PARTIAL SUMS OF HYPERGEOMETRIC SERIES
OF UNIT ARGUMENT

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Abstract. The asymptotic behaviour of partial sums of generalized hypergeometric series of unit argument is investigated.

1. Introduction

This paper deals with finite sums

\[ \sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(1 + l)} \] (1)

which are partial sums of the first \( m \) terms of hypergeometric series \( _{p+1}F_p \) of unit argument \( z = 1 \) multiplied by appropriate gamma function factors,

\[ \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)}_{p+1}F_p \left( \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{array} \right) \bigg| z \bigg) \] (2)

\[ = \sum_{l=0}^{\infty} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(1 + l)} z^l. \]

An important quantity in this context is

\[ s_p = b_1 + \cdots + b_p - a_1 - a_2 - \cdots - a_{p+1}, \] (3)

the non-trivial characteristic exponent of the underlying differential equation at \( z = 1 \).

If \( \Re(s_p) > 0 \), then the hypergeometric series converges at \( z = 1 \), and its value is given by the Gaussian summation formula for \( p = 1 \) or by the corresponding generalized formulas [5] for \( p > 1 \). The terms of the series behave asymptotically as

\[ l^{a_1+a_2+\cdots+a_{p+1}-b_1-\cdots-b_p-1} \] for \( l \to \infty \), and if we consider the partial sum of the first \( m \) terms, then the contribution of the missing tail is \( O(m^{-s_p}) \) as \( m \to \infty \). Therefore we have

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Theorem 1. If \( \Re(s_1) > 0 \), then

\[
(4) \quad \sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)}{\Gamma(b_1 + l)\Gamma(1 + l)} = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(s_1)}{\Gamma(a_1 + s_1)\Gamma(a_2 + s_1)} + O(m^{-s_1})
\]
as \( m \to \infty \).

Or more generally,

Theorem 2. If \( \Re(s_p) > 0 \), then

\[
(5) \quad \sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(1 + l)} = g_0(0) + O(m^{-s_p})
\]
as \( m \to \infty \), where

\[
(6) \quad g_0(0) = \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(s_p)}{\Gamma(a_1 + s_p)\Gamma(a_2 + s_p)} \sum_{k=0}^{\infty} \frac{(s_p)_k}{(a_1 + s_p)_k(a_2 + s_p)_k} A_k^{(p)}
\]
and the sum converges provided that \( \Re(a_j) > 0 \) for \( j = 3, \ldots, p + 1 \).

Here use is made of the Pochhammer symbol

\[
(x)_n = x(x + 1)\cdots(x + n - 1) = \Gamma(x + n)/\Gamma(x),
\]
and the coefficients \( A_k^{(p)} \) for \( p = 2, 3, \ldots \) are given in [5], but a few of them are displayed here again for convenience:

\[
(7) \quad A_k^{(2)} = \frac{(b_2 - a_3)_k(b_1 - a_3)_k}{k!},
\]

\[
(8) \quad A_k^{(3)} = \sum_{k_2=0}^{k} \frac{(b_3 + b_2 - a_4 - a_3 + k_2)_k}{(k - k_2)!} \frac{(b_1 - a_3)_k(b_3 - a_4)_k(b_2 - a_4)_k}{k_2!} ,
\]

\[
(9) \quad A_k^{(4)} = \sum_{k_2=0}^{k} \frac{(b_4 + b_3 + b_2 - a_5 - a_4 - a_3 + k_2)_k}{(k - k_2)!} \frac{(b_1 - a_3)_k(b_3 - a_4)_k(b_2 - a_4)_k}{(k_2 - k_3)!} \frac{(b_4 - a_5)_k(b_3 - a_5)_k}{k_3!}.
\]

For \( p = 3, 4, \ldots \) several other representations are possible [5].

In the case \( p = 2 \), the sum in (6) is an \( 3F_2 \) and so (10)–(13) reduce, when \( m \to \infty \), to a well-known transformation formula for \( 3F_2 \) of unit argument [1], [10], [14].

The simple formula (11) above corresponding to \( p = 1 \) can be recovered from the general formula (5)–(6) if we define

\[
(10) \quad A_0^{(1)} = 1, \quad A_k^{(1)} = 0 \quad \text{for} \quad k > 0,
\]
so that then the sum over \( k \) is equal to 1 and disappears. In a similar way the formulas in the other theorems below simplify for \( p = 1 \).

It is the purpose of this work to investigate the asymptotic behaviour of the partial sums (13) for situations when \( \Re(s_p) \leq 0 \) and to find a more detailed formula in cases like \( s_p = 1 \), when a few additional terms of higher order might be desirable.
2. The case when \( s_p \) is not equal to an integer

If \( \Re(s_p) \leq 0 \), then the hypergeometric series diverges at \( z = 1 \), and the contribution of the missing tail of the series does not asymptotically vanish. Responsible for this behaviour are the singular terms of the hypergeometric function at \( z = 1 \). We have to explicitly subtract the contribution from the most important singular terms in order to get the tail to vanish as \( m \to \infty \). The behaviour of the hypergeometric function at \( z = 1 \) is given by the continuation formula

\[
\frac{\Gamma(a_1) \Gamma(a_2) \cdots \Gamma(a_{p+1})}{\Gamma(b_1) \cdots \Gamma(b_p)} {}_pF_q \left( \begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{array} \right| z \right) \\
= \sum_{n=0}^{\infty} g_n(0) (1 - z)^n + (1 - z)^{s_p} \sum_{n=0}^{\infty} g_n(s_p)(1 - z)^n,
\]

which is valid provided that \( s_p \) is not equal to an integer. Here the coefficient \( g_0(0) \) defined in (6) above enters, and, while the other \( g_n(0) \) are not needed here, the coefficients \( g_n(s_p) \) of the singular term are [5]

\[
g_n(s_p) = (-1)^n \frac{(a_1 + s_p) \cdots (a_2 + s_p)n \Gamma(-s_p - n)}{(1)_n} \\
\times \sum_{k=0}^{n} \frac{(-n)_k}{(a_1 + s_p)_k(a_2 + s_p)_k} A^{(p)}_k.
\]

The singular terms on the right-hand side of (11) have, with \( x = s_p + n \), the \( z \)-dependence

\[
(1 - z)^x = \sum_{l=0}^{\infty} \frac{(-x)_l}{\Gamma(1 + l)} z^l.
\]

With \( z = 1 \), we shall need the partial sum of this series,

\[
\sum_{l=0}^{m-1} \frac{(-x)_l}{\Gamma(1 + l)} = -\frac{1}{x} \frac{(-x)_m}{\Gamma(m)} \quad (m = 1, 2, \ldots, x \neq 0).
\]

Treating the first \( N + 1 \) singular terms on the right-hand side of (11) in this way, we may obtain

\[
\sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l) \Gamma(a_2 + l) \cdots \Gamma(a_{p+1} + l)}{\Gamma(b_1 + l) \cdots \Gamma(b_p + l) \Gamma(1 + l)} = g_0(0) \\
- \sum_{n=0}^{N} g_n(s_p) \frac{1}{s_p + n} \frac{(-s_p - n)_m}{\Gamma(m)} + T_1 + T_2,
\]

where

\[
T_1 = -\sum_{l=m}^{\infty} \frac{\Gamma(a_1 + l) \Gamma(a_2 + l) \cdots \Gamma(a_{p+1} + l)}{\Gamma(b_1 + l) \cdots \Gamma(b_p + l) \Gamma(1 + l)} \\
+ \sum_{n=0}^{N} g_n(s_p) \sum_{l=m}^{\infty} \frac{(-s_p - n)_l}{\Gamma(1 + l)},
\]

\[
T_2 = -\lim_{m \to \infty} \sum_{n=N+1}^{\infty} \frac{g_n(s_p)}{(s_p + n) \Gamma(-s_p - n)} \frac{\Gamma(-s_p - n + m)}{\Gamma(m)}.
\]
We now have to show that the contributions $T_1$ and $T_2$ of the tails of the series are small. The terms in the series in $T_2$ are $O((m^{s_p}n)^{N+1})$ as $m \to \infty$, and so the terms, in the worst case when $n = N + 1$, converges provided that $\Re(-s_p - N) < 0$. The sum of the series then is $O(m^{-s_p-N})$, and $T_2$ exists and vanishes. The term $T_1$ may be written

$$T_1 = -\sum_{l=m}^{\infty} \frac{\Gamma(-s_p + l)}{\Gamma(1 + l)} S_l,$$

where

$$S_l = \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(-s_p + l)} - \sum_{n=0}^{N} (-1)^n g_n(s_p) \frac{1}{\Gamma(-s_p - n)(1 + s_p - l)^n}.$$

In the second term, use has been made of the reflection formula of the gamma function. It can be shown [6] that $S_l$ is $O(l^{-N-1})$ as $l \to \infty$, so the terms of the sum are $O(l^{-s_p-N-2})$, and $T_1$ is $O(m^{-s_p-N-1})$ as $m \to \infty$. In this way we arrive at

**Theorem 3.** If $s_p$ is not equal to an integer and $N = 0, 1, 2, \ldots$ is chosen to be greater than $\Re(-s_p)$, then

$$\sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(1 + l)} = g_0(0)$$

$$-\sum_{n=0}^{N} (-1)^n \frac{1}{s_p + n} \frac{(a_1 + s_p)_n(a_2 + s_p)_n}{(1)_n} \Gamma(-s_p - n + m)$$

$$\times \sum_{k=0}^{n} \frac{(-n)_k}{(a_1 + s_p)_k(a_2 + s_p)_k} A_k^{(p)} + O(m^{-s_p-N-1})$$

as $m \to \infty$.

This theorem is applicable in the case of $\Re(s_p) > 0$, too, and then may give more details of the asymptotic behaviour than Theorem 2.

3. THE CASE WHEN $s_p$ IS EQUAL TO AN INTEGER

If $s_p$ is equal to an integer, then the procedure of proof remains essentially the same, except that the starting point is a more complicated continuation formula, in place of (11), containing logarithmic terms which have to be expanded, too.

3.1. THE CASE WHEN $s_p$ IS EQUAL TO ZERO. If $s_p$ is equal to zero, the hypergeometric series is called zero-balanced. It is this case which seems to be most interesting. While a few early results can be found in the monographs [1], [13], [10], the topic had received renewed attention, with emphasis on the zero-balanced series, in connection with Ramanujan’s notebooks [11], [2], [3], [9], [8], [4], [15], [12], [13], [7].
The required continuation formula for the zero-balanced hypergeometric function is [5]

\[
\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)}{F}_{p+1}\left(\begin{array}{c} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{array}\right| z) = \sum_{n=0}^{\infty} d_n (1 - z)^n + \sum_{n=0}^{\infty} e_n (1 - z)^n \ln(1 - z),
\]

where

\[
e_n = \frac{(a_1)_{n}(a_2)_{n}}{(1)_{n}(1)_{n}} \sum_{k=0}^{n} \frac{(-n)_k}{(a_1)_k(a_2)_k} A_k^{(p)},
\]

\[
d_0 = 2 \psi(1) - \psi(a_1) - \psi(a_2) + \sum_{k=1}^{\infty} \frac{\Gamma(k)}{(a_1)_k(a_2)_k} A_k^{(p)},
\]

and the other coefficients are not needed here. Again, we have to expand the singular terms on the right-hand side,

\[
(1 - z)^n \ln(1 - z) = \sum_{l=1}^{\infty} c_l^{(n)} z^l,
\]

where

\[
c_l^{(n)} = \frac{(-1)^n}{l} \frac{\Gamma(1 + n)\Gamma(l - n)}{\Gamma(l)}
\]

for \( l > n \), while the coefficients for \( l \leq n \) are

\[
c_1^{(1)} = -1, \quad c_1^{(2)} = -1, \quad c_2^{(2)} = \frac{3}{2}.
\]

With \( z = 1 \), we need the partial sums of these series,

\[
\sum_{l=1}^{m-1} c_l^{(0)} = \psi(1) - \psi(m),
\]

\[
\sum_{l=1}^{m-1} c_l^{(1)} = -\frac{1}{m - 1},
\]

\[
\sum_{l=1}^{m-1} c_l^{(2)} = \frac{1}{(m - 1)(m - 2)}.
\]

Keeping the contribution from the singular terms up to and including \( n = 2 \) together with the constant term on the right-hand side of (21) we arrive at

**Theorem 4.** If \( s_p \) is equal to zero, then

\[
m^{-1} \sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(1 + l)} \frac{\Gamma(k)}{(a_1)_k(a_2)_k} A_k^{(p)}
\]

\[
+ \psi(1) - \psi(a_1) - \psi(a_2) + \psi(m) + [a_1a_2 - A_1^{(p)}](m - 1)^{-1}
\]

\[
- \frac{1}{4}[a_1(a_1 + 1)a_2(a_2 + 1) - 2(a_1 + 1)(a_2 + 1)A_1^{(p)} + 2A_2^{(p)}]
\]

\[
\times[(m - 1)(m - 2)]^{-1} + O(m^{-3})
\]

as \( m \to \infty \), where the infinite sum over \( k \) converges if \( \Re(a_j) > 0 \) for \( j = 3, \ldots, p+1 \).
This theorem gives more details of the asymptotic behaviour than our earlier formula \[7\]. If desired, equation (30) may be rewritten using
\[\psi(m) = \ln(m) - \frac{1}{2}m^{-1} - \frac{1}{12}m^{-2} + O(m^{-3})\]
in order to exhibit the logarithmic dependence on \(m\).

3.2. The case when \(s_p\) is equal to a positive integer. For \(s_p\) equal to a positive integer \(t\), Theorems 1 or 2 are applicable, but for \(s_p\) equal to 1 or 2 a few more terms of the asymptotic expansion might be desirable. In these cases the required continuation formula reads \[5\]
\[
\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)}_{p+1}F_p\left(\begin{array}{c}a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p\end{array}\right| z) = \sum_{n=0}^{t-1} l_n (1-z)^n + \sum_{n=0}^{\infty} [w_n + q_n \ln(1-z)](1-z)^n,
\]
where
\[
q_n = -(-1)^t \frac{(a_1 + t)_n(a_2 + t)_n}{\Gamma(1+t+n)} \sum_{k=0}^{n} \frac{(-n)_k}{(a_1 + t)_k(a_2 + t)_k} A_k^{(p)},
\]
and the other coefficients are not needed here. Using the expansions of the logarithmic terms and their partial sums from above and keeping the contributions up to and including \(n = 1\) in case of \(t = 1\) or \(n = 0\) in case of \(t = 2\), we may arrive at the following theorems.

**Theorem 5.** If \(s_p\) is equal to 1, then
\[
\sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(1+l)} = \frac{1}{a_1 a_2} \sum_{k=0}^{\infty} \frac{(1)_k}{(a_1 + 1)_k(a_2 + 1)_k} A_k^{(p)} - (m-1)^{-1} + \frac{1}{2}[(a_1 + 1)A_1^{(p)}][(m-1)(m-2)]^{-1} + O(m^{-3})
\]
as \(m \to \infty\), where the infinite sum converges if \(\Re(a_j) > 0\) for \(j = 3, \ldots, p+1\).

**Theorem 6.** If \(s_p\) is equal to 2, then
\[
\sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(1+l)} = \frac{1}{(a_1)_2(a_2)_2} \sum_{k=0}^{\infty} \frac{(2)_k}{(a_1 + 2)_k(a_2 + 2)_k} A_k^{(p)} - \frac{1}{2}[(m-1)(m-2)]^{-1} + O(m^{-3})
\]
as \(m \to \infty\), where the infinite sum converges if \(\Re(a_j) > 0\) for \(j = 3, \ldots, p+1\).

We may observe that Theorem 5 and Theorem 6 are not unexpected, but (35) and (36) are special cases of (20). Indeed, all the quantities in (20) remain well-defined when \(s_p\) approaches a positive integer, and so Theorem 5 is valid even for positive integer values of \(s_p\).
3.3. **The case when \( s_p \) is equal to a negative integer.** When \( s_p \) is equal to a negative integer \(-t\), then the required continuation formula is \([5]\):

\[
\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)} F_p^{p+1}
\left(\begin{array}{c}
a_1, a_2, \ldots, a_{p+1} \\
b_1, \ldots, b_p
\end{array} \middle| z\right)
= (1-z)^{-t} \sum_{n=0}^{\infty} h_n (1-z)^n + \sum_{n=0}^{\infty} [u_n + v_n \ln(1-z)](1-z)^n,
\]

where

\[
(38) \quad h_n = (-1)^n \frac{(a_1 - t)_n(a_2 - t)_n \Gamma(t - n)}{(1 + n)} \sum_{k=0}^{n} \frac{(-n)_k}{(a_1 - t)_k(a_2 - t)_k} A_k^{(p)},
\]

\[
(39) \quad v_n = (-1)^t \frac{(a_1 - t)_t(a_2 - t)_t}{(1 + t)} \sum_{n=0}^{t} \frac{(-t - n)_n}{(a_1 - t)_n(a_2 - t)_n} A_k^{(p)},
\]

\[
(40) \quad u_0 = \sum_{k=1}^{\infty} \frac{\Gamma(k)}{(a_1)_k(a_2)_k} A_k^{(p)}
+ (-1)^t \frac{(a_1 - t)_t(a_2 - t)_t}{(1 + t)} \sum_{k=0}^{t} \frac{(-t)_k}{(a_1 - t)_k(a_2 - t)_k} A_k^{(p)}
\times [\psi(1 + t - k) + \psi(1) - \psi(a_1) - \psi(a_2)],
\]

and the other \( u_n \) are not needed here. Proceeding as above and keeping the contributions from the logarithmic terms up to and including \( n = 2 \), we then may get

**Theorem 7.** If \( s_p \) is equal to a negative integer \(-t\), then

\[
(41) \quad \sum_{l=0}^{m-1} \frac{\Gamma(a_1 + l)\Gamma(a_2 + l)\cdots\Gamma(a_{p+1} + l)}{\Gamma(b_1 + l)\cdots\Gamma(b_p + l)\Gamma(1 + l)}
= \sum_{n=0}^{t-1} (-1)^n \frac{1}{t - n} \frac{(a_1 - t)_n(a_2 - t)_n \Gamma(t - n) \Gamma(t - n + m)}{(1 + n) \Gamma(m)}
\times \sum_{k=0}^{n} \frac{(-n)_k}{(a_1 - t)_k(a_2 - t)_k} A_k^{(p)}
+ \sum_{k=1}^{\infty} \frac{\Gamma(k)}{(a_1)_k(a_2)_k} A_k^{(p)}
+ (-1)^t \frac{(a_1 - t)_t(a_2 - t)_t}{(1 + t)} \sum_{k=0}^{t} \frac{(-t)_k}{(a_1 - t)_k(a_2 - t)_k} A_k^{(p)}
\]
Corollary 1. Using this information, we obtain from (30) of [5] for this purpose is the alternative representation (4.2) of [5]. The constant term CT of (42) can then be reduced to

\[
CT = -\psi(1) + 4 \ln(2) - \frac{1}{18} \sum_{l=0}^{\infty} (1 + l) \frac{(2)_l(2)_l}{(\gamma^2)_l} F_2 \left( \begin{array}{c} \frac{3}{2}, \frac{3}{2} - l \\ 2, 3 \end{array} \middle| 1 \right).
\]

Here the infinite sum seems to be difficult to evaluate analytically, although it must simply be equal to \(18 \ln(2)\). For we know from Berndt [3] that

\[
CT = -\psi(1) + 3 \ln(2).
\]

Using this information, we obtain from (30)

**Corollary 1.**

\[
\frac{1}{4} \pi^2 \sum_{l=0}^{m-1} \frac{(\frac{1}{2})_l(\frac{1}{2})_l(l\frac{1}{2})(\frac{5}{4})_l}{(1)_l(1)_l(1)_l(l\frac{1}{2})_l} = \psi(m) - \psi(1) + 3 \ln(2) + \frac{1}{4} (m - 1)^{-1} - \frac{1}{4} (m - 1)(m - 2)^{-1} + O(m^{-3}).
\]

Our second example is a zero-balanced \(F_4\) which depends only on three independent parameters \(a, b, c\) according to \(a_1 = a, a_2 = b, a_3 = c, a_4 = a + b + c + 1, a_5 = (a + b + c + 1)/2, b_1 = b + c, b_2 = c + a, b_3 = a + b, b_4 = (a + b + c - 1)/2\). Here again, the infinite sum in the constant term of (30) is difficult to evaluate analytically, but fortunately we know the constant term from a different source [8], [9], [11]. Thus we obtain

**Corollary 2.**

\[
\sum_{l=0}^{m-1} \frac{\Gamma(a + l)\Gamma(b + l)\Gamma(c + l)\Gamma(a + b + c - 1 + l)\Gamma(\frac{1}{2}[a + b + c + 1] + l)}{\Gamma(b + c + l)\Gamma(c + a + l)\Gamma(a + b + l)\Gamma(\frac{1}{2}[a + b + c - 1] + l)\Gamma(1 + l)}
= \psi(m) + \frac{1}{2} [\psi(1) - \psi(a) - \psi(b) - \psi(c)] + \frac{1}{2} (a + b + c - 1)(m - 1)^{-1} + \frac{1}{4} [2abc - (a + b + c)(a + b + c - 1)] \middle| (m - 1)(m - 2)^{-1} + O(m^{-3}).
\]
As a special case of this formula, with \( a = b = c = 1/2 \), we can again get Corollary 1 above.

Other partial sums with interrelated parameters have been obtained by A. K. Srivastava [15].

**References**

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