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MARCINKIEWICZ'S THEOREM ON OPERATOR MULTIPLIERS OF FOURIER SERIES

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Abstract. We give some sufficient conditions on the operators $A_m \in \mathcal{B}(L^p(0,1))$ which for each $\Phi_m \in L^p(0,1)$ imply the inequality

$$\int_0^1 \int_0^{2\pi} \left| \sum_m e^{imx} \cdot A_m \Phi_m(y) \right|^p dxdy \leq c_p^p \int_0^1 \int_0^{2\pi} \left| \sum_m e^{imx} \cdot \Phi_m(y) \right|^p dxdy,$$

$1 < p < \infty$.

1. Introduction

Let D denote the unit disc $\{z : |z| < 1\}$ and let $dA(z) = dx\, dy$ $(z = x + iy)$ be the Lebesgue measure on D. In investigations of boundedness of the integral operators on $L^p(w(|z|)\, dA(z))$ (w is weight function) it is often necessary to consider inequalities of the form

$$\int_0^1 \int_0^{2\pi} \left| \sum_m e^{imx} \cdot A_m \Phi_m(y) \right|^p dxdy \leq c_p^p \int_0^1 \int_0^{2\pi} \left| \sum_m e^{imx} \cdot \Phi_m(y) \right|^p dxdy \tag{1}$$

(c_p is a constant which depends only on p), where $A_m \in \mathcal{B}(L^p(0,1))$ and $\Phi_m \in L^p(0,1), m \in \mathbb{Z}$. Here $\mathcal{B}(L^p(0,1))$ denotes the space of bounded operators on $L^p(0,1)$. For example, the problem of a precise two-sided estimate of the norm of the Cauchy operator on $L^p(D)$ reduces to an inequality of the form (1) with concrete operators A_m. Also, the problem of boundedness of the Bergman projection on $L^p(w(|z|)\, dA(z))$ (in terms of the weight w) reduces to an inequality of the form (1) and studying of the corresponding operators A_m. The results concerning the above-mentioned problems will appear in forthcoming papers.

For $A \in \mathcal{B}(L^p(0,1)), ||A||_{\mathcal{B}(L^p(0,1))}$ denotes its norm. On the other hand, when $A_m = \lambda_m I$ and $\Phi_m \equiv a_m$, inequality (1) becomes

$$\int_0^{2\pi} \left| \sum_m e^{imx} \cdot \lambda_m a_m \right|^p dx \leq c_p^p \int_0^{2\pi} \left| \sum_m e^{imx} \cdot a_m \right|^p dx, \tag{2}$$

which is precisely Marcinkiewicz's inequality for the multipliers of Fourier's series of functions from $L^p(0, 2\pi), 1 < p < \infty$.

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Sufficient conditions for inequality (2) are given by Marcinkiewicz's theorem (see [4], pp. 346-348) on multipliers of Fourier's series:

(1) $\sup\limits_{m\in\mathbb{Z}} |\lambda_m| < +\infty$,

(2) $\sup\limits_{m\geq 0} \sum\limits_{|j|=2^m+1}^{2^{m+1}} |\lambda_j - \lambda_{j+1}| < +\infty$.

In this paper we will give some conditions for operators $A_m \in \mathcal{B}(L^p(0,1))$, $m \in \mathbb{Z}$, which are sufficient for the inequality (1) to hold.

2. Result

Theorem 1. *Let* $(\lambda_m)_{m\in\mathbb{Z}}$ *be a sequence of complex numbers such that*

$$\sup_{m\in\mathbb{Z}} |\lambda_m| < +\infty \quad \text{and} \quad \sup_{m\geq 0} \sum_{|j|=2^m+1}^{2^{m+1}} |\lambda_j - \lambda_{j+1}| < +\infty. \tag{3}$$

If $(A_m)_{m\in\mathbb{Z}}$ *is a sequence of operators in* $\mathcal{B}(L^p(0,1))$, $1<p<\infty$, *such that*

$$\sum_{m\in\mathbb{Z}} \left|\left| \frac{A_m}{\lambda_m} - \frac{A_{m+1}}{\lambda_{m+1}} \right|\right|_{\mathcal{B}(L^p(0,1))} < +\infty, \tag{4}$$

then there exists a constant c_p *(depending only on* p*) such that* (1) *holds for arbitrary functions* $\Phi_m \in L^p(0,1)$. *(All sums in* (1) *are finite,* $x\in[0,2\pi]$, $y\in[0,1]$.*)*

Remark 1. If $A_m = \lambda_m I$ (I is a unit operator), then (4) is obviously satisfied and thus the previous theorem gives the inequality

$$\int_0^1\int_0^{2\pi} \left| \sum_m e^{imx}\cdot \lambda_m \Phi_m(y) \right|^p dxdy \leq c_p^p \int_0^1\int_0^{2\pi} \left| \sum_m e^{imx}\cdot \Phi_m(y) \right|^p dxdy, \tag{5}$$

which holds under assumption (3).

In particular, if $\Phi_m(y) = a_m$, we get Marcinkiewicz's theorem.

If $\lambda_m = 0$ for $m \leq 0$ and $\lambda_m = 1$ for $m \geq 1$, we get Riesz's Theorem on projection (see [1], pp. 113-117).

Before the proof the theorem we recall some facts about tensor products of normed spaces and tensor product of operators defined on these spaces (see e.g. [3] for more details).

Denote $E = L^p(0,2\pi)$, $F = L^p(0,1)$ $(p \geq 1)$.

Let $E \otimes F$ denote the vector space generated by all functions of the form $u(x)v(y)$, $u\in E, v\in F$ (tensor product). For

$$z = \sum_{i=1}^n u_i(x) v_i(y) \in E\otimes F$$

set

$$||z||_{E\otimes F} = \inf \sum_i ||\alpha_i||_p \, ||\beta_i||_p,$$

where $||\alpha_i||_p = \left(\int\limits_0^{2\pi} |\alpha_i|^p \, dx\right)^{\frac{1}{p}}$, $||\beta_i||_p = \left(\int\limits_0^1 |\beta_i|^p \, dy\right)^{\frac{1}{p}}$ and inf is taken over all possible (finite) representations of z as $z = \sum\limits_{i=1}^n \alpha_i(x)\beta_i(y)$, $\alpha_i\in E$, $\beta_i\in F$. It is easy to check that $||\cdot||_{E\otimes F}$ is a norm on $E\otimes F$. Completing $E\otimes F$ with respect to this norm we get the Banach space $E\widehat{\otimes} F$. It is well known that $E\widehat{\otimes} F$ is isomorphic to $L^p\left((0,2\pi)\times(0,1)\right)$. This means that there exists a constant $c\,(p)$ such that

$$(c\,(p))^{-1}\cdot ||z||_{L^p((0,2\pi)\times(0,1))} \le ||z||_{E\otimes F} \le c\,(p)\cdot ||z||_{L^p((0,2\pi)\times(0,1))} \tag{6}$$

for each $z\in E\otimes F$.

If $A\in\mathcal{B}\left(L^p\,(0,2\pi)\right)$ and $B\in\mathcal{B}\left(L^p\,(0,1)\right)$ we define the operator $A\otimes B: E\otimes F\longrightarrow E\otimes F$ as

$$(A\otimes B)\left(\sum_{i=1}^{n} u_i\,(x)\,v_i\,(y)\right)=\sum_{i=1}^{n}(Au_i)\,(x)\cdot\,(Bv_i)\,(y)\,.$$

It is well known that

$$||A\otimes B||_{\mathcal{B}(E\otimes F)} = ||A||_{\mathcal{B}(L^p(0,2\pi))}\cdot ||B||_{\mathcal{B}(L^p(0,1))} \tag{7}$$

and thus the operator $A\otimes B$ extends to the bounded operator on $E\widehat{\otimes} F$. We keep the same notation $A\otimes B$ for this extension.

Obviously

$$\begin{cases} A\otimes(B+C)=A\otimes B+A\otimes C,\\ (B+C)\otimes A=B\otimes A+C\otimes A.\end{cases} \tag{8}$$

Proof of Theorem 1. For $1<p<\infty$ let P_k, $k\in\mathbb{Z}$, be the linear operator on $L^p\,(0,2\pi)$ defined by

$$P_k f\,(x)=\frac{1}{2\pi}\int_0^{2\pi} f\,(y)\;e^{ik(x-y)}\,dy.$$

Then the "scalar" Marcinkiewicz's theorem implies the existence of constant $c_1\,(p)$ such that

$$\left|\left|\sum_{k=N_1}^{N_2}\lambda_k P_k\right|\right|_{\mathcal{B}(L^p(0,2\pi))}\le c_1\,(p) \tag{9}$$

$(N_1,N_2$ are arbitrary integers, $c_1\,(p)$ a constant which depends only on $p)$.

Consider the operator

$$T=\sum_m P_m\otimes A_m.$$

We will show that T is bounded on $E\widehat{\otimes} F$. To do this it is enough to show that it is bounded on $E\otimes F$.

Let $r\le m\le s$, $r,s\in\mathbb{Z}$. From the representation

$$T=\sum_{m=r}^{s}\lambda_m P_m\otimes\frac{A_m}{\lambda_m},$$

by putting

$$S_{r,\nu}=\sum_{m=r}^{\nu}\lambda_m P_m,\qquad \nu=r,r+1,\ldots,s,$$

and using Abel's summation formula, we get

$$T=S_{r,s}\otimes\frac{A_s}{\lambda_s}+\sum_{m=r}^{s-1}S_{r,m}\otimes\left(\frac{A_m}{\lambda_m}-\frac{A_{m+1}}{\lambda_{m+1}}\right).$$

From (7),(8),(4),(9) and the previous equality we obtain

$$||T||_{\mathcal{B}(E\otimes F)} \leq c_2(p) \tag{10}$$

where $c_2(p)$ is a constant depending on p only. Note that if $\Phi_m \in L^p(0,1)$, then

$$z = \sum_m e^{imx}\Phi_m(y) \in E \otimes F$$

and

$$Tz = \sum_m e^{imx} A_m \Phi_m(y) \in E \otimes F.$$

Hence, (10) gives

$$||Tz||_{E\otimes F} \leq c_2(p)\ ||z||_{E\otimes F}\,.$$

This inequality together with (6) yields

$$\begin{aligned}(c(p))^{-1}\,||Tz||_{L^p((0,2\pi)\times(0,1))} &\leq ||Tz||_{E\otimes F}\\ &\leq c_2(p)\ ||z||_{E\otimes F} \leq c(p)\ c_2(p)\ ||z||_{L^p((0,2\pi)\times(0,1))}\,,\end{aligned}$$

i.e.

$$||Tz||_{L^p((0,2\pi)\times(0,1))} \leq c_p^p\ ||z||_{L^p((0,2\pi)\times(0,1))}$$

where $c_p = \left((c(p))^2\ c_2(p)\right)^{\frac{1}{p}}$, which completes the proof. □

Remark 2. Putting

$$\Phi_m(y) = \begin{cases} r_m(t)\,f_m(y)\,; & m \geq 1, f_m \in L^p(0,1)\,,\\ 0\quad ; & m \leq 0,\end{cases}$$

in (1) (r_m are Rademacher's functions) we get

$$\int_0^1\int_0^{2\pi}\left|\sum_{m\geq 1} e^{imx} r_m(t)\,A_m f_m(y)\right|^p dxdy \leq c_p^p \int_0^1\int_0^{2\pi}\left|\sum_{m\geq 1} e^{imx} r_m(t)\,f_m(y)\right|^p dxdy,$$

whence by integration on $t \in [0,1]$, and using Khinchin's inequality, we get

$$\int_0^1 \left(\sum_{m\geq 1} |A_m f_m(y)|^2\right)^{\frac{p}{2}} dy \leq d_p^p \int_0^1 \left(\sum_{m\geq 1} |f_m(y)|^2\right)^{\frac{p}{2}} dy \tag{11}$$

(d_p a constant depending only on p). Therefore, if there holds (4), then we have (11). Inequality (11) was proved in [4], p. 336, in the case where all the operators A_m are mutually equal. From Theorem 1 it follows that (11) holds in the case where A_m need not be equal but are "mutually near" in the sense that there holds in (4). Now suppose that (1) holds when we replace (4) by the conditions analogous to Marcinkiewicz's conditions:

$$\begin{cases} \sup\limits_{m\in\mathbb{Z}} \|A_m\|_{\mathcal{B}(L^p(0,1))} < +\infty,\\ \sup\limits_{m\geq 0} \sum\limits_{|j|=2^m+1}^{2^{m+1}} \|A_j - A_{j+1}\|_{\mathcal{B}(L^p(0,1))} < +\infty.\end{cases} \tag{12}$$

Then (11) holds as well.

Let $B_m \in \mathcal{B}(L^p(0,1))$ be a sequence of operators such that $\sup\limits_{m\geq 1} \|B_m\|_{\mathcal{B}(L^p(0,1))} < +\infty$ and let

$$A_m = \begin{cases} 0 & ; \quad m \neq 2^k, \\ B_k & ; \quad m = 2^k. \end{cases}$$

Clearly $(A_m)_{m=1}^{\infty}$ satisfies (12), so (11) holds for an arbitrary (finite) choice of functions $f_m \in L^p(0,1)$. Putting

$$f_m = \begin{cases} 0 & ; \quad m \neq 2^k, \\ \varphi_k & ; \quad m = 2^k, \varphi_k \in L^p(0,1), \end{cases}$$

we get

$$\int\limits_0^1 \left(\sum_{k\geq 1} |B_k \varphi_k(y)|^2 \right)^{\frac{p}{2}} dy \leq d_p^p \int\limits_0^1 \left(\sum_{k\geq 1} |\varphi_k(y)|^2 \right)^{\frac{p}{2}} dy,$$

i.e. an inequality of the form (11) for which it is not clear whether it holds only under the assumption $\sup\limits_{k\geq 1} \|B_k\|_{\mathcal{B}(L^p(0,1))} < +\infty$.

If A_m are integral operators on $L^p(0,1), 1 < p < \infty$, then (4) can sometimes be replaced by the condition imposed on the kernel of A_m.

Namely we have the following

Corollary 1. *Let $K_m(x,y), m \in \mathbb{Z}$, be homogeneous functions of order -1 and let A_m' be the bounded operators on $L^p(0,\infty)$ defined by*

$$A_m' f(x) = \int\limits_0^\infty K_m(x,y)\, f(y)\, dy$$

such that

$$\sum_m \int\limits_0^\infty y^{-\frac{1}{p}} \left| \frac{K_m(1,y)}{\lambda_m} - \frac{K_{m+1}(1,y)}{\lambda_{m+1}} \right| dy < +\infty.$$

If $A_m : L^p(0,1) \longrightarrow L^p(0,1)$ are the operators defined by

$$A_m f(x) = \int\limits_0^1 K_m(x,y)\, f(y)\, dy,$$

then (1) *holds.*

Proof. Let $P : L^p(0,\infty) \longrightarrow L^p(0,\infty)$ be the linear operator defined by

$$Pf(x) = \mathcal{X}_{[0,1]}(x)\, f(x),$$

where

$$\mathcal{X}_{[0,1]}(x) = \begin{cases} 1, & x \in [0,1], \\ 0, & x \notin [0,1]. \end{cases}$$

Applying the Hardy-Littlewood inequality (see [2]) we get

$$\begin{aligned} \left\| \frac{A_m}{\lambda_m} - \frac{A_{m+1}}{\lambda_{m+1}} \right\|_{\mathcal{B}(L^p(0,1))} &= \left\| P \left(\frac{A_m'}{\lambda_m} - \frac{A_{m+1}'}{\lambda_{m+1}} \right) P \right\|_{\mathcal{B}(L^p(0,\infty))} \\ &\leq \left\| \frac{A_m'}{\lambda_m} - \frac{A_{m+1}'}{\lambda_{m+1}} \right\|_{\mathcal{B}(L^p(0,\infty))} \end{aligned}$$

and

$$\left\| \frac{A_m^{'}}{\lambda_m} - \frac{A_{m+1}^{'}}{\lambda_{m+1}} \right\|_{\mathcal{B}(L^p(0,\infty))} \leq \int\limits_0^\infty y^{-\frac{1}{p}} \left| \frac{K_m(1,y)}{\lambda_m} - \frac{K_{m+1}(1,y)}{\lambda_{m+1}} \right| dy,$$

which implies the corollary. □

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