(x,t) \in Q'} \left| \int_0^a u_\delta^p dx \right| \cdot |x^{-q} - (\tilde{x})^{-q}|
\]
\[
+ \sup_{(x,t) \in Q'} \left| (\tilde{x})^{-q} \cdot \int_0^a p(u_\delta(x, \tilde{t}) + \tau(u_\delta(x, t) - u_\delta(x, \tilde{t})))^{p-1}(u_\delta(x, t) - u_\delta(x, \tilde{t})) dx \right|
\]
\[
\leq (\alpha^*)^{-q} \|\int_0^a h^p(x,t) dx\|_{\infty} + \|\int_0^a h^p(x,t) dx\|_{\infty} \|x^{-q}\|_{H^{\alpha}(a^\prime, a^\prime)}
\]
\[
+ (\alpha^*)^{-q} \|\int_0^a ph^{p-1}(x,t) dx\|_{\infty} \|u_\delta\|_{H^{\alpha}(Q')}
\]
\[
\leq k_6
\]
for some positive constant \( k_6 \) which is independent of \( \delta \), where \( \tau \in (0, 1) \). By Theorem 4.10.1 of Ladyzenskaya et al. [11, pp. 351-352], we have
\[
\|u_\delta\|_{H^{2+\alpha,1+\alpha}(Q')} \leq k_7
\]
for some positive constant \( k_7 \) independent of \( \delta \). This implies that \( u_\delta, u_\delta t, u_\delta x \) and \( u_\delta xx \) are equicontinuous in \( Q' \). By the Ascoli-Arzel\'a theorem, we know that
\[
\|u\|_{H^{2+\alpha,1+\alpha}(Q')} \leq k_8
\]
for some \( \alpha' \in (0, \alpha) \) and some positive constant \( k_8 \) independent of \( \delta \), and that the derivatives of \( u \) are the uniform limits of the corresponding partial derivatives of \( u_\delta \). Hence \( u \) satisfies (1.1) and \( \lim_{t \to 0} u(x, t) = \lim_{t \to 0} u_\delta(x, t) = \lim_{t \to 0} u_\delta(x) = u_0(x) \). Also from 0 \( \leq u(x, t) \leq h(x, t) \) and \( h(x, t) \to 0 \) as \( x \to 0 \) or \( x \to a \), we have \( \lim_{x \to 0} u(x, t) = \lim_{x \to a} u(x, t) = 0 \), thus \( u \in C(\overline{\Omega_0}) \cap C^{2,1}(\Omega_0) \) is the solution of (1.1) in \( \Omega_{t_0} \). We complete the proof. \( \square \)

By using Lemma 2.1, there exists at most one nonnegative solution \( u \) of (1.1). A proof similar to that of Theorem 2.5 of Floater [7] gives the following continuation result.

**Theorem 2.5.** Let \( T \) be the supremum over \( t_0 \) for which there is a unique nonnegative solution \( u(x, t) \in C(\overline{\Omega_0}) \cap C^{2,1}(\Omega_0) \) of (1.1). Then (1.1) has a unique nonnegative solution \( u(x, t) \in C([0, a] \times [0, T]) \cap C^{2,1}((0, a) \times (0, T)) \) if \( T < +\infty \), then \( \limsup_{t \to T} \max_{x \in [0, a]} u(x, t) = +\infty \).

### 3. Blow-up of Solution

In this section, we will give some global existence and blow-up results of the solution of (1.1).

#### 3.1. Existence and nonexistence of the global solution

In this subsection, we would assume that \( q > \gamma - 1 \).

First, the solution of the following elliptic boundary value problem
\[
-(x^\gamma \psi'(x))' = 1, \quad x \in (0, a); \quad \psi(0) = \psi(a) = 0
\]
is given by \( \psi(x) = \frac{a^{\gamma}}{2\gamma}(\frac{x}{a})^{1-\gamma}(1 - \frac{x}{a}) \). By direct computation,
\[
\int_0^a \psi^p(x) dx = a^{(2-\gamma)p+1} B(p(1-\gamma) + 1, p + 1)/(2-\gamma)^p,
\]
where \( B \) is the Beta function.
Proof. Since $a_1 = [a(2-\gamma)p+1B(p(1-\gamma)+1,p+1)/(2-\gamma)p]^{-\frac{1}{p+1}}$, we have

Let $\tilde{u}(x,t) = a_1\psi(x)$. Then we have

$t^q\tilde{u}_t(x,t) - (x^q\tilde{u}_x(x,t)) = -(x^q a_1\psi'(x))' = a_1$

$x^q\tilde{u}_t(x,t) = x^q a_1\psi(x) |^pdx = \int_0^a \tilde{u}'(x,t)dx,$

$\tilde{u}(0,t) = \tilde{u}(a,t) = 0,$

$\tilde{u}(x,0) = a_1\psi(x) \geq u_0(x),$ this is an upper solution of problem (1.1). By Theorem 2.5, $T = +\infty$, i.e., $u(x,t)$ exists globally.

Next, we consider the following eigenvalue problem:

$$-(x^q\varphi'(x))' = \lambda x^q\varphi(x), \quad x \in (0,a),$$

(3.1)

$$\varphi(0) = \varphi(a) = 0.$$  

By transformation $\varphi(x) = x^{\frac{1-\gamma}{q}}y(x),$ the above differential equation becomes

$$x^2y''(x) + xy'(x) - \frac{(1-\gamma)^2}{4}y(x) + \lambda x^{q+2-\gamma}y(x) = 0, \quad x \in (0,a).$$

Again by transformation $x = z^{\frac{1}{q+2-\gamma}},$ problem (3.1) becomes

$$z^2y''(z) + zy'(z) + \left[\frac{4\lambda z^2}{(q+2-\gamma)^2} - \frac{(1-\gamma)^2}{(q+2-\gamma)}\right]y(z) = 0, \quad z \in (0,b),$$

$$y(0) = y(b) = 0,$$

where $b = a^{\frac{q+2-\gamma}{2-\gamma}}$. Equation (3.2) is a Bessel equation. Its general solution is given by

$$y(z) = AJ_{\frac{1-\gamma}{q+2-\gamma}} \left( \frac{2\sqrt{\lambda}}{q+2-\gamma}z \right) + BJ_{-\frac{1-\gamma}{q+2-\gamma}} \left( -\frac{2\sqrt{\lambda}}{q+2-\gamma}z \right),$$

where $A$ and $B$ are arbitrary constants, and $J_{\frac{1-\gamma}{q+2-\gamma}}$ and $J_{-\frac{1-\gamma}{q+2-\gamma}}$ denote Bessel functions of the first kind of orders $\frac{1-\gamma}{q+2-\gamma}$ and $-\frac{1-\gamma}{q+2-\gamma}$, respectively. Let $\mu$ be the first root of $J_{\frac{1-\gamma}{q+2-\gamma}} (\frac{2\sqrt{\lambda}}{q+2-\gamma}b)$. By Mclachlan [12 pp. 29 and 75], it is positive. It is obvious that $\mu$ is the first eigenvalue of problem (3.1); also we can easily obtain the corresponding eigenfunction

$$\varphi(x) = kx^{\frac{1-\gamma}{q+2-\gamma}} J_{\frac{1-\gamma}{q+2-\gamma}} \left( \frac{2\sqrt{\lambda}}{q+2-\gamma}x^{\frac{q+2-\gamma}{2-\gamma}} \right),$$

(3.3)

which is positive for $x \in (0,a)$. Since $q > \gamma - 1$, we can choose $k > 0$ such that $\max_{x \in [0,a]} x^q \varphi(x) = 1$. Then, we have
Theorem 3.2. Let \( u(x,t) \) be the solution of the problem (1.1). If

\[
\int_0^a x^q \varphi(x) u_0(x) dx > \left( \frac{\mu}{\int_0^a \varphi(x) dx} \right)^{\frac{1}{p-1}} \int_0^a x^q \varphi(x) dx,
\]
then \( u(x,t) \) blows up in finite time.

Proof. We set

\[ U(t) = \int_0^a x^q \varphi(x) u(x,t) dx. \]

Multiplying (1.1) by \( \varphi(x) \) and integrating it over \( x \) from 0 to \( a \) leads to

\[
\int_0^a x^q u_t \varphi dx = \int_0^a (x^q u_x) \varphi dx + \int_0^a \varphi dx \int_0^a u^p dx.
\]

Integration by parts and Jensen’s inequality imply that

\[
U'(t) = \int_0^a x^q \varphi(x) u_t dx \\
\geq -\mu \int_0^a x^q \varphi(x) u(x,t) dx + \int_0^a \varphi(x) dx \int_0^a x^q \varphi(x) u^p(x,t) dx \\
\geq -\mu U(t) + \int_0^a \varphi(x) dx (\int_0^a x^q \varphi(x) dx)^{1-p} (\int_0^a x^q \varphi(x) u(x,t) dx)^p \\
= -\mu U(t) + \int_0^a \varphi(x) dx (\int_0^a x^q \varphi(x) dx)^{1-p} U^p(t).
\]

Therefore \( U(t) \) satisfies the following relation:

\[
U'(t) \geq U(t)(-\mu + \int_0^a \varphi(x) dx U^{p-1}(t)/(\int_0^a x^q \varphi(x) dx)^{p-1}),
\]

\[
U(0) = \int_0^a x^q \varphi(x) u_0(x) dx.
\]

By the hypothesis,

\[
U(0) = \int_0^a x^q \varphi(x) u_0(x) > \left( \frac{\mu}{\int_0^a \varphi(x) dx} \right)^{\frac{1}{p-1}} \int_0^a x^q \varphi(x) dx,
\]

hence \( U(t) \) tends to infinity in finite time. Therefore, \( u(x,t) \) ceases to exist at some finite time; that is to say, \( u(x,t) \) blows up in finite time.

3.2. Global blow-up. In this subsection, we would assume that \( q > 0 \) and \( \gamma = 0 \). Chan & Chan [3] showed that the Green’s function \( G(x,\xi,t-\tau) \) associated with the operator \( L = x^q \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \), subject to the first boundary condition, exists. For ease of reference, we state their Lemmas 2 and 4 in the following lemma.

Lemma 3.3. (a) For \( t > \tau \), \( G(x,\xi,t-\tau) \) is continuous for \( (x,\xi,t,\tau) \in ([0,a] \times (0,T)) \times ((0,a) \times [0,T]) \).

(b) For each fixed \( (\xi,\tau) \in (0,a) \times [0,T) \), \( G_t(x,\xi,t-\tau) \in C([0,a] \times (\tau,T]) \).

(c) In \( \{ (x,\xi,\tau) : x and \xi are in (0,a), T \geq t > \tau \geq 0 \} \), \( G(x,\xi,t-\tau) \) is positive.

In Lemma 7 of [3], Chan & Yang gave some additional properties of \( G(x,\xi,t-\tau) \).

Lemma 3.4. For fixed \( x_0 \in (0,a) \), given any \( x \in (0,a) \) and any finite time \( T \), there exist positive constants \( C_1 \) (depending on \( x \) and \( T \)) and \( C_2 \) (depending on \( T \)) such that

\[
\int_0^a G(x,\xi,t) d\xi > C_1 \quad \text{for} \quad 0 \leq t \leq T,
\]

\[
\int_0^a G(x_0,\xi,t) d\xi < C_2 \quad \text{for} \quad 0 \leq t \leq T.
\]

Now we give the global blow-up result.

Theorem 3.5. If the solution \( u(x,t) \) of (1.1) blows up at the point \( x_0 \in (0,a) \) and in finite time \( T \), then the blow-up set of \( u(x,t) \) is \([0,a]\)
Proof. By Green’s second identity, we have
\[
\frac{\partial u}{\partial t} = f(x,t) - \int_0^a \frac{\partial G}{\partial \xi}(x,\xi, t) u_0(\xi)d\xi - \int_0^a \frac{\partial G}{\partial t}(x,\xi, t) u(\xi)d\xi - \int_0^a \frac{\partial G}{\partial \tau}(x,\xi, \tau) u(\xi)d\xi - \int_0^a g(x,\xi) d\xi
\]
for any \((x,t) \in (0,a) \times (0,T)\). According to the conditions given in this theorem, the solution \(u(x,t)\) of (1.1) blows up at \(x = x_0\) and in finite time \(T\). Then \(\limsup_{t \to T} u(x_0,t) = +\infty\). By (3.4) and Lemma 3.4, we have
\[
\frac{\partial u}{\partial t} = f(x,t) - \int_0^a \frac{\partial G}{\partial \xi}(x,\xi, t) u_0(\xi)d\xi - \int_0^a \frac{\partial G}{\partial t}(x,\xi, t) u(\xi)d\xi - \int_0^a \frac{\partial G}{\partial \tau}(x,\xi, \tau) u(\xi)d\xi - \int_0^a g(x,\xi) d\xi
\]
Since \(\limsup_{t \to T} u(x_0,t) = +\infty\), we have
\[
\lim_{t \to T} \int_0^a u(x,t) dx = +\infty.
\]
On the other hand, for any given \(x \in (0,a)\), we have
\[
\frac{\partial u}{\partial t} \geq f(x,t) - \int_0^a \frac{\partial G}{\partial \xi}(x,\xi, t) u_0(\xi)d\xi - \int_0^a \frac{\partial G}{\partial t}(x,\xi, t) u(\xi)d\xi - \int_0^a \frac{\partial G}{\partial \tau}(x,\xi, \tau) u(\xi)d\xi - \int_0^a g(x,\xi) d\xi
\]
\[
\geq C_1 \int_0^t \int_0^a u(x,t) dx dt + C_2 \int_0^t \int_0^a u(x,t) dx dt,
\]
It follows from the above inequality and (3.5) that \(\limsup_{t \to T} u(x, t) = +\infty\).

For any \(\tilde{x} \in (0,a)\), we can choose a sequence \(\{x_n, t_n\}\) such that \((x_n, t_n) \to (\tilde{x}, T)\) as \(n \to +\infty\) and \(\lim_{n \to +\infty} u(x_n, t_n) = +\infty\). Thus the blow-up set is the whole domain \([0,a]\), and we complete the proof.

References


Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China
E-mail address: youpengchen@263.sina.net
Current address: Department of Mathematics, Yancheng Teachers College, Yancheng 224002, Jiangsu, People’s Republic of China
E-mail address: youpchen@yahoo.com.cn

Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China

Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China