

**WEAKLY SEQUENTIAL COMPLETENESS  
 OF THE PROJECTIVE TENSOR PRODUCT**

$$L^p[0, 1] \hat{\otimes} X, \quad 1 < p < \infty$$

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**ABSTRACT.** D. R. Lewis (1977) proved that for a Banach space  $X$  and  $1 < p < \infty$ ,  $L^p[0, 1] \hat{\otimes} X$ , the projective tensor product of  $L^p[0, 1]$  and  $X$ , is weakly sequentially complete whenever  $X$  is weakly sequentially complete. In this note, we give a short proof of Lewis's result, based on our sequential representation (2001) of  $L^p[0, 1] \hat{\otimes} X$ .

For any Banach space  $X$ , we will denote its topological dual by  $X^*$  and its closed unit ball by  $B_X$ . From [4, page 3] and [5, page 155], we know that the Haar system  $\{\chi_i\}_{i=1}^\infty$  is an unconditional basis of  $L^p[0, 1]$ ,  $1 < p < \infty$ . Let us use  $K_p$  to denote the unconditional basis constant of the basis  $\{\chi_i\}_{i=1}^\infty$ . Now renorm  $L^p[0, 1]$  by

$$\|f\|_p^{new} = \sup \left\{ \left\| \sum_{i=1}^{\infty} \theta_i a_i \chi_i \right\|_p : \theta_i = \pm 1, i = 1, 2, \dots \right\}, \quad f = \sum_{i=1}^{\infty} a_i \chi_i \in L^p[0, 1].$$

Then

$$\|\cdot\|_p \leq \|\cdot\|_p^{new} \leq K_p \cdot \|\cdot\|_p.$$

With this new norm,  $L^p[0, 1]$  is also a Banach space. Furthermore,  $\{\chi_i\}_{i=1}^\infty$  is a monotone, unconditional basis with respect to this new norm. Now let

$$e_i = \frac{\chi_i}{\|\chi_i\|_p^{new}}, \quad i = 1, 2, \dots$$

Then  $\{e_i\}_{i=1}^\infty$  is a normalized, unconditional basis of  $(L^p[0, 1], \|\cdot\|_p^{new})$  whose unconditional basis constant is 1. From [4, pp. 18–19] we have the following

**Proposition 1.** *Let  $u = \sum_{i=1}^\infty \alpha_i e_i \in L^p[0, 1]$ ,  $1 < p < \infty$ . Then*

- (i) *For each subset  $\sigma$  of  $\mathbb{N}$ ,  $\|\sum_{i \in \sigma} \alpha_i e_i\|_p^{new} \leq \|u\|_p^{new}$ .*
- (ii) *For each choice of signs  $\theta = \{\theta_i\}_1^\infty$ ,  $\|\sum_{i=1}^\infty \theta_i \alpha_i e_i\|_p^{new} \leq \|u\|_p^{new}$ .*
- (iii) *For each  $\lambda = (\lambda_i)_i \in \ell_\infty$ ,  $\|\sum_{i=1}^\infty \lambda_i \alpha_i e_i\|_p^{new} \leq 2 \cdot \|\lambda\|_{\ell_\infty} \cdot \|u\|_p^{new}$ .*

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For any Banach space  $X$  and  $1 < p < \infty$  with  $1/p + 1/p' = 1$ , define

$$L_{weak}^p(X) = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_i x^*(x_i)e_i \text{ converges in } L^p[0, 1] \text{ for } \forall x^* \in X^* \right\},$$

$$L^p\langle X \rangle = \left\{ \bar{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i^*(x_i)| < \infty \forall (x_i^*)_i \in L_{weak}^{p'}(X^*) \right\},$$

and define norms on  $L_{weak}^p(X)$  and  $L^p\langle X \rangle$ , respectively, to be

$$\|\bar{x}\|_{weak} = \sup \left\{ \left\| \sum_{i=1}^{\infty} x^*(x_i)e_i \right\|_p^{new} : x^* \in B_{X^*} \right\}, \quad \bar{x} \in L_{weak}^p(X),$$

$$\|\bar{x}\|_{L^p\langle X \rangle} = \sup \left\{ \sum_{i=1}^{\infty} |x_i^*(x_i)| : (x_i^*)_i \in B_{L_{weak}^{p'}(X^*)} \right\}, \quad \bar{x} \in L^p\langle X \rangle.$$

With their own norm respectively,  $L_{weak}^p(X)$  and  $L^p\langle X \rangle$  are Banach spaces [1], [2]. Moreover, from [1] we have the following

**Proposition 2.** (i) For each  $\bar{x} = (x_i)_i \in L^p\langle X \rangle$ ,

$$\lim_n \|(0, \dots, 0, x_n, x_{n+1}, \dots)\|_{L^p\langle X \rangle} = 0.$$

(ii)  $L^p[0, 1] \hat{\otimes} X$  is isomorphic to  $(L^p[0, 1], \|\cdot\|_p^{new}) \hat{\otimes} X$  which is isometrically isomorphic to  $L^p\langle X \rangle$ .

**Proposition 3.**  $(L^p\langle X \rangle)^* = L_{weak}^{p'}(X^*)$ , where the dual operation is defined by

$$\langle \bar{x}, \bar{x}^* \rangle = \sum_{i=1}^{\infty} x_i^*(x_i),$$

for each  $\bar{x} = (x_i)_i \in L^p\langle X \rangle$  and each  $\bar{x}^* = (x_i^*)_i \in L_{weak}^{p'}(X^*)$ .

*Proof.* Let  $F \in (L^p\langle X \rangle)^*$ . For each  $i \in \mathbb{N}$ , define an  $x_i^* \in X^*$  by

$$x_i^*(x) = \langle (0, \dots, 0, \overset{(i)}{x}, 0, 0, \dots), F \rangle.$$

Then one can associate a linear operator  $\tilde{F} : (L^p[0, 1], \|\cdot\|_p^{new}) \longrightarrow X^*$  by

$$\tilde{F}(f)(x) = \langle (e_i^*(f)x)_i, F \rangle,$$

for each  $f \in (L^p[0, 1], \|\cdot\|_p^{new})$  and  $x \in X$ , i.e.,

$$\tilde{F}(f)(x) = \sum_{i=1}^{\infty} e_i^*(f)x_i^*(x).$$

The operator  $\tilde{F}$  is bounded, in fact

$$\|\tilde{F}\| \leq \|F\|,$$

and it is easy to check that

$$\|(x_i^*)_i\|_{L_{weak}^{p'}(X^*)} = \|\tilde{F}\|.$$

This shows that  $F = (x_i^*)_i \in L_{weak}^{p'}(X^*)$ . The other inclusion is obvious.  $\square$

**Lemma 4.** For each  $\bar{x}^* \in (L^p\langle X \rangle)^* = L^p{}'_{weak}(X^*)$ , define

$$\begin{aligned} I_{\bar{x}^*} : L^p\langle X \rangle &\longrightarrow \ell_1 \\ \bar{x} &\longmapsto (x_i^* x_i)_i. \end{aligned}$$

Then  $I_{\bar{x}^*}$  is a continuous linear map.

*Proof.* For each  $(s_i)_i \in \ell_\infty$ , by Proposition 1,

$$\begin{aligned} \left| \sum_{i=1}^{\infty} s_i x_i^* x_i \right| &= |\langle (s_i x_i)_i, \bar{x}^* \rangle| \leq \| (s_i x_i)_i \|_{L^p\langle X \rangle} \cdot \| \bar{x}^* \|_{(L^p\langle X \rangle)^*} \\ &\leq 2 \cdot \| (s_i)_i \|_{\ell_\infty} \cdot \| \bar{x} \|_{L^p\langle X \rangle} \cdot \| \bar{x}^* \|_{(L^p\langle X \rangle)^*}. \end{aligned}$$

So

$$\| (x_i^* x_i)_i \|_{\ell_1} \leq 2 \cdot \| \bar{x} \|_{L^p\langle X \rangle} \cdot \| \bar{x}^* \|_{(L^p\langle X \rangle)^*}.$$

Therefore,  $I_{\bar{x}^*}$  is continuous. The proof is complete.  $\square$

**Theorem 5.**  $L^p[0, 1] \hat{\otimes} X$  is weakly sequentially complete if  $X$  is weakly sequentially complete.

*Proof.* By Proposition 2, it is enough to show that  $L^p\langle X \rangle$  is weakly sequentially complete if  $X$  is weakly sequentially complete. Let  $\{\bar{x}^{(n)}\}_1^\infty$  be a weakly Cauchy sequence in  $L^p\langle X \rangle$ . It is easy to see that  $\{x_i^{(n)}\}_{n=1}^\infty$  are weakly Cauchy sequences in  $X$  for each  $i \in \mathbb{N}$ . Therefore there are  $x_i \in X$  such that

$$(1) \quad \text{weak-}\lim_n x_i^{(n)} = x_i, \quad i = 1, 2, \dots.$$

Denote  $M = \sup_n \| \bar{x}^{(n)} \|_{L^p\langle X \rangle} < \infty$ . For each  $\bar{x}^* = (x_i^*)_i \in L^p{}'_{weak}(X^*) = (L^p\langle X \rangle)^*$  and each fixed  $m \in \mathbb{N}$ , from (1) there exists an  $n_0 \in \mathbb{N}$  such that

$$|x_i^*(x_i^{(n_0)} - x_i)| \leq 1/m, \quad i = 1, 2, \dots, m.$$

Thus,

$$\begin{aligned} \sum_{i=1}^m |x_i^*(x_i)| &\leq \sum_{i=1}^m |x_i^*(x_i^{(n_0)} - x_i)| + \sum_{i=1}^m |x_i^*(x_i^{(n_0)})| \\ &\leq 1 + \sum_{i=1}^{\infty} |x_i^*(x_i^{(n_0)})| \\ &\leq 1 + \| \bar{x}^* \|_{(L^p\langle X \rangle)^*} \cdot \| \bar{x}^{(n_0)} \|_{L^p\langle X \rangle} \\ &\leq 1 + M \| \bar{x}^* \|_{(L^p\langle X \rangle)^*}. \end{aligned}$$

Letting  $m \longrightarrow \infty$ ,

$$\sum_{i=1}^{\infty} |x_i^*(x_i)| \leq 1 + M \| \bar{x}^* \|_{(L^p\langle X \rangle)^*} < \infty.$$

Therefore  $\bar{x} = (x_i)_i \in L^p\langle X \rangle$ . Next, we want to show that  $\bar{x}^{(n)}$  converges to  $\bar{x}$  weakly in  $L^p\langle X \rangle$ .

Fix  $\bar{x}^* \in (L^p\langle X \rangle)^*$ . By Lemma 4,  $I_{\bar{x}^*}$  is continuous and hence, weakly – weakly continuous. Since  $\{\bar{x}^{(n)}\}_1^\infty$  is a weakly Cauchy sequence in  $L^p\langle X \rangle$ ,  $\{I_{\bar{x}^*}(\bar{x}^{(n)})\}_1^\infty$  is a weakly Cauchy sequence in  $\ell_1$ , and hence, relatively weakly sequentially compact.

By the Schur property,  $\{I_{\bar{x}^*}(\bar{x}^{(n)})\}_1^\infty$  is a relatively sequentially compact subset of  $\ell_1$ . Thus, for each  $\varepsilon > 0$ , there exists an  $m_1 \in \mathbb{N}$  such that

$$(2) \quad \sum_{i=m_1+1}^{\infty} |x_i^*(x_i^{(n)})| \leq \varepsilon/3, \quad n = 1, 2, \dots$$

By Proposition 3, there exists an  $m_2 > m_1$  such that

$$(3) \quad \sum_{i=m_2+1}^{\infty} |x_i^*(x_i)| \leq \varepsilon/3.$$

Now, from (1), there exists an  $n_0 \in \mathbb{N}$  such that for each  $n > n_0$ ,

$$(4) \quad |x_i^*(x_i^{(n)} - x_i)| < \varepsilon/3, \quad i = 1, 2, \dots, m_2.$$

Thus, from (2), (3), and (4), for each  $n > n_0$ ,

$$\begin{aligned} |\langle \bar{x}^{(n)} - \bar{x}, \bar{x}^* \rangle| &= \left| \sum_{i=1}^{\infty} x_i^*(x_i^{(n)} - x_i) \right| \\ &\leq \sum_{i=1}^{m_2} |x_i^*(x_i^{(n)} - x_i)| \\ &\quad + \sum_{i=m_2+1}^{\infty} |x_i^*(x_i^{(n)})| + \sum_{i=m_2+1}^{\infty} |x_i^*(x_i)| \\ &\leq \varepsilon. \end{aligned}$$

Hence,  $\bar{x}^{(n)}$  converges to  $\bar{x}$  weakly in  $L^p\langle X \rangle$ . Therefore,  $L^p\langle X \rangle$  is weakly sequentially complete. The proof is complete.  $\square$

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