

## ISOMETRIC COPIES OF $l^1$ AND $l^\infty$ IN ORLICZ SPACES EQUIPPED WITH THE ORLICZ NORM

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ABSTRACT. Criteria in order that an Orlicz space equipped with the Orlicz norm contains a linearly isometric copy (or an order linearly isometric copy) of  $l^1$  (or  $l^\infty$ ) are given.

### 1. INTRODUCTION

Let  $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{R}_+$  stand for the set of natural numbers, the set of reals and the set of nonnegative reals, respectively. Let  $(T, \Sigma, \mu)$  be a  $\sigma$ -finite measure space that does not reduce to a finite number of atoms, where all atoms that appear in  $\Sigma$  have measure 1, and let  $L^0 = L^0(T, \Sigma, \mu)$  be the space of all (equivalence classes of)  $\Sigma$ -measurable functions defined on  $T$ .

A function  $\Phi : \mathbb{R} \rightarrow [0, +\infty]$  is said to be an Orlicz function if  $\Phi(0) = 0$ ,  $\Phi(u) \rightarrow \infty$  as  $u \rightarrow \infty$ ,  $\Phi$  is even and convex and  $\lim_{u \rightarrow b(\Phi)_-} \Phi(u) = \Phi(b(\Phi))$ , where  $b(\Phi) = \sup\{u > 0 : \Psi(u) < \infty\} > 0$ . Note that the case  $\Phi(b(\Phi)) = \infty$  is not excluded. The function  $\Psi$  complementary to  $\Phi$  in the sense of Young is defined by

$$\Psi(u) = \sup_{v>0} \{u|v - \Phi(v)\}.$$

It is obvious that the Young inequality

$$uv \leq \Phi(u) + \Psi(v)$$

holds for all  $u, v \in \mathbb{R}$  and that in the case when  $b(\Phi) = \infty$  or  $\Phi(b(\Phi)) < \infty$  and  $\Phi'_-(b(\Phi)) < \infty$ , we have the equality

$$uv = \Phi(u) + \Psi(v)$$

for all  $u \in [0, b(\Phi))$  (resp.  $u \in [0, b(\Phi)]$ ) and  $v \in [\Phi'_-(u), \Phi'_+(u)]$ , where  $\Phi'_-(u)$  and  $\Phi'_+(u)$  denote the left (resp. the right) derivative of  $\Phi$  at the point  $u$ . For an example of an Orlicz function  $\Phi$  with  $0 < b(\Phi) < \infty$  and  $\Phi'_-(b(\Phi)) = \infty$  such that the equality

$$b(\Phi)v = \Phi(b(\Phi)) + \Psi(v)$$

holds for no  $v \in \mathbb{R}_+$  we refer to [6].

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Let us define for an Orlicz function  $\Phi$  another important parameter,

$$a(\Phi) = \sup\{u \geq 0 : \Phi(u) = 0\}.$$

Given an Orlicz function  $\Phi$  we define on  $L^0$  the convex modular

$$I_\Phi(x) = \int_T \Phi(x(t))d\mu$$

and the Orlicz space

$$L^\Phi = L^\Phi(T, \Sigma, \mu) := \{x \in L^0 : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\}.$$

The most important norms in  $L^\Phi$  are the following two:

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\},$$

called the Luxemburg norm (see [1] and [7] - [12]), and

$$\|x\|_\Phi^0 = \sup\{|\int_T x(t)y(t)d\mu| : I_\Psi(y) \leq 1, y \in L^0\},$$

called the Orlicz norm (see [1] and [7] - [12]). The Amemiya formula

$$\|x\|_\Phi^0 = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

for the Orlicz norm is very useful because it does not use the function  $\Psi$  complementary to  $\Phi$ . For Orlicz functions  $\Phi$  that are  $N$ -functions at infinity, that is,  $(\Phi(u)/u) \rightarrow \infty$  as  $u \rightarrow \infty$ , this formula was well known from the beginning of the theory of Orlicz spaces (see [8]) and for arbitrary Orlicz functions it was proved in [6]. For any  $x \in L^\Phi$ , we define  $\text{supp } x = \{t \in T : x(t) \neq 0\}$ .

It is well known that for any Orlicz function the quotient  $\Phi(u)/u$  is nondecreasing on  $\mathbb{R}_+$ . So the limit (finite or infinite)  $A(\Phi) = \lim_{u \rightarrow \infty} (\Phi(u)/u)$  always exists. Let us define the function  $R_\Phi(u) = A(\Phi)|u| - \Phi(u)$ . As we will see below, this function will be of great importance.

We say that an Orlicz function  $\Phi$  satisfies the  $\Delta_2$ -condition at zero (at infinity) [on  $\mathbb{R}_+$ ] if there are positive constants  $K \geq 2$  and  $u_0$  with  $0 < \Phi(u_0) < \infty$  such that the inequality  $\Phi(2u) \leq K\Phi(u)$  holds for all  $u \in [0, u_0]$  ( $u \in [u_0, \infty)$ ) [ $u \in \mathbb{R}_+$ ]. We then write  $\Phi \in \Delta_2(0)$  ( $\Phi \in \Delta_2(\infty)$ ) [ $\Phi \in \Delta_2$ ] for short. It is obvious that  $\Phi \in \Delta_2 \iff \Phi \in \Delta_2(0)$  and  $\Phi \in \Delta_2(\infty)$ . Moreover,  $b(\Phi) = \infty$  whenever  $\Phi \in \Delta_2(\infty)$  and  $a(\Phi) = 0$  whenever  $\Phi \in \Delta_2(0)$ .

An Orlicz space  $L^\Phi$  equipped with the Orlicz norm  $\|\cdot\|_\Phi^0$  will be denoted by  $L_0^\Phi$ . The unit ball and the unit sphere of  $L_0^\Phi$  will be denoted by  $B(L_0^\Phi)$  and  $S(L_0^\Phi)$ , respectively. For any  $x \in L_0^\Phi \setminus \{0\}$  we denote by  $K(x)$  the set of these  $k > 0$  such that  $\|x\|_\Phi^0 = \frac{1}{k}(1 + I_\Phi(kx))$ . In the case when  $\|x\|_\Phi^0 = \lim_{k \rightarrow \infty} \frac{1}{k}(1 + I_\Phi(kx))$ , we write  $\infty \in K(x)$ . If  $A(\Phi) < \infty$ , then it can happen that  $K(x) = \emptyset$ .

Since in the case when  $a(\Phi) = 0$  the Orlicz space  $L_0^\Phi$  is strictly monotone (it does not matter if  $\Phi$  satisfies or not the suitable  $\Delta_2$ -condition; see [5]),  $L_0^\Phi$  cannot contain an order isometric copy of  $l^\infty$  (in contrast to the case when the Luxemburg norm is considered). However, as we will see below in the case when  $a(\Phi) = 0$ ,  $L_0^\Phi$  can contain an order isometric copy of  $l^\infty$ .

In this paper we present criteria for the existence in Orlicz spaces  $L_0^\Phi$  equipped with the Orlicz norm a linearly isometric copy or an order linearly isometric copy of  $X$ , where  $X$  is equal to  $l^\infty$  or  $l^1$ . Such criteria are important when we are looking for criteria of other important topological and geometrical properties of  $L_0^\Phi$ . Our results on  $l^\infty$ -copies do not follow from [4] although they are connected

with those results. In the special case of Orlicz spaces our results are more precise. The results of [13] are also connected with our Theorems 1 and 2. A class of Orlicz spaces isomorphic or isomorphically isometric to  $l^\infty$  is distinguished in [13].

## 2. RESULTS

We start with criteria for the existence of an order linearly isometric copy of  $l^\infty$  in  $L_0^\Phi$ .

**Theorem 1.** *Let  $\Phi$  be an Orlicz function with  $b(\Phi) = \infty$ . Then  $L_0^\Phi$  contains an order linearly isometric copy of  $l^\infty$  if and only if  $\mu$  is infinite and  $a(\Phi) > 0$ .*

*Proof. Sufficiency.* Assume that  $b(\Phi) = \infty$ ,  $a(\Phi) > 0$  and  $\mu$  is infinite. Divide  $T$  into a sequence  $(T_n)_{n=1}^\infty$  of pairwise disjoint sets such that  $\mu(T_n) = \infty$  for any  $n \in \mathbb{N}$ . Define  $x_n = a(\Phi)\chi_{T_n}$  ( $n \in \mathbb{N}$ ). Then

$$\begin{aligned} 1 + I_\Phi(x_n) &= 1, \\ \frac{1}{k}(1 + I_\Phi(kx_n)) &= \frac{1}{k} > 1 \quad (\forall k \in (0, 1), \quad n \in \mathbb{N}), \\ \frac{1}{k}(1 + I_\Phi(kx_n)) &= \infty \quad (\forall k > 1, \quad n \in \mathbb{N}). \end{aligned}$$

Therefore,  $\|x_n\|_\Phi^0 = 1$  for any  $n \in \mathbb{N}$ . Moreover, in the same way we can prove that  $\|\sum_{n=1}^\infty x_n\|_\Phi^0 = 1$ . Hence it follows that the operator

$$Py = \sum_{n=1}^\infty y_n x_n \quad (\forall y = (y_n) \in l^\infty),$$

which is obviously linear and positive, is an order isometry of  $l^\infty$  onto the closed subspace  $P(l^\infty)$  of  $L_0^\Phi$  (cf [4]).

*Necessity.* Note that the inequality  $a(\Phi) > 0$  is necessary, since if  $a(\Phi) = 0$ , then  $L_0^\Phi$  is strictly monotone (see [5]). Since strict monotonicity is preserved by linear order isometries and  $l^\infty$  is not strictly monotone,  $L_0^\Phi$  cannot contain an order linearly isometric copy of  $l^\infty$  if  $a(\Phi) = 0$ . Therefore, we may assume in the remaining part of the proof of necessity that  $a(\Phi) > 0$ .

In order to prove the necessity of the condition  $\mu(T) = \infty$ , assume to the contrary that  $\mu(T) < \infty$ ,  $a(\Phi) > 0$  and  $L_0^\Phi$  contains an order linearly isometric copy of  $l^\infty$ . Since any order linear isometry preserves the orthogonality of elements (see [4]), there is in  $L_0^\Phi$  a sequence  $(x_n)_{n=1}^\infty$  such that  $x_n \geq 0$ ,  $\|x_n\|_\Phi^0 = 1$  for all  $n \in \mathbb{N}$  and  $\|\sum_{n=1}^\infty x_n\|_\Phi^0 = 1$ . It is enough to take  $e_n = (0, \dots, 0, 1, 0, \dots) \in l^\infty$  and  $x_n = Pe_n$ , where  $P$  is a linear order isometry of  $l^\infty$  onto the closed subspace  $P(l^\infty)$  of  $L_0^\Phi$ . Then the equality  $\|x_n\|_\Phi^0 = 1$  for any  $n \in \mathbb{N}$  is obvious. To prove the equality

$$\left\| \sum_{n=1}^\infty x_n \right\|_\Phi^0 = 1,$$

note first that

$$\left\| \sum_{n=1}^k x_n \right\|_\Phi^0 = \left\| \sum_{n=1}^k Pe_n \right\|_\Phi^0 = \left\| P \left( \sum_{n=1}^k e_n \right) \right\|_\Phi^0 = \left\| \sum_{n=1}^k e_n \right\|_\infty = 1$$

for any  $k \in \mathbb{N}$ .

Since  $L_0^\Phi$  has the Fatou property (see [6]), we get  $\sum_{n=1}^\infty x_n \in L_0^\Phi$  and

$$\left\| \sum_{n=1}^\infty x_n \right\|_\Phi^0 = \lim_{k \rightarrow \infty} \left\| \sum_{n=1}^k x_n \right\|_\Phi^0 = 1.$$

Since  $\mu(T) < \infty$  and  $\text{supp } x_n \cap \text{supp } x_m = \emptyset$  for  $n \neq m$ , we get  $\mu(\text{supp } x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . By the assumption that all atoms that appear in  $T$  have measure 1 and  $\Sigma$  does not reduce to a finite number of atoms, there is no loss of generality in assuming that  $T$  is nonatomic. Therefore, we may assume that  $\Phi \notin \Delta_2(\infty)$ , because otherwise  $L_0^\Phi$  is order continuous and so  $L_0^\Phi$  cannot even contain an order linearly isomorphic copy of  $l^\infty$ . In consequence we get that  $A := A(\Phi) = \infty$ . This implies that  $K(x) \neq \emptyset$  for any  $x \in L_0^\Phi \setminus \{0\}$  (see [3]). We claim that

- (1) for any  $c > 0$  and  $m \in \mathbb{N}$  there is  $n \in \mathbb{N}$  such that  $n > m$  and  $\mu\{t \in T : x_n(t) > c\} > 0$ .

Otherwise, there are  $c > 0$  and  $m \in \mathbb{N}$  such that  $x_n(t) \leq c$   $\mu$ -a.e. in  $T$  for all  $n \in \mathbb{N}$ ,  $n > m$ . Since  $\mu(T) < \infty$ ,  $b(\Phi) = \infty$  and  $\mu(\text{supp } x_n) \rightarrow 0$  as  $n \rightarrow \infty$ , we conclude that  $I_\Phi(\lambda x_n) \rightarrow 0$  as  $n \rightarrow \infty$  for any  $\lambda > 0$ . Consequently, for any  $k > 1$  taking  $n \rightarrow \infty$ , we get

$$1 = \|x_n\|_\Phi^0 \leq \frac{1}{k}(1 + I_\Phi(kx_n)) \rightarrow \frac{1}{k} < 1,$$

a contradiction. So, the proof of the claim is finished.

Let  $n \in \mathbb{N}$  be such that

$$\mu\{t \in T : x_n(t) > a(\Phi)\} > 0$$

and choose an arbitrary  $k \in K(x_1 + x_n)$ . Since  $\|x_1 + x_n\|_\Phi^0 = 1$ , we have  $k \geq 1$ , whence  $\mu(A) > 0$  for  $A = \{t \in T : kx_n(t) > a(\Phi)\}$ . Therefore,

$$\begin{aligned} \|x_1 + x_n\|_\Phi^0 &= \frac{1}{k}(1 + I_\Phi(k(x_1 + x_n))) \\ &> \frac{1}{k}(1 + I_\Phi(kx_1)) \geq \|x_1\|_\Phi^0 \\ &= 1, \end{aligned}$$

a contradiction, which finishes the proof. □

**Theorem 2.** *Let  $\Phi$  be an Orlicz function with  $b(\Phi) < \infty$ . Then  $L_0^\Phi$  contains an order linearly isometric copy of  $l^\infty$  if and only if  $a(\Phi) > 0$  and either (a)  $\mu(T) = \infty$  or (b)  $\mu(T) < \infty$  and  $a(\Phi) = b(\Phi)$ .*

*Proof.* If  $a(\Phi) > 0$ ,  $b(\Phi) < \infty$  and  $\mu(T) = \infty$ , we can repeat the appropriate part of the proof of Theorem 1 to prove the sufficiency.

Assume now that  $b(\Phi) < \infty$ ,  $a(\Phi) > 0$ ,  $a(\Phi) = b(\Phi)$  and  $\mu(T) < \infty$ . Then  $L_0^\Phi = L^\infty$  and  $\|x\|_\Phi^0 = \frac{1}{a(\Phi)}\|x\|_\infty$  for any  $x \in L_0^\Phi$ , where  $\|x\|_\infty := \text{esssup}_{t \in T} |x(t)|$ .

Therefore,  $L_0^\Phi$  is order linearly isometric to  $(L^\infty, \|\cdot\|_\infty)$ , so it contains an order linearly isometric copy of  $l^\infty$ . □

*Necessity.* If  $a(\Phi) = 0$ , then  $L_0^\Phi$  is strictly monotone; so it cannot contain an order linearly isometric copy of  $l^\infty$ . So, under the assumptions that  $\mu(T) < \infty$ ,  $a(\Phi) > 0$  and  $b(\Phi) < \infty$ , we need to prove the necessity of the condition  $a(\Phi) = b(\Phi)$ . Assume to the contrary that  $a(\Phi) < b(\Phi)$  and  $L_0^\Phi$  contains an order linearly isometric copy of  $l^\infty$ . Then there is a sequence  $(x_n)_{n=1}^\infty$  in  $S(L_0^\Phi)$  of disjointly supported elements

such that  $\|\sum_{n=1}^{\infty} x_n\|_{\Phi}^0 = 1$  (see the proof of the necessity part of Theorem 1). Therefore, we have  $\mu(\text{supp } x_n) \rightarrow 0$  as  $n \rightarrow \infty$  and  $K(x) \neq \emptyset$  for any  $x \in L_0^{\Phi} \setminus \{0\}$ . Let  $m, n \in \mathbb{N}$ ,  $m \neq n$ , and  $k_{m,n} \in K(x_m + x_n)$ . Then

$$(2) \quad 1 = \frac{1}{k_{m,n}}(1 + I_{\Phi}(k_{m,n}x_m)) + \frac{1}{k_{m,n}}I_{\Phi}(k_{m,n}x_n).$$

Since  $\frac{1}{k_{m,n}}(1 + I_{\Phi}(k_{m,n}x_m)) \geq 1$ , equality (1) yields

$$(3) \quad \frac{1}{k_{m,n}}(1 + I_{\Phi}(k_{m,n}x_m)) = 1 \quad \text{and} \quad I_{\Phi}(k_{m,n}x_n) = 0.$$

In the same way we can get

$$(4) \quad \frac{1}{k_{m,n}}(1 + I_{\Phi}(k_{m,n}x_n)) = 1 \quad \text{and} \quad I_{\Phi}(k_{m,n}x_m) = 0.$$

By (2), (3) and (4), we get  $k_{m,n} = 1$  for all  $m, n \in \mathbb{N}$ ,  $m \neq n$ , and  $|x_n| \leq a(\Phi)$   $\mu$ -a.e. in  $T$  for any  $n \in \mathbb{N}$ .

For any  $k > 1$  satisfying  $ka(\Phi) < b(\Phi)$ , since  $\mu(\text{supp } x_n) \rightarrow 0$ , we get  $I_{\Phi}(kx_n) \rightarrow 0$ , and consequently

$$1 = \|x_n\|_{\Phi}^0 \leq \frac{1}{k}(1 + I_{\Phi}(kx_n)) \rightarrow \frac{1}{k} < 1 \quad \text{as} \quad n \rightarrow \infty,$$

a contradiction. This proves the necessity of the equality  $a(\Phi) = b(\Phi)$  whenever  $\mu(T) < \infty$ .  $\square$

Theorems 1 and 2 can be summarized into the following result.

**Theorem 3.** *Let  $\Phi$  be an Orlicz function. Then we have:*

(i) *if  $\mu(T) = \infty$ , then  $L_0^{\Phi}$  contains an order linearly isometric copy of  $l^\infty$  if and only if  $a(\Phi) > 0$ ;*

(ii) *if  $\mu(T) < \infty$ , then  $L_0^{\Phi}$  contains an order linearly isometric copy of  $l^\infty$  if and only if  $a(\Phi) > 0$ ,  $b(\Phi) < \infty$  and  $a(\Phi) = b(\Phi)$ , that is,  $L_0^{\infty}$  is order linearly isometric to  $(L^\infty, \|\cdot\|_\infty)$ .*

In the next theorem we will use the function  $R(u) = A|u| - \Phi(u)$ , where  $A = A(\Phi) = \lim_{u \rightarrow \infty} (\Phi(u)/u)$ .

**Theorem 4.** *For any Orlicz function  $\Phi$  with  $a(\Phi) = 0$  and  $b(\Phi) = +\infty$  and any nonatomic  $\sigma$ -finite measure space  $(T, \Sigma, \mu)$ , the following assertions are equivalent:*

- (1)  *$L_0^{\Phi}$  has a subspace order linearly isometric to  $l^1$ ;*
- (2)  *$L_0^{\Phi}$  has a subspace linearly isometric to  $l^1$ ;*
- (3) *There exists a nonzero  $x \in L_0^{\Phi}$  such that  $K(x) = \emptyset$ ;*
- (4) *The function  $R(u)$  is upper bounded.*

*Proof.* The implication **1**  $\Rightarrow$  **2** is obvious. Let us now prove the implication **2**  $\Rightarrow$  **3**. If  $K(x) \neq \emptyset$  for all nonzero  $x$  in  $L_0^{\Phi}$ , then  $L_0^{\Phi}$  is non-square by the proof of Theorem 3.26 in [1]. Therefore, assertion **2** is not true if assertion **3** is not true, which finishes the proof of the implication **2**  $\Rightarrow$  **3**. Assume that assertion **3** holds and  $R(u)$  is not upper bounded. Let

$$f(k) := \frac{1}{k}(1 + \int_T \Phi(kx(t))d\mu) = A \int_T |x(t)|d\mu + \frac{1}{k}(1 - \int_T R(kx(t))d\mu)$$

for  $k > 0$ . Assertion **3** implies that  $\|x\|_{\Phi}^0 = A \int_T |x(t)| d\mu$  (see [1] and [2]). Since  $R$  is not upper bounded, we conclude that

$$(5) \quad \frac{1}{k} \left(1 - \int_T R(kx(t)) d\mu\right) < 0$$

for  $k > 0$  large enough. Therefore (5) implies that

$$\|x\|_{\Phi}^0 \leq f(k) < A \int_T |x(t)| d\mu = \|x\|_{\Phi}^0$$

for  $k > 0$  large enough, a contradiction.

**4**  $\Rightarrow$  **1**. Let  $\sup\{R(u) : u \in \mathbb{R}_+\} =: c < \infty$ . Pick pairwise disjoint subsets  $E_n$  ( $n = 1, 2, \dots$ ) of  $T$  such that  $\mu(E_n) > 0$  and  $c \sum_{n=1}^{\infty} \mu(E_n) < 1$ . Choose  $a_n > 0$  such that  $\|a_n \chi_{E_n}\|_{\Phi}^0 = 1$ . Define  $x_n = a_n \chi_{E_n}$ . Then for any  $(b_n)_{n=1}^{\infty} \in l^1$  and any  $\varepsilon > 0$ , pick  $k > 0$  such that

$$\left\| \sum_{n=1}^{\infty} b_n x_n \right\|_{\Phi}^0 > \frac{1}{k} \left(1 + I_{\Phi} \left(k \sum_{n=1}^{\infty} b_n x_n\right)\right) - \varepsilon.$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} |b_n| &\geq \left\| \sum_{n=1}^{\infty} b_n x_n \right\|_{\Phi}^0 \\ &> \frac{1}{k} \left(1 + \sum_{n=1}^{\infty} \Phi(k b_n a_n) \mu(E_n)\right) - \varepsilon \\ &= A \sum_{n=1}^{\infty} |b_n a_n| \mu(E_n) + \frac{1}{k} \left(1 - \sum_{n=1}^{\infty} R(k |b_n| a_n) \mu(E_n)\right) - \varepsilon \\ &\geq A \sum_{n=1}^{\infty} |b_n a_n| \mu(E_n) + \frac{1}{k} \left(1 - c \sum_{n=1}^{\infty} \mu(E_n)\right) - \varepsilon \\ &\geq A \sum_{n=1}^{\infty} |b_n a_n| \mu(E_n) - \varepsilon. \end{aligned}$$

Combining this with the fact that

$$\begin{aligned} \|x_n\|_{\Phi}^0 &\leq \lim_{k \rightarrow \infty} \frac{1}{k} (1 + I_{\Phi}(k x_n)) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} (1 + \Phi(k a_n) \mu(E_n)) \\ &= A a_n \mu(E_n), \end{aligned}$$

we get

$$\begin{aligned} \sum_{n=1}^{\infty} |b_n| &\geq A \sum_{n=1}^{\infty} |b_n| a_n \mu(E_n) \\ &\geq \sum_{n=1}^{\infty} |b_n| \|x_n\|_{\Phi}^0 = \sum_{n=1}^{\infty} |b_n|. \end{aligned}$$

This shows that the map  $P : l^1 \rightarrow L_{\Phi}^{\Phi}$  defined by  $P((b_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} b_n x_n$  is a linear isometry. Since the map  $P$  is positive, it is an order isometry.

**Theorem 5.** Let  $\Phi$  be an Orlicz function and the measure space be nonatomic and  $\sigma$ -finite. Then the following are equivalent:

- (1) The space  $L_0^\Phi$  contains an order isometric copy of  $l^1$ ;
- (2) There exists a nonzero  $x \in L_0^\Phi$  such that  $K(x) = \emptyset$ ;
- (3) The function  $R(u)$  is upper bounded.

*Proof.* The implications  $2 \Rightarrow 3 \Rightarrow 1$  can be proved in the same way as in Theorem 4.

$1 \Rightarrow 2$ . Assume that  $2$  does not hold. Then  $K(x) \neq \emptyset$  for any  $x \in L_0^\Phi \setminus \{0\}$ . Take any  $x, y \in S(L_0^\Phi)$  with  $\text{supp } x \cap \text{supp } y = \emptyset$ . Let  $k \in K(x)$ ,  $h \in K(y)$ . We may assume, without loss of generality, that  $k \leq h$ . Then

$$\begin{aligned} 2 &= \|x\|_\Phi^0 + \|y\|_\Phi^0 = \frac{1}{k}(1 + I_\Phi(kx)) + \frac{1}{h}(1 + I_\Phi(hy)) \\ &\geq \frac{1}{k}(1 + I_\Phi(kx)) + \frac{1}{h} + \frac{1}{k}I_\Phi(ky) \\ &\geq \frac{1}{k}(1 + I_\Phi(k(x+y))) + \frac{1}{h} \geq \|x+y\|_\Phi^0 + \frac{1}{h}, \end{aligned}$$

whence  $\|x+y\|_\Phi^0 \leq 2 - \frac{1}{h} < 2$ . This yields that  $1$  cannot hold. Indeed, otherwise, by the fact that order isometry preserves disjointness of supports of functions (up to a set of measure zero), see [4], taking  $e_1 = (1, 0, 0, \dots)$ ,  $e_2 = (0, 1, 0, \dots)$  in  $l^1$  and an order linear isometry  $P$  of  $l^1$  onto a closed subspace of  $L_0^\Phi$ , we get  $\mu(\text{supp } Px \cap \text{supp } Py) = 0$  and  $Px, Py \in S(L_0^\Phi)$ . Therefore, we have by the above

$$2 = \|e_1 + e_2\|_{l^1} = \|Px + Py\|_\Phi^0 < 2,$$

a contradiction which finishes the proof of the implication  $1 \Rightarrow 2$  as well as the proof of the theorem.  $\square$

*Remark 1.* The condition  $\lim_{u \rightarrow \infty} (\Phi(u)/u) = A < \infty$  need not imply that  $L_0^\Phi$  contains a linearly isometric copy of  $l^1$ , i.e.  $A = A(\Phi) < \infty$  need not imply that  $R(u)$  is bounded.

To show this, define for example the function  $\Phi(u) = A(u+c) - K(u+c)^\alpha$  for  $u \geq 0$  and  $\Phi(-u) = \Phi(u)$ , where  $A > 0$ ,  $K > 0$ ,  $0 < \alpha < 1$ , and  $c = (K/A)^{1/(1-\alpha)}$ . It is obvious that  $\lim_{u \rightarrow \infty} (\Phi(u)/u) = A$ . Moreover,

$$R(u) = Au - \Phi(u) = K(u+c)^\alpha - Ac,$$

whence we get  $\sup\{R(u) : u \geq 0\} = \infty$ .

*Remark 2.* It should be worth noting that the function  $\Phi(u) = |u| - 1 + e^{-|u|}$  is strictly convex but, by Theorem 5,  $L_0^\Phi$  contains an order linearly isometric copy of  $l^1$ . This contrasts nicely with what happens if  $\Phi$  were an N-function, where the strict convexity of  $L_0^\Phi$  and  $\Phi$  go hand-in-hand for nonatomic measure spaces (see Milnes theorem, that is, Theorem 6 on page 274 in [12]).

*Remark 3.* It is well known that  $l^\infty$  contains a linearly isometric copy of any separable Banach space  $X$ . Namely if  $(x_n)_{n=1}^\infty$  is the sequence which is dense in  $X$  and  $(x_n^*)_{n=1}^\infty \subset S(X^*)$  is such that  $x_n^*(x_n) = \|x_n\|$  for any  $n \in \mathbb{N}$ , then the isometry  $P : X \rightarrow l^\infty$  is defined by  $Px = (x_n^*(x))_{n=1}^\infty \in l^\infty$  for any  $x \in X$ .

From Theorems 3 and 4 and Remark 3, we get the following.

**Corollary 1.** There are Orlicz spaces  $L_0^\Phi$  containing a linearly isometric copy of  $l^1$  but not containing an order linearly isometric copy of  $l^1$ .

Indeed, if  $a(\Phi) = b(\Phi)$ , then by Theorem 3 and Remark 3,  $L_0^\Phi$  contains a linearly isometric copy of  $l^1$ . However, the equality  $a(\Phi) = b(\Phi)$  yields  $a(\Phi) > 0$  and  $b(\Phi) < \infty$ , whence  $K(x) \neq \emptyset$  for any  $x \in L_0^\Phi \setminus \{0\}$ . Consequently, by Theorem 5,  $L_0^\Phi$  does not contain an order linearly isometric copy of  $L_0^\Phi$ . This phenomena is possible only in Banach lattices  $X$  which are not strictly monotone. Otherwise (see [14]), if  $X$  contains a linearly isometric copy of  $l^1$ , then it also contains an order linearly isometric copy of  $l^1$ .

## REFERENCES

- [1] S. T. Chen, *Geometry of Orlicz Spaces*, Dissertationes Math. **356** (1996), 1-204. MR **97i**:46051
- [2] Y. A. Cui, H. Hudzik, M. Nowak and R. Pluciennik, *Some geometric properties in Orlicz sequence spaces equipped with the Orlicz norm*, J. Convex Analysis **6(1)** (1999), 91-113. MR **2000j**:46037
- [3] R. Grzaślewicz and H. Hudzik, *Smooth points of Orlicz spaces equipped with Luxemburg norm*, Math. Nachr. **155** (1992), 31-45. MR **94k**:46057
- [4] H. Hudzik, *Banach lattices with order isometric copies of  $l^\infty$* , Indag. Math. **9 (4)** (1998), 521-527. MR **2000d**:46025
- [5] H. Hudzik and W. Kurc, *Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices*, J. Approx. Theory **95** (1998), 353-368. MR **99k**:46044
- [6] H. Hudzik and L. Maligranda, *Amemiya norm equals Orlicz norm in general*, Indag. Mathem. **11 (4)** (2000), 573-585.
- [7] A. Kamińska, *Flat Orlicz-Musielak sequence spaces*, Bull. Acad. Polon. Sci. Math. **30** (1982), no. 7-8, 347-352. MR **84h**:46013
- [8] M. A. Krasnoselskii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, P. Noordhoff Ltd., Groningen 1961 (translation). MR **23**:A4016
- [9] W. A. J. Luxemburg, *Banach Function Spaces*, Thesis, Technische Hogeschoolte Delft, 1955. MR **17**:285a
- [10] L. Maligranda, *Orlicz Spaces and Interpolation*, Seminars in Math. **5**, Univ. Estadual de Campinas, Campinas, SP, Brasil 1989.
- [11] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. **1034**, Springer-Verlag, Berlin, 1983. MR **85m**:46028
- [12] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1991. MR **92e**:46059
- [13] M. Wisła, *On a class of Orlicz sequence spaces isomorphic to  $l^\infty$* , to appear.
- [14] M. Wójtowicz, *Contractive projections onto isometric copies of  $L^1(\nu)$  in strictly monotone Banach lattices*, to appear.

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