ON THE ERROR TERM IN AN ASYMPTOTIC FORMULA FOR THE SYMMETRIC SQUARE $L$-FUNCTION

YUK-KAM LAU

(Communicated by Wen-Ching Winnie Li)

Abstract. Recently Wu proved that for all primes $q$, 

$$\sum_f L(1, \text{sym}^2 f) = \frac{\pi^2}{432} q + O(q^{27/28} \log^6 q)$$

where $f$ runs over all normalized newforms of weight 2 and level $q$. Here we show that $27/28$ can be replaced by $9/10$.

1. Introduction

Let $q$ be a prime and 

$$\Gamma_0(q) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) : q|c \right\}.$$ 

We denote by $S_2(q)$ the space of all holomorphic cusp forms for $\Gamma_0(q)$ of weight 2. With respect to the inner product 

$$\langle f, g \rangle = \int_{\Gamma_0(q) \backslash \mathbb{H}} f(z) \overline{g(z)} \, dx \, dy,$$

$S_2(q)$ is a finite-dimensional Hilbert space, and there is an orthogonal basis $\mathcal{B}_2(q)$ (which is the set of all normalized newforms in $S_2(q)$) such that 

(i) each $f \in \mathcal{B}_2(q)$ is a common eigenvector of all Hecke operators $T_n$ with $(n, q) = 1$, i.e. when $f \in \mathcal{B}_2(q)$ and $(n, q) = 1$, 

$$T_n f = \lambda_f(n) f;$$

(ii) the Fourier expansion of $f \in \mathcal{B}_2(q)$ is 

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) e(nz)$$

where $e(\alpha) = e^{2\pi i \alpha}$, $\lambda_f(n)$ is the eigenvalue in (i) if $(n, q) = 1$ and $\lambda_f(n)^2 = l^{-1}\lambda_f(m)^2$ if $n = lm$ where $l$ is a power of $q$ and $(m, q) = 1$ (see [3, (2.19) and (2.24)]).
For the properties of $\lambda_f(n)$, it is known that they are all real and satisfy the Deligne bound $|\lambda_f(n)| \leq \tau(n)$. (Here and in the sequel $\tau(n) = \sum_{d|n} 1$ is the divisor function.) Moreover we have

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \epsilon_q(d)\lambda_f\left(\frac{mn}{d^2}\right)$$

where $\epsilon_q$ is the principal character mod $q$. In particular, we see that $\lambda_f(1) = 1.$ 

Associated to each $f \in B_2(q)$, we define the symmetric square $L$-function by

$$L(s, \text{sym}^2 f) = \zeta_q(2s) \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s} \quad \text{for } \Re s > 1,$$

where $\zeta_q(s) = \prod_{p \mid q} (1 - p^{-s})^{-1}$. This $L$-function extends to an entire function over $\mathbb{C}$ and it satisfies a functional equation; more precisely, let us write

$$\Lambda(s, \text{sym}^2 f) = \left(\frac{q}{\pi^3/2}\right)^s \Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\right)L(s, \text{sym}^2 f).$$

Then we have $\Lambda(s, \text{sym}^2 f) = \Lambda(1-s, \text{sym}^2 f)$. Analogous to the Riemann zeta function, the values attained by $L(s, \text{sym}^2 f)$ in the critical strip are interesting. Particularly for $s = 1$ and all large prime $q$, we have the asymptotic formula

$$\sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{\pi^4}{432} q + O(q^{7/10} \log^{16} q)$$

for some constants $0 < \alpha < 1$ and $\beta > 0$. Here, we are concerned with the size of the error term. In [1], Akbary proved that $\alpha = 45/46$ is admissible and recently Wu gave an improvement to $\alpha = 27/28$ (see [2]). Our purpose is to show the refinement below.

**Theorem.** Let $q$ be a prime. There is an absolute constant $c > 0$ such that

$$\sum_{f \in B_2(q)} L(1, \text{sym}^2 f) = \frac{\zeta(2)^3}{2\pi^2} q + O(q^{9/10} \log^{16} q).$$

(Note that $\zeta(2)^3/(2\pi^2) = \pi^4/432$.)

**Remark.** In decimal form we have $\frac{45}{46} \approx 0.978$, $\frac{27}{28} \approx 0.964$ and $\frac{9}{10} = 0.9$.

2. SOME PREPARATION

**Lemma 1.** Let $A > 1$ be any fixed constant and $q < y < q^A$ but $y \notin \mathbb{Z}$. We have

$$L(1, \text{sym}^2 f) = \zeta_q(2) \sum_{n \leq y} \frac{\lambda_f(n^2)}{n} + O(q^{7/10} (y^{-1} + (\frac{q}{y})^{2/7}))$$

where $\epsilon > 0$ is an arbitrarily small constant and the implied constant in the $O$-term depends on $\epsilon$.

**Proof.** This follows from the truncated Perron’s formula. Using the estimate

$$\Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\right) \asymp |t|^{(3|s+1|)/2} e^{-3\pi|t|/4}$$

for $s = \sigma + it$ where $\sigma \ll 1$ and $|t| \gg 1$, we can derive from the functional equation the convexity bound: for $0 \leq \sigma \leq 1$,

$$L(\sigma + it, \text{sym}^2 f) \ll (|t|^{3/2})^{1-\sigma+\epsilon}.$$
By [2, Lemma 12.1], we see that for any $T \gg 1$,
\[
\zeta_q(2) \sum_{n \leq y} \frac{\lambda_f(n^2)}{n} \leq \frac{\zeta_q(2)}{2\pi i} \int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^{1+s}} \frac{y^s ds}{s} + O(y^\infty \sum_{n=1}^{\infty} \frac{n^{1+\varepsilon} \min(1, (T|\log \frac{y}{n}|)^{-1}))}
\]
\[(5) \quad \text{To evaluate the O-term, we split the summation over n into three pieces: } n \leq y/2, n \geq 3y/2 \text{ and } y/2 < n < 3y/2. \text{ As } |\log(y/n)| \gg 1 \text{ in the first two pieces, these two sums are } O(T^{-1}y^\varepsilon). \text{ The third one is}
\]
\[
\ll y^{-1} \sum_{y/2 < n < 3y/2} |y - n|^{-1} + y^{-1+\varepsilon} \ll y'(T^{-1} + y^{-1}).
\]
Thus the overall contribution is absorbed in the O-term in our lemma.

From (2), we can replace $\sum_{n=1}^{\infty} \lambda_f(n^2)n^{-(1+s)}$ in (5) by
\[
\zeta_q(2+2s)^{-1}L(1+s,\text{sym}^2f).
\]
Then we apply the residue theorem to the rectangular contour with vertices at $\pm iT$ and $-1/2 + \pm iT$. The integral in (5) equals a sum of two terms: the main term $L(1,\text{sym}^2f)$ from the pole at $s = 0$, and the remainder term which is
\[
\ll \int_{-1/2+\varepsilon}^{1/2-\varepsilon} \frac{|L(1+\alpha + iT,\text{sym}^2f)|y^\alpha}{\zeta_q(2+2\alpha+i2T)} \frac{d\alpha}{T} + \int_{-T}^{T} \frac{|L(1/2 + \varepsilon + it,\text{sym}^2f)|}{\zeta_q(1+2\varepsilon+i2t)} \frac{dt}{1+|t|}
\]
Using the bound $\zeta(\sigma+it)^{-1} \ll \log(1+|t|)$ for $\sigma \geq 1$ and $|t| \gg 1$, the two O-terms are $\ll (yT)^{\varepsilon}(y^{-1/2}q^{1/2}T^{3/4} + T^{-1})$. The proof is complete after setting $T = (y/q)^{2/7}$.

Our next task is to extend the admissible range in [3, Lemma 2]. To this end, we modify the mean square estimate result in [4, Corollary 1]. Suppose $M \leq q^9$ and $\{a_n\}_{1 \leq n \leq M}$ is a sequence of complex numbers. Then by taking $a_n = 0$ for $M < n \leq q^9$, [3, Proposition 1] with $N = q^9$ gives
\[
(6) \quad \sum_{f \in B_2(q)} \left| \sum_{n \leq M} a_n \rho_f(n) \right|^2 \ll q^9(\log q)^{15} \sum_{n \leq M} |a_n|^2
\]
where $\rho_f(n) = \sum_{r^2 = n} e_q(m)\lambda_f(l^2)$. (Note that $B_2(q) = S_2(q)^*$ in [4] for prime $q$.)

Lemma 2. Let $M \gg 1$ and suppose that $\{a(n)\}_{M < n \leq 2M}$ satisfies
\[
a(n) \ll \frac{(\tau(n) \log n)^A}{n}
\]
for some constant $A > 0$. There exists a constant $B = B(A) \geq 0$ such that
\[
\sum_{f \in B_2(q)} \left| \sum_{M < n \leq 2M} a(n)\lambda_f(n^2) \right|^2 \ll \max(1, q^9M^{-1}) \log^B(qM).
\]
The implied constant depends on $A$. 

Proof. When $M \geq q^3$, it follows immediately from [4, Corollary 1] (by taking $N = M$). Consider the case $M < q^3$. From [4, (16)], we have

$$S := \sum_{f \in B_2(q)} \left| \sum_{M < n \leq 2M} a(n) \lambda_f(n^2) \right|^2 = \sum_{f \in B_2(q)} \left| \sum_{l < 2M} a_l \rho_f(l) \right|^2$$

where

$$a_l = \sum_{\sqrt{M/l} < m \leq \sqrt{2M/l}} \mu(m) \epsilon_q(m) a(lm^2)$$

$$\ll \frac{(\tau(l) \log 2l)^A}{l} \sum_{\sqrt{M/l} < m \leq \sqrt{2M/l}} \frac{(\tau(m) \log 2m)^2 A}{m^2}$$

$$\ll (Ml)^{-1/2} (\log ML)^B$$

(see the proof of [4, Corollary 1] as well). $B$ denotes an unspecified positive constant depending on $A$ and its value may differ at each occurrence in the proof. By (6),

$$S \ll q^3 (\log q)^{15} \sum_{l < 2M} (Ml)^{-1} (\log ML)^B < q^3 M^{-1} \log^B (qM).$$

Define for $1 \leq x < y$,

$$\omega_f(x, y) = \sum_{x \leq n < y} \lambda_f(n^2) \frac{\mu(n)}{n}.$$ 

**Lemma 3.** Let $x > 0$ and $x < y \ll q^A$ for some constant $A > 0$. Suppose $r \geq 1$ is a fixed integer satisfying $x^r \geq q^3$. Then there exists a constant $D = D(r) > 0$ such that

$$\sum_{f \in B_2(q)} \omega_f(x, y)^{2r} \ll (\log q)^D$$

where the implied constant depends on $A$ and $r$.

**Proof.** Following the argument in the proof of [4, Lemma 4], one can show that

$$\omega_f(x, y)^r = \sum_{x^r \leq mn < y^r} \lambda_f(m^2) \frac{c(m, n)}{mn}$$

where $c(m, n)$ is independent of $f$ and $c(m, n) = 0$ if $n$ is not of the form $n = dn_1$ where $d|m$ and $n_1$ is squarefull. Moreover, $|c(m, n)| \leq \tau(mn)^\gamma$ for some integer $\gamma = \gamma(r) > 0$ depending on $r$. Then we write

$$\omega_f(x, y)^r = \sum_{H = 2^k} \sum_{x^r \leq mn < y^r} \lambda_f(m^2) \frac{c(m, n)}{mn}$$

where the first summation runs over all nonnegative integers $k$. Define

$$c_H(m) = \sum_{H \leq n < 2H} \sum_{x^r m^{-r} \leq n < y^r m^{-1}} \frac{c(m, n)}{n}.$$
Then, using \( \sum_{n \leq z} \tau(n)^\gamma \ll z^{1/2} (\log z)^{2\gamma} \), we have
\[
\chi_H(m) \ll \tau(m)^\gamma \sum_{d|m} \frac{1}{d} \sum_{H \leq d H < 2H \text{ squarefull}} \tau(n)^\gamma \sum_{n \leq z} \tau(n)^\gamma \sum_{d|m} \frac{1}{d} \sum_{H \leq d H < 2H \text{ squarefull}} \tau(n)^\gamma \sum_{n \leq z} \tau(n)^\gamma
\]
\[
\ll \tau(m)^\gamma \cdot \left( \sum_{d|m} \frac{1}{d} \sum_{d > \sqrt{H}} \frac{1}{d} \sum_{H \leq d H < 2H \text{ squarefull}} \tau(n)^\gamma \sum_{n \leq z} \tau(n)^\gamma \right)
\]
\[
(7) \quad \ll H^{-1/2} (\tau(m)(\log m)(\log H))^D.
\]
Here we use \( D \) to denote a positive constant (depending on \( r \)) which may assume different values at other places. Making use of (6) for \( H \geq q \),
\[
\omega_f(x, y)^r = \sum_{H=2^k < q} \sum_{x''/(2H) < m \leq y''/H} \lambda_f(m^2) \frac{\chi_H(m)}{m} + O(q^{-1/2} \log D q).
\]
Squaring both sides and averaging over all \( f \in B_2(q) \) yields
\[
\sum_{f \in B_2(q)} \omega_f(x, y)^{2r} \ll \left( \sum_{H=2^k < q} H^{-1} \sum_{x''/(2H) < m \leq y''/H} \lambda_f(m^2) \frac{\chi_H(m)\sqrt{H}}{m} \right)^2 + 1 \log D q
\]
\[
(8) \quad \ll \left( \sum_{H=2^k < q} H^{-1} \sum_{x''/(2H) < m \leq y''/H} \lambda_f(m^2) \frac{\chi_H(m)\sqrt{H}}{m} \right)^2 + 1 \log D q
\]
as \( \sum_{i \in I} a_i^2 \ll |I| \sum_{i \in I} a_i^2 \) and \( |B_2(q)| \ll q \). For each \( H \), we split the range of the summation over \( m \) into dyadic intervals \( M < m \leq 2M \) where \( M \geq x''/(2H) \). It follows from Lemma [2] and [7] that
\[
\sum_{f \in B_2(q)} \left( \lambda_f(m^2) \frac{\chi_H(m\sqrt{H})}{m} \right)^2 \ll \max(1, q^9 x^{-r} H) \log D q.
\]
Inserting it into (3), we conclude that
\[
\sum_{f \in B_2(q)} \omega_f(x, y)^{2r} \ll \log D q \sum_{H=2^k < q} \max(H^{-1}, q^9 x^{-r}),
\]
and our result follows in view of the condition \( x'' \geq q^9 \).

3. Proof of the Theorem

Define for \( f \in B_2(q) \), \( w_f = 4\pi(f, f) \), which is a positive real number. We have from [3] Lemma 2.5 that \( w_f = (2\pi^2)^{-1} q L(1, \text{sym}^2 f) \) and from [3] Corollary 2.2 (with \( \tau_3((m, n)) \leq \tau((m, n))^2 \leq \tau(m)\tau(n) \)) that
\[
(9) \quad \sum_{f \in B_2(q)} w_f^{-1} \lambda_f(m^2) \lambda_f(n^2) = \delta(m, n) + O(q^{-1} (mn)^{1/2} (\tau(m)\tau(n))^2 \log 2mn)
\]
for \( \min(m, n) < q \), where \( \delta(\cdot, \cdot) \) is the Kronecker delta. (Note that \( w_f^{-1} = w_f \) in [3].) In particular, \( \sum_{f \in B_2(q)} w_f^{-1} \ll 1 \) as \( \lambda_f(1) = 1 \).

We split the sum over \( n \) in Lemma [1] into two subsums \( \sum_{n \leq x} + \sum_{x < n < y} \) where \( 1 < x < q < y \). (Our choice will be \( x = q^{9/10} \) and \( y = q^{173/110} \).) Squaring the
Applying the argument in (12), we get that
\[
\sum_{f \in \mathcal{B}_2(q)} L(1, \text{sym}^2 f) = \frac{q}{2\pi^2} \sum_{f \in \mathcal{B}_2(q)} w_f^{-1} L(1, \text{sym}^2 f)^2
\]
(10)
\[
= \frac{q}{2\pi^2} \zeta_q(2)(S_1 + 2S_2 + S_3) + O(q^{-1} + (q^2/7))
\]
where
\[
S_1 = \sum_f w_f^{-1} \left( \sum_{n \leq x} \frac{\lambda_f(n^2)}{n} \right)^2,
\]
\[
S_2 = \sum_f w_f^{-1} \left( \sum_{n \leq x} \frac{\lambda_f(n^2)}{n} \right) \left( \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} \right),
\]
\[
S_3 = \sum_f w_f^{-1} \left( \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} \right)^2.
\]

It follows from the bound \(w_f^{-1} \ll q^{-1} \log q\) (see [4, (20)]) and Lemma 3 that if \(x^r \geq q^9\),
\[
S_3 \ll \frac{\log q}{q} \sum_f \omega_f(x, y)^2 \ll (\sum_f \omega_f(x, y)^2)^{1/r} |\mathcal{B}_2(q)|^{1-1/r} q^{-1} \log q
\]
(11)
\[
\ll q^{-1/r} \log^{c_1} q.
\]

Throughout \(c_i, i = 1, 2, \cdots\), denote unspecified positive constants. Using (10), we obtain that for \(x < q\),
\[
S_1 = \sum_{n \leq x} n^{-2} + O(q^{-1} \sum_{m, n \leq x} (mn)^{-1/2} \tau(m)^2 \tau(n)^2 \log 2mn)
\]
(12)
\[
= \zeta(2) + O(x^{-1} + q^{-1} x \log^{c_1} q).
\]

To treat \(S_2\), we split it into two parts: let \(z = qx^{-1}\),
\[
S_2 = \sum_f w_f^{-1} \sum_{n \leq z} \frac{\lambda_f(n^2)}{n} \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n} + \sum_f w_f^{-1} \sum_{z \leq n \leq x} \frac{\lambda_f(n^2)}{n} \sum_{x < n \leq y} \frac{\lambda_f(n^2)}{n}
\]
(13)
\[
= S_{21} + S_{22}, \text{ say.}
\]

By (9), we have, provided that \(z \leq x\) (or equivalently \(x \geq q^{1/2}\)),
\[
S_{21} \ll q^{-1}(\log^{c_1} q) \sum_{m \leq z \leq n \leq y} \tau(m)^2 \tau(n)^2 (mn)^{-1/2} \ll \frac{y}{q^x} \log^{c_1} q.
\]

Applying the argument in (12), we get that
\[
 \sum_f w_f^{-1} \left( \sum_{z < n \leq x} \frac{\lambda_f(n^2)}{n} \right)^2 \ll z^{-1} + q^{-1} x \log^{c_1} q \ll q^{-1} x \log^{c_1} q.
\]
By \( ab \ll |a|^2 + |b|^2 \) and (11), we have \( S_{22} \ll (q^{-1/r} + q^{-1}x) \log^{c_1} q \). Hence, by (13),
\[
S_2 \ll (q^{-1/r} + q^{-1}x + \left( \frac{y}{qx} \right)^{1/2}) \log^{c_1} q.
\]
Putting this estimate, (11) and (12) into (10), we infer that as \( q(2) = q(2) + O(q^{-2}) \),
\[
\sum_{f \in \mathcal{B}_2(q)} L(1, \operatorname{sym}^2 f) = \frac{\zeta(2)^3}{2\pi^2} q + qO((q^{-1/r} + q^{-1}x) \log^{c_1} q
+ q^r (x^{-1} + (\frac{y}{qx})^{1/2} + (\frac{q}{y})^{2/7})).
\]
Subject to the condition \( x^r \geq q^9 \), we take \( x = q^{9/r} \) and select \( r = 10 \), \( x = q^{9/10} \)
and \( y = q^{173/110} \) by equating \( q^{-1/r} = q^{-1}x \). This ends the proof.

References


Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong

E-mail address: yklau@maths.hku.hk