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THE APPROXIMATION ORDER OF POLYSPLINES

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Abstract. We show that the scaling spaces defined by the polysplines of order p provide approximation order $2p$. For that purpose we refine the results on one-dimensional approximation order by L-splines obtained by de Boor, DeVore, and Ron (1994).

1. Introduction

In the last decade the approximation order of shift-invariant subspaces of the space $L^2(\mathbb{R}^n)$ of all square-integrable functions on the euclidean space $\mathbb{R}^n$ has been investigated extensively, e.g., in the survey paper [10] approximately 100 references are given. The problem can be formulated in a rather general way: suppose that $(V_h)_{h\in I}$ is a family of subspaces of $L^2(\mathbb{R}^n)$ (not necessarily shift-invariant) where I is a subset of $(0,\infty)$ having 0 as an accumulation point. One has to estimate the rates of decay of the approximation error

$$E(f,V_h) := \inf\left\{\|f-s\|_{L^2(\mathbb{R}^n)} : s\in V_h\right\} \tag{1}$$

for h tending to 0. If W is a subspace of $L^2(\mathbb{R}^n)$ endowed with a norm $\|\cdot\|_W$, we say that $(V_h)_{h\in I}$ *provides approximation order* m with respect to the norm $\|\cdot\|_W$ if there exists a constant c_W such that for every $f\in W$ and for every $h\in I$

$$E(f,V_h) \le c_W \cdot h^m \cdot \|f\|_W \ . \tag{2}$$

Usually W is the potential space $W_2^m(\mathbb{R}^n)$ for $m\in(0,\infty)$ defined as the subspace of those $f\in L^2(\mathbb{R}^n)$ such that

$$\|f\|_{W_2^m(\mathbb{R}^n)} := (2\pi)^{-\frac{n}{2}} \left\|(1+|\xi|)^m \widehat{f}(\xi)\right\|_{L^2(\mathbb{R}^n)} < \infty. \tag{3}$$

In this note we want to prove that cardinal polysplines of order p provide approximation order $2p$.

The motivation for the present work comes from the fact that polysplines are useful for solving multivariate interpolation problems [4], [5], [6] and they are of importance for the multivariate Wavelet Analysis; cf. the monograph [9]. Recall that a function $S:\mathbb{R}^n\setminus\{0\}\to\mathbb{C}$ is called a *cardinal polyspline*[1] *(on annuli) of*

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[1] The first author introduced polysplines in 1991 in a more general setting with arbitrary interfaces; see [3] and [9].

order p if S is $(2p-2)$-times continuously differentiable and the restriction of S to each open annulus $\left\{x \in \mathbb{R}^n : e^l < |x| < e^{l+1}\right\}$ is a polyharmonic function[2] of order p for $l \in \mathbb{Z}$. The reason for calling such polysplines "cardinal" is found in Theorem 3, where it is seen that after expanding S in a Fourier–Laplace series of spherical harmonics, the coefficients $S_{k,l}(\log r)$ are cardinal L-splines in the usual sense of the word; cf. Micchelli's paper of 1976 [8].

Introducing a parameter $h > 0$, by P_h we denote the set of all functions $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ which are $(2p-2)$-times continuously differentiable and whose restriction to each open annulus $A_{h,l} := \left\{x \in \mathbb{R}^n : e^{hl} < |x| < e^{h(l+1)}\right\}$ is a polyharmonic function of order p for $l \in \mathbb{Z}$. Then the *scaling spaces of polysplines of order p*, in short PV_h, are defined as the L^2-closure of $P_h \cap L^2(\mathbb{R}^n), h > 0$.

The main result is the following:

Theorem 1. *The sequence $(PV_h)_{h>0}$ provides approximation order $2p$ where p denotes the order of the polysplines. More precisely, there exists a constant $C > 0$ such that for all h with $0 < h < 1$ and $f \in L^2(\mathbb{R}^n)$ the following inequality holds:*

$$\inf\left\{\|f-g\|_{L^2(\mathbb{R}^n)} : g \in PV_h\right\} \leq C \cdot h^{2p} \cdot \left(\int_{\mathbb{R}^n} \left| \, |x|^{2p} \cdot \Delta^p f(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

Note that in place of the norm (3) we have a semi–norm on the right–hand side which is zero on the polyharmonic functions of order p.

The paper is organized as follows: in Section 2 we discuss the approximation order of cardinal L-splines by using important results from [2]. In Section 3 the main result will be proven.

2. APPROXIMATION ORDER OF CARDINAL L-SPLINES

Let us recall Theorem 4.3 in the fundamental paper [2]: Suppose that for every $h > 0$, the space S_h is the $L^2(\mathbb{R}^n)$-closure of the linear space generated by the shifts $\varphi_h(\cdot - m), m \in \mathbb{Z}^n$ of the function $\varphi_h \in L^2(\mathbb{R}^n)$ (so S_h is the shift-invariant space generated by φ_h) and that $V_h = \left\{s\left(\frac{\cdot}{h}\right) : s \in S_h\right\}$. Then the family $(V_h)_{h\in I}$ provides approximation order m with respect to the norm $\|\cdot\|_{W_2^m(\mathbb{R}^n)}$ defined in (3) if and only if there exists $D > 0$ such that for all $h \in I$ and for almost all $x \in C := [-\pi, \pi]^n$

$$|\Lambda_{\varphi_h}(x)| \leq D \cdot (h + |x|^m), \tag{4}$$

where

$$(\Lambda_{\varphi_h}(\xi))^2 := \frac{\sum_{\alpha\in\mathbb{Z}^n,\alpha\neq 0} \left|\widehat{\varphi_h}(\xi + 2\pi\alpha)\right|^2}{\sum_{\beta\in\mathbb{Z}^n} \left|\widehat{\varphi_h}(\xi + 2\pi\beta)\right|^2} \leq 1.$$

We will need a refinement of that result. For our purposes it will be useful to consider, instead of (3), different norms. In the following we replace the function $(1 + |x|)^m$ by a measurable function $Q(x)$ with the following properties: (i) the zero set $Q^{-1}(0)$ of Q is a set of Lebesgue measure zero and (ii) there exists a constant $D_1 > 0$ such that

$$\left|Q\left(\frac{x}{h}\right)\right| \geq D_1 \frac{1}{h^m} \qquad \text{for all } x \notin C := [-\pi, \pi]^n. \tag{5}$$

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[2] Recall that a function f defined on an open set U in the euclidean space $\mathbb{R}^n$ is *polyharmonic of order p* if f is $2p$-times continuously differentiable and $\Delta^p f(x) = 0$ for all $x \in U$ where Δ is the Laplace operator and Δ^p its p-th iterate.

Suppose further that there exists a constant $D_2 > 0$ such that for all $x \in C$ and for all $0 < h < 1$

$$|\Lambda_{\varphi_h}(hx)| \leq h^m D_2 |Q(x)|. \tag{6}$$

An analysis of the proof in [2] shows that then the following inequality holds (for us the constants D_1 and D_2 defined in the formula will be very important!)

$$E(f, V_h) \leq \left(D_2 (2\pi)^{\frac{n}{2}} + \frac{1}{D_1 (2\pi)^{\frac{n}{2}}} \right) \cdot h^m \cdot \left\| Q(\xi) \widehat{f}(\xi) \right\|_{L_2(\mathbb{R}^n)}. \tag{7}$$

We recall some facts about L-splines: Let L be a linear differential operator with constant coefficients of order $N+1$, say

$$L = M_\Lambda := \prod_{j=1}^{N+1} \left(\frac{d}{dv} - \lambda_j \right) \text{ where } \Lambda := (\lambda_1, ..., \lambda_{N+1}). \tag{8}$$

Then a function $u : \mathbb{R} \to \mathbb{R}$ is called a *cardinal L-spline on the mesh $h\mathbb{Z}$* $(h > 0)$ if u is $(N-1)$-times continuously differentiable and if for every $l \in \mathbb{Z}$ there exists $f_l \in U_L := \{f \in C^\infty(\mathbb{R}) : Lf = 0\}$ such that $u(t) = f_l(t)$ for all $t \in (lh, (l+1)h)$. The set of all cardinal L-splines for the operator $L = M_\Lambda$ on $h\mathbb{Z}$ will be denoted by $\mathcal{S}_{h\mathbb{Z}}(\Lambda)$. The scaling spaces $V_h(\Lambda)$ are defined by

$$V_h(\Lambda) = L^2(\mathbb{R})\text{-closure of } \mathcal{S}_{h\mathbb{Z}}(\Lambda) \cap L^2(\mathbb{R}). \tag{9}$$

Let Q_Λ be the basic spline which can be defined by its Fourier transform by

$$\widehat{Q_\Lambda}(\xi) = \frac{\prod_{j=1}^{N+1} \left(e^{-\lambda_j} - e^{-i\xi} \right)}{\prod_{j=1}^{N+1} (i\xi - \lambda_j)}. \tag{10}$$

Theorem 2. *Let $N \in \mathbb{N}$ be fixed. Then there exists a constant $D > 0$ such that for all $\Lambda = (\lambda_1, ..., \lambda_{N+1}) \in \mathbb{R}^{N+1}$ and for all $f \in L_2(\mathbb{R})$ the following inequality holds:*

$$E(f, V_h(\Lambda)) \leq h^{N+1} \cdot D \left\| P_\Lambda(\xi) \widehat{f}(\xi) \right\|_{L_2(\mathbb{R})}, \tag{11}$$

where the polynomial $P_\Lambda(x) = \prod_{j=1}^{N+1}(ix - \lambda_j)$.

Remark 3. Note that if we used the usual Sobolev norm (3), then we would not be able to obtain the sharp constant D of inequality (11); the last is the main virtue of Theorem 2.

Proof. By the above we have to check (5) and (6). Note that for $Q := P_\Lambda$ we have the estimate

$$\left| P_\Lambda\left(\frac{x}{h}\right) \right|^2 = \prod_{j=1}^{N+1} \left(\left(\frac{x}{h}\right)^2 + \lambda_j^2 \right) \geq \pi^{2(N+1)} \frac{1}{h^{2(N+1)}} \tag{12}$$

for all $|x| \geq \pi$ and for all $h > 0$. Hence it suffices to show that

$$|\Lambda_{\varphi_h}(h\xi)|^2 \leq h^{2(N+1)} |P_\Lambda(\xi)|^2 \sum_{\alpha \in \mathbb{Z}, \alpha \neq 0} \frac{1}{(\pi |\alpha|)^{2(N+1)}}. \tag{13}$$

The trivial inequality $(\Lambda_{\varphi_h}(\xi))^2 \leq \frac{\sum_{\alpha\in\mathbb{Z},\alpha\neq 0}|\widehat{\varphi_h}(\xi+2\pi\alpha)|^2}{|\widehat{\varphi_h}(\xi)|^2}$ and the estimate

$$\frac{|\widehat{\varphi_h}(\xi+2\pi\alpha)|^2}{|\widehat{\varphi_h}(\xi)|^2} = \frac{\left|\widehat{Q_{h\Lambda}}(\xi+2\pi\alpha)\right|^2}{\left|\widehat{Q_{h\Lambda}}(\xi)\right|^2} = \prod_{j=1}^{N+1}\left|\frac{i\xi - h\lambda_j}{i(\xi+2\pi\alpha)-h\lambda_j}\right|^2$$

yield

$$|\Lambda_{\varphi_h}(h\xi)|^2 \leq h^{2(N+1)}\prod_{j=1}^{N+1}\left(\xi^2+\lambda_j^2\right)\sum_{\alpha\in\mathbb{Z},\alpha\neq 0}\prod_{j=1}^{N+1}\frac{1}{(h\xi+2\pi\alpha)^2+h^2\lambda_j^2}.$$

Since $(h\xi+2\pi\alpha)^2+h^2\lambda_j^2 \geq (h\xi+2\pi\alpha)^2 \geq (2\pi|\alpha|-|h\xi|)^2$ we obtain for $0<h<1$ and $|\xi|\leq\pi$ the estimate $2\pi|\alpha|-|h\xi| \geq \pi|\alpha|$ (since $\alpha\neq 0$) arriving at (13). □

3. THE APPROXIMATION ORDER OF POLYSPLINES

Let $\mathbb{S}^{n-1} = \{x\in\mathbb{R}^n; |x|=1\}$ be the unit sphere. Each $x\in\mathbb{R}^n$ will be written in spherical coordinates $x=r\theta$ with $r\geq 0$ and $\theta\in\mathbb{S}^{n-1}$. Recall that a function $Y:\mathbb{S}^{n-1}\to\mathbb{C}$ is a *spherical harmonic* of degree $k\in\mathbb{N}_0$ if there exists a homogeneous harmonic polynomial $P(x)$ of degree k such that $P(\theta)=Y(\theta)$ for all $\theta\in\mathbb{S}^{n-1}$. The set $\mathfrak{H}_k$ of all spherical harmonics of degree exactly k is a linear space of dimension $a_k := \dim\mathfrak{H}_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$. We denote by $Y_{k,l}$ with $l=1,2,...,a_k$ a basis for $\mathfrak{H}_k$. For a detailed account we refer to Stein and Weiss [12].

Let $u:(R_1,R_2)\to\mathbb{C}$ be infinitely differentiable and $Y_k\in\mathfrak{H}_k$. Then it is well known that $\Delta(u(r)Y_k(\theta)) = Y_k(\theta)L_{(k)}u(r)$ where we have put

$$L_{(k)} = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} - \frac{k(k+n-2)}{r^2}. \tag{14}$$

By iteration we have $\Delta^p u = Y_k(\theta)\cdot\left[L_{(k)}\right]^p u(r)$. For convenience, we write

$$\Lambda_+(k,p) := \{k, k+2, ..., k+2p-2\},$$
$$\Lambda_-(k,p) := \{-k-n+2, -k-n+4, ..., -k-n+2p\}.$$

The space of solutions of the equation $L_{(k)}^p f(r) = 0$ which are C^∞ for $r>0$ is generated by a simple basis: for $j\in\Lambda_+(k,p)\cup\Lambda_-(k,p)$ the function r^j is clearly a solution, while for $j\in\Lambda_+(k,p)\cap\Lambda_-(k,p)$ we obtain a second solution $r^j\log r$. It will be convenient to make a transform of the variable r to $v=\log r$. Then a solution of the form r^j will be transformed to e^{jv} and a solution of the form $r^j\log r$ is transformed to ve^{jv}. We see immediately that all solutions to the equation $L_{(k)}^p f(r) = 0$ are transformed to solutions of the equation $M_{\Lambda(k)}g(v) = 0$ where $M_{\Lambda(k)}$ is defined by (8) with respect to the vector

$$\Lambda_k := (k, k+2, ..., k+2(p-1), -(k+n)+2, ..., -(k+n)+2p). \tag{15}$$

The dependence on the parameter p and n will be suppressed throughout the paper.

A proof of the following can be found in [6] and [9, Theorem 9.7].

Theorem 4. *Let $S:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be a polyspline of order p. Then the Laplace-Fourier coefficient $S_{k,l}:\mathbb{R}\to\mathbb{R}$ defined by*

$$S_{k,l}(v) := \int_{\mathbb{S}^{n-1}} S(e^v\theta)Y_{k,l}(\theta)\,d\theta \tag{16}$$

is a cardinal L-spline with respect to the linear differential operator $M_{\Lambda(k)}$.

We want to characterize the $L^2(\mathbb{R}^n)$-closure PV_h. It is a temptation to assume that for $S \in PV_h$ the Fourier-Laplace coefficient defined through formula (16) will be in $V_h(\Lambda_k)$, i.e., in the closure of $\mathcal{S}_{h\mathbb{Z}}(\Lambda_k) \cap L_2(\mathbb{R})$. This is *not true* since the transformation rule will give us an additional weight for $f \in L_2(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_0^\infty \int_{\mathbb{S}^{n-1}} |f(r\theta)|^2 r^{n-1} d\theta dr. \tag{17}$$

Fortunately, this problem can be easily solved; see e.g. [7].

Theorem 5. *Define* $\overline{\Lambda_k} = \left(\frac{n}{2}, ..., \frac{n}{2}\right) + \Lambda_k$. *Then for each* $k \in \mathbb{N}_0, l = 1, ..., a_k$, *the following map, defined on* $P_h \cap L^2(\mathbb{R}^n)$ *by*

$$S \longmapsto \overline{S}_{k,l}(v) := e^{\frac{n}{2}v} \int_{\mathbb{S}^{n-1}} S(e^v\theta) \, Y_{k,l}(\theta) \, d\theta, \tag{18}$$

maps onto $\mathcal{S}_{h\mathbb{Z}}\left(\overline{\Lambda_k}\right) \cap L^2(\mathbb{R}, dv)$, *and, by continuity, it can be extended to a map from* PV_h onto $V_h\left(\overline{\Lambda_k}\right)$. *Furthermore,* PV_h *is isomorphic to*

$$V_h := \bigoplus_{k\in\mathbb{N}_0, l=1,\dots,a_k} V_h\left(\overline{\Lambda_k}\right).$$

Proof of Theorem 1. Let $f \in L^2(\mathbb{R}^n)$ and $g \in PV_h$. Then by the transformation rule (17),

$$\|f - g\|^2_{L^2(\mathbb{R}^n)} = \int_0^\infty \int_{\mathbb{S}^{n-1}} |f(r\theta) - g(r\theta)|^2 r^{n-1} d\theta dr. \tag{19}$$

Let $f_{k,l}$ and $g_{k,l}$ be the Laplace-Fourier coefficients of f and g respectively as defined in (16). Note that $v \longmapsto \overline{g_{k,l}}(e^v) := e^{\frac{n}{2}v} g_{k,l}(e^v)$ is in $V_h\left(\overline{\Lambda_k}\right)$. Since $Y_{k,l}(\theta)$ constitutes an orthonormal basis, we obtain

$$\|f - g\|^2_{L^2(\mathbb{R}^n)} = \sum_{k=0}^\infty \sum_{l=1}^{a_k} \int_{-\infty}^\infty |f_{k,l}(e^v) - g_{k,l}(e^v)|^2 e^{nv} dv. \tag{20}$$

Minimizing the expression $g \longmapsto \|f - g\|^2_{L^2(\mathbb{R}^n)}$ for $g \in PV_h$ is equivalent to minimizing the expression

$$\int_{-\infty}^\infty \left| e^{\frac{n}{2}v} f_{k,l}(e^v) - \overline{g_{k,l}}(e^v) \right|^2 dv$$

for each $k \in \mathbb{N}$, $l = 1, ..., a_k$, where $\overline{g_{k,l}} \in V_h\left(\overline{\Lambda_k}\right)$. Theorem 2 applied to $\Lambda = \overline{\Lambda_k}$ (hence $N + 1 = 2p$) shows that there exists a constant $C_p > 0$ which only depends on p (and not on the values λ_j in $\overline{\Lambda_k}$) such that

$$E\left(e^{\frac{n}{2}v} f_{k,l}(e^v), V_h\left(\overline{\Lambda_k}\right)\right) \le h^{2p} \cdot C_p \left\| P_{\overline{\Lambda_k}} \cdot \widehat{e^{\frac{n}{2}v} f_{k,l}(e^v)} \right\|_{L_2(\mathbb{R})}. \tag{21}$$

Write $G_{k,l}(v) := e^{\frac{n}{2}v} f_{k,l}(e^v)$. A simple computation (using Parseval's identity and the fact that differentiation becomes multiplication via Fourier transform) shows that

$$\frac{1}{2\pi} \left\| P_{\overline{\Lambda_k}} \cdot \widehat{G_{k,l}} \right\|^2_{L_2(\mathbb{R})} = \int_{-\infty}^\infty \left| M_{\overline{\Lambda_k}} G_{k,l}(v) \right|^2 dv.$$

A calculation shows that $M_{\overline{\Lambda_k}}\left(e^{\frac{n}{2}v}f_{k,l}\left(e^v\right)\right) = e^{\frac{n}{2}v}M_{\Lambda_k}\ \left(f\left(e^v\right)\right)$. Then (20) and (21) yield

$$E\left(f, PV_h\right)^2 \leq h^{4p}\cdot 2\pi C_p^2 \sum_{k=0}^{\infty}\sum_{l=1}^{a_k}\left\| e^{\frac{n}{2}v}M_{\Lambda_k}\ \left(f\left(e^v\right)\right)\right\|^2_{L_2(\mathbb{R}^n)}.$$

The next theorem applied to the case $p = q$ finishes the proof. ☐

Theorem 6. *Let* $p, q \in \mathbb{N}_0$ *and define* $\left\|f\left(x\right)\right\|^2_{q,p} := \int \left|\left|x\right|^{2q}\cdot\Delta^p f\left(x\right)\right|^2 dx$ *for* $f \in L_2\left(\mathbb{R}^n\right)$. *Then*

$$\left\|f\left(x\right)\right\|^2_{q,p} = \sum_{k=0}^{\infty}\sum_{l=1}^{a_k} \int \left|e^{v\left(2q-2p+\frac{n}{2}\right)}M_{\Lambda_k}\ \left(f_{k,l}\left(e^v\right)\right)\right|^2 dv$$

where $f_{k,l}(r)$ *are the Laplace-Fourier coefficients of* f *defined as in equality* (16).

Proof. Assume that $f(r\theta) = f_{k,\ell}\left(r\right)Y_{k,\ell}\left(\theta\right)$. Since $\Delta^p f\left(x\right) = L_k^p f_{k,\ell}\left(r\right)Y_{k,\ell}\left(\theta\right)$, we obtain

$$\left\|f\left(x\right)\right\|^2_{q,p} = \int_0^{\infty}\int_{\mathbb{S}^{n-1}}\left|r^{2q}L^p_{(k)}f_{k,\ell}\left(r\right)Y_{k,\ell}\left(\theta\right)\right|^2 r^{n-1}drd\theta.$$

The integration over θ only gives a factor 1. Now we change the variable $r = e^v$ and apply the identity $\left(L_k^p f_{k,l}\right)\left(e^v\right) = e^{-2vp}M_{\Lambda_k}\ \left(f_{k,l}\left(e^v\right)\right)$; see e.g. Theorem 10.34 in [9]. Then

$$\left\|f\left(x\right)\right\|^2_{q,p} = \int \left|e^{2vq}e^{-2vp}M_{\Lambda_k}\ \left(f_{k,l}\left(e^v\right)\right)\right|^2 e^{nv}dv.$$

Finally, we see that for arbitrary $f \in L^2\left(\mathbb{R}^n\right)$ the result follows via the orthogonal decomposition of f in spherical harmonics. ☐

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