ON SCHOTTKY GROUPS ARISING FROM
THE HYPERGEOMETRIC EQUATION
WITH IMAGINARY EXponents

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Abstract. In an article by Sasaki and Yoshida (2000), we encountered Schot-
tky groups of genus 2 as monodromy groups of the hypergeometric equation
with purely imaginary exponents. In this paper we study automorphic func-
tions for these Schottky groups, and give a conjectural infinite product formula
for the elliptic modular function \( \lambda \).

1. Introduction

When the three exponents of the hypergeometric differential equation are purely
imaginary, its monodromy group is a Schottky group of genus 2. We give a set of
generators of the monodromy group in Proposition 1; these are chosen so that they
reflect the symmetry of the hypergeometric equation with respect to the three ex-
ponents. The main result of this paper is Proposition 2, which gives an automorphic
function with respect to the Schottky group as an absolutely convergent infinite
product. This automorphic function maps the Riemann surface of genus 2 to \( \mathbb{P}^1 \).
Its inverse map has an interesting property as stated in Proposition 3. By letting
the three purely imaginary exponents go to zero in the formula in Proposition 2,
we are led to an infinite product which would hopefully converge in some sense and
represent the elliptic modular function \( \lambda \); this conjectural formula is given in the
last section.

2. Schwarz map for the hypergeometric equations

Let us consider the hypergeometric differential equation

\[
x(1-x)\frac{d^2u}{dx^2} + \{c - (a + b + 1)x\} \frac{du}{dx} - abu = 0
\]

with purely imaginary exponents

\[
1 - c = i\theta_0, \quad c - a - b = i\theta_1, \quad a - b = i\theta_2,
\]

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where we assume $\theta_0, \theta_1, \theta_2 > 0$ for simplicity. For (any) two linearly independent solutions $u_1$ and $u_2$, the (multi-valued) map

$$s : \mathbb{C} \setminus \{0, 1\} \ni x \mapsto u_1(x) : u_2(x) \in \mathbb{P}^1 := \mathbb{C} \cup \{\infty\}$$

is often called a Schwarz map (or Schwarz’s $s$-map).

### 3. The fundamental domains

We found in [SY] a domain $F_x$ in the $x$-plane and a domain $F_s$ in the $s$-plane so that the restricted map

$$s|_{F_x} : F_x \rightarrow F_s$$

is conformally isomorphic and the whole $s$ can be recovered by $s|_{F_x}$ through Schwarz’s reflection principle; they are called fundamental domains for the Schwarz map.

Let us denote by $C(c, r)$ the circle on the $s$-plane with center $c$ and radius $r$, and consider the three disjoint circles on the $s$-plane

$$C_1 = C(0, 1), \quad C_2 = C(0, T), \quad C_3 = C(-C, R),$$

where $T = e^{\theta_1}, r = e^{-\theta_0\pi}$,

$$C = \frac{\xi(1 - r^2)}{\xi^2 - r^2}, \quad R = \frac{r(1 - \xi^2)}{\xi^2 - r^2}, \quad \xi = \left(\frac{\cosh \theta_2 \pi + \cosh(\theta_0 - \theta_1) \pi}{\cosh \theta_2 \pi + \cosh(\theta_0 + \theta_1) \pi}\right)^{\frac{1}{2}}.$$

Since $\xi$, as a function of $\theta_2 \geq 0$, increases monotonically to 1, and

$$1 > \xi|_{\theta_2=0} = \frac{T r + 1}{T + r} > r,$$

we have

$$C - R - 1 = (1 - r)(1 - \xi)/(\xi + r) > 0,$$

$$T - C - R = \frac{(T + r) \xi - (T r + 1)}{\xi - r} > 0,$$

and so

$$-T < -C - R < -C + R < -1 < 1 < T.$$ 

The domain, in the upper half-plane, bounded by $C_1, C_2, C_3$ and the real axis can serve as a fundamental domain $F_s$, and has the shape of a two-arched bridge as in Figure 1. The fundamental domain $F_s$ also has the shape of a two-arched bridge as in Figure 11 and is bounded by three real segments and three curves, which are not (part of) circles.

### 4. The monodromy group

Thanks to these fundamental domains and Schwarz’s reflection principle applied along their sides, the monodromy group of the differential equation can be described as follows.

The reflection with respect to the circle $C(c, r)$ ($c$ : real) is given by

$$\varphi(c, r) : s \mapsto \frac{r^2}{s - c} + c.$$
Let $\Lambda$ be the group generated by the three reflections $\varphi_1, \varphi_2, \varphi_3$ with respect to the circles $C_1, C_2, C_3$, respectively. The monodromy group $\Lambda_0$ of the hypergeometric equation is the subgroup of $\Lambda$, of index 2, consisting of the even words of $\varphi_1, \varphi_2, \varphi_3$.

On the other hand, for the circle $C(c, r)$, we define the fractional linear transformation of order 2 which fixes the two intersection points of the circle and the real axis:

$$\gamma(c, r) : s \mapsto \frac{r^2}{s - c} + c.$$ 

Let $\Gamma_0 (\theta = (\theta_0, \theta_1, \theta_2))$ be the group generated by the three involutions $\gamma_1, \gamma_2, \gamma_3$ with respect to the circles $C_1, C_2, C_3$, respectively. The monodromy group $\Lambda_0$ is the subgroup of $\Gamma_0$, of index 2, consisting of the even words of $\gamma_1, \gamma_2, \gamma_3$. Let $\Omega(\subset \mathbb{P}^1)$ be the domain of discontinuity of $\Gamma_0$ and the Schottky group $\Lambda_0$.

The presentation above has some problems. Though the hypergeometric differential equation is symmetric with respect to $\theta_0, \theta_1, \theta_2$, the three circles $C_1, C_2, C_3$ are not so; for example, if we let $\theta_2 \to 0$, then the circles $C_2$ and $C_3$ kiss, and if we let $\theta_1 \to 0$, then $C_3$ tends to a point and $C_1$ and $C_2$ coincide. Moreover, since the two circles $C_1$ and $C_2$ are concentric, the infinite product in the next section does not converge. So we make a linear fractional change of the coordinate $s$ as

$$s \mapsto \frac{(3 + T^2)s + 1 + 3T^2}{4(s + T^2)}.$$ 

Then the diameters of the three circles on the real axis are given as

$$C_1 : [s_4, s_5], \quad C_2 : [s_1, s_6], \quad C_3 : [s_2, s_3],$$ 

(see Figure 2) where

$$s_1 = -\frac{(1 - T)^2}{4T}, \quad s_2 = -\frac{(T - 1)^3 - (3 + T^2)(T - C - R)}{4(T^2 - T + T - C - R)},$$ 

$$s_3 = -\frac{(1 + T^2)(C - R - 1)}{4(T^2 - 1 - (C - R - 1))} + \frac{1}{2}, \quad s_4 = \frac{1}{2},$$ 

$$s_5 = 1, \quad s_6 = \frac{(1 + T)^2}{4T}.$$
Note that we have $s_1 < \cdots < s_6$. Now it is easy to see

**Proposition 1.** If $\theta_1 = 0$, then $C_1$ and $C_2$ kiss; if $\theta_2 = 0$, then $C_2$ and $C_3$ kiss; and if $\theta_3 = 0$, then $C_3$ and $C_1$ kiss.

**Proof.** If $\theta_1 = 0$, then $T = 1$, and so $s_5 = s_6$; if $\theta_2 = 0$, then $T - C - R = 0$, and so $s_1 = s_2$; if $\theta_3 = 0$, then $r = 1, C - R - 1 = 0$, and so $s_3 = s_4$. \qed

5. **Schottky automorphic functions**

From now on, we regard that the groups defined above are represented with respect to this new coordinate $s$. The following proposition and Corollary 1 are shown in [GP], IX, §2, for Schottky groups over nonarchimedean local fields which are called Whittaker groups.

**Proposition 2.** For $p, q \in \Omega$ with $\Gamma_0 \cdot p \neq \Gamma_0 \cdot q$, the infinite product

$$f_\theta(p, q; s) := \prod_{\gamma \in \Gamma_\theta} \frac{s - \gamma(p)}{s - \gamma(q)}$$

converges uniformly on any compact subset of $\Omega - \Gamma_\theta \cdot q$ and defines a $\Gamma_\theta$-automorphic function, which induces an isomorphism

$$f_\theta(p, q) : \Omega/\Gamma_\theta \to \mathbb{P}^1, \ p \mapsto 0, \ q \mapsto \infty, \ \infty \mapsto 1.$$ 

**Proof.** The Schottky group $\Lambda_\theta$ of rank 2 has a fundamental domain bounded by the circles $C_2, C_3$ and their reflections with respect to the circle $C_1$; hence $\Lambda_\theta$ is circle decomposable in the sense of [BBEIM], 5.2. Then by a result of Schottky [S], $\sum_{\gamma \in \Lambda_\theta} |\gamma'(z)|$ is convergent for any $z \in \Omega$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, we have

$$|g(p) - g(q)| = \left| \frac{p - q}{(cp + d)(cq + d)} \right| \leq \frac{|p - q|}{2} (|g'(p)| + |g'(q)|);$$
hence if \( p, q \in \Omega \), then the infinite products
\[
\prod_{\gamma \in \Lambda_\theta} \frac{s - \gamma(p)}{s - \gamma(q)} = \prod_{\gamma \in \Lambda_\theta} \left( 1 - \frac{\gamma(p) - \gamma(q)}{s - \gamma(q)} \right),
\]
\[
f_\theta(p, q; s) = \prod_{\gamma \in \Lambda_\theta} \left( \frac{s - \gamma(p)}{s - \gamma(q)} \frac{s - \gamma(q_1)}{s - \gamma(q_1)} \right)
\]
are convergent absolutely and uniformly on any compact subset of \( \Omega - \Gamma_\theta \cdot q \). Since
\[
\left( \frac{\rho(s) - \rho(\gamma(p))}{\rho(s) - \rho(\gamma(q))} \right) \left( \frac{s - \gamma(p)}{s - \gamma(q)} \right)^{-1}
\]
is independent of \( s \), there is a map \( \psi : \Gamma_\theta \rightarrow \mathbb{C}^\times \) such that
\[
f_\theta(p, q; \rho(s)) = \psi(\rho)f_\theta(p, q; s) \ (\rho \in \Gamma_\theta).
\]
Then from the fact that \( \psi \) is a group homomorphism and that \( \Gamma_\theta \) is generated by the elements \( \gamma_i \) of order 2, \( \text{Im}(\psi) \) is contained in \( \{ \pm 1 \} \), hence is independent of \( p, q \in \Omega \). Since \( \Omega \) is connected, \( \text{Im}(\psi) \) is, in fact, \( \{ 1 \} \) which implies that \( f_\theta(p, q; s) \) gives a meromorphic function on \( \Omega/\Gamma_\theta \) having only one pole at \( \Gamma_\theta \cdot q \) and this pole is of order 1. Therefore, \( f_\theta(p, q; s) \) induces an isomorphism \( \Omega/\Gamma_\theta \cong \mathbb{P}^1 \).

**Corollary 1.** The curve \( \Omega/\Lambda_\theta \) of genus two is represented as the double cover of the line branching at the six points \( f_\theta(p, q; s_j) \), where \( s_1, \ldots, s_6 \) are fixed points of \( \gamma_1, \gamma_2, \gamma_3 \), which are the intersection points of the circles \( C_1, C_2, C_3 \) and the real axis.

**Remark 1.** If we choose other \( p', q' \), then \( f_\theta(p', q'; s) \) and \( f_\theta(p, q; s) \) are related linear fractionally with coefficients independent of \( s \).

**Corollary 2.** If we take \( p \) and \( q \) reals, then \( f_\theta(p, q) \) maps the fundamental domain \( F_s \) conformally onto the upper half-plane.

**Proof.** Since \( f = f_\theta(p, q) \) is real, the three real segments on the boundary \( \partial F_s \) of \( F_s \) are mapped on the real axis. Let us see that the hemicircles on \( \partial F_s \) are also mapped on the real axis. Suppose \( f \) is invariant under the involution \( s \mapsto r^2/(s - c) + c \). The image of a point \( s = c + re^{i\phi} \) on the circle \( C(c, r) \) is given as follows:
\[
f(s) = f(r^2/(s - c) + c) = f(c + re^{-i\phi}) = \overline{f(s)}.
\]

\[\square\]

6. Fuchsian equations with six singular points

For real \( p, q \), put
\[
t_1 = f_\theta(p, q; s_1) < \cdots < t_6 = f_\theta(p, q; s_6),
\]
where \( s_1 < \cdots < s_6 \) are as above. According to the theory of Schwarzian derivatives there is a unique second-order linear differential equation
\[
E_\theta : \frac{d^2v}{dt^2} + R_\theta(t)v = 0
\]
with regular singular points at \( t = t_j \) of exponent 1/2 such that the ratio of two suitable linearly independent solutions (the Schwarz’s \( s \)-map for \( E_\theta \)) is the inverse
function of $t = f_\theta(p, q; s)$. The coefficient $R_\theta(t)$ can be, assuming $s_6 = \infty$, expressed as

$$R_\theta(t) = P_\theta(t) / \prod_{j=1}^{5} (t - t_j)^2,$$

where $P_\theta(t)$ is a polynomial of degree eight. Among the nine coefficients of $P_\theta(t)$, six are determined by the local condition (exponent is $1/2$). The remaining three are not determined by local data (in this sense, these are classically called the accessory parameters), are functions of $\theta = (\theta_1, \theta_2, \theta_3)$. Though the authors have no idea what kind of functions they are, we can tell the very specific monodromy behavior of this equation. Let us take a point $p$ on the upper half $t$-space, and a path $\rho_j$ starting from $p$, going straight near to $t_j$, turning once around $t_j$, and traveling straight back to $p$.

**Proposition 3.** Let $M_j$ be the projective local monodromy of the equation $E_\theta$ along the loop $\rho_j$ ($j = 1, \ldots, 6$). Then we have

$$M_3 \circ M_2 = M_5 \circ M_4 = M_1 \circ M_6 = id.$$

This proposition can be readily shown if we trace these loops and their inverse images under $f_\theta$ in Figure 3.
7. A conjectural formula for the $\lambda$ function

We recall Jacobi’s theta functions

$$
\vartheta_{00}(v, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1}z^2)(1 - q^{2n-1}z^{-2}),
$$

$$
\vartheta_{01}(v, \tau) = \vartheta_{00}(v + \frac{1}{2}, \tau),
$$

and theta constants

$$
\vartheta_{00}(\tau) = \vartheta_{00}(0, \tau), \quad \vartheta_{01}(\tau) = \vartheta_{01}(0, \tau),
$$

where $q = e^{\pi i \tau}, z = e^{\pi i v}$, and the elliptic modular function, the $\lambda$ function

$$
\lambda(\tau) = \left( \frac{\vartheta_{01}(\tau)}{\vartheta_{00}(\tau)} \right)^4, \quad \tau \in \mathbf{H},
$$

which gives an isomorphism

$$
\mathbf{H}/\Gamma(2) \longrightarrow \mathbb{P}^1 - \{0, 1, \infty\}, \quad 0 \mapsto 0, \ 1 \mapsto \infty, \ \infty \mapsto 1,
$$

where $\mathbf{H}$ is the upper half-plane $\{\tau \in \mathbb{C} | \Im(\tau) > 0\}$, and $\Gamma(2)$ is the principal congruence subgroup of level 2 of the elliptic modular group.

Now we go back to the situation of §4 and §5. Letting $\theta_1, \theta_2, \theta_3 \to 0$, the generating involutions $\gamma_1, \gamma_2, \gamma_3$ of $\Gamma_0 \cap \Gamma(2)$ tend to the three involutions with respect to the three kissing circles

$$
C(3/4, 1/4), \ C(1/2, 1/2), \ C(1/4, 1/4),
$$

respectively; we denote by $\Gamma_0$ the group generated by these three involutions. Accordingly, the group $\Lambda_0$ tends to the modular group $\Gamma(2)$. Proposition 2 suggests the following conjecture.

**Conjecture.** As $\theta_1, \theta_2, \theta_3 \to 0$, the function $f_0(0, 1; \tau)$ converges uniformly on any compact set of $\mathbf{H}$ to $\lambda(\tau)$. The infinite product

$$
\prod_{\gamma \in \Gamma_0} \frac{\tau - \gamma(0)}{\tau - \gamma(1)}
$$

converges, in some sense, to $\lambda(\tau)$.

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**References**


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