THERE IS NO SEPARABLE UNIVERSAL II$_1$-FACTOR

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Abstract. Gromov constructed uncountably many pairwise nonisomorphic discrete groups with Kazhdan’s property (T). We will show that no separable II$_1$-factor can contain all these groups in its unitary group. In particular, no separable II$_1$-factor can contain all separable II$_1$-factors in it. We also show that the full group C$^*$-algebras of some of these groups fail the lifting property.

1. Results

We recall that a discrete group $\Gamma$ is said to have Kazhdan’s property (T) if the trivial representation is isolated in the dual $\hat{\Gamma}$ of $\Gamma$, equipped with the Fell topology. This is equivalent to saying that there exist a finite subset $E$ of generators in $\Gamma$ and a decreasing function $f: \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ such that the following is true: if $\pi$ is a unitary representation of $\Gamma$ on a Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ is a unit vector with $\varepsilon = \max_{s \in E} \|\pi(s)\xi - \xi\|$, then there is a vector $\eta \in \mathcal{H}$ with $\|\xi - \eta\| < f(\varepsilon)$ (in particular, $\eta \neq 0$ when $\varepsilon$ is small enough) such that $\pi(s)\eta = \eta$ for all $s \in \Gamma$. We refer the reader to [HV] and [V] for the information of Kazhdan’s property (T). We recall that a discrete group $\Gamma$ is said to be quasi-finite if all of its proper subgroups are finite, and it is said to be of infinite conjugacy classes (abbreviated to ICC) if all nontrivial conjugacy classes in $\Gamma$ are infinite. We note that a discrete group $\Gamma$ is ICC if and only if its group von Neumann algebra $L\Gamma$ is a factor. We also observe that a group which is quasi-finite and ICC has to be simple.

Gromov (Corollary 5.5.E in [G]) claimed that any torsion-free noncyclic hyperbolic group has a quotient group all of whose proper subgroups are cyclic of prescribed orders (cf. Theorem 3.4 in [Y]). This claim was partly confirmed by Olshanskii (Corollary 4 in [O]). Actually, what Olshanskii proved there is that any torsion-free noncyclic hyperbolic group has a nontrivial quasi-finite quotient group. We observe that Olshanskii’s argument gives us the following.

Theorem 1 (Gromov-Olshanskii). Any torsion-free noncyclic hyperbolic group has uncountably many pairwise nonisomorphic quotient groups all of which are quasi-finite and ICC. In particular, there is a discrete group $\Gamma$ with Kazhdan’s property (T) which has uncountably many pairwise nonisomorphic quotient groups $\{\Gamma_\alpha\}_{\alpha \in I}$ all of which are simple and ICC.

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Connes conjectured that a discrete group $\Delta$ with Kazhdan’s property (T) and
the ICC property is uniquely determined by its group von Neumann algebra $L\Delta$.
The following theorem and its corollary, which was suggested by S. Popa, confirm
Connes’ conjecture for $\{\Gamma_\alpha\}_{\alpha \in I}$ “modulo countable sets” and solve Problem 4.4.29
in [S], Conjecture 4.5.5 in [P1] and Problem III.45 in [H]. See also Theorem 1 in
[P2] and its remarks.

Theorem 2. Let $\Gamma$ and $\{\Gamma_\alpha\}_{\alpha \in I}$ be as in Theorem 1 and let $M$ be a separable
$\Pi_1$-factor. Then, the set
\[
\{\alpha \in I : \text{the unitary group } U(M) \text{ of } M \text{ contains a subgroup isomorphic to } \Gamma_\alpha\}
\]
is at most countable.

Recall that two $\Pi_1$-factors $M$ and $N$ are said to be stably equivalent if there are
$n \in \mathbb{N}$ and a projection $p \in M_n(M)$ such that $pM_n(M)p$ is isomorphic to $N$.

Corollary 3. Let $\Gamma$ and $\{\Gamma_\alpha\}_{\alpha \in I}$ be as in Theorem 1 and let $M$ be a separable
$\Pi_1$-factor. Then, the set
\[
\{\alpha \in I : M \text{ contains a subfactor which is stably equivalent to } L\Gamma_\alpha\}
\]
is at most countable.

In connection with Connes’ embedding problem [C], it would be interesting to
know whether all (or at least one of) the $\Gamma_\alpha$’s are embeddable into the unitary group
$U(R^\omega)$ of the ultrapower $R^\omega$ of hyperfinite $\Pi_1$-factors. Since each $\Gamma_\alpha$ arises as a limit
of hyperbolic groups, we observe that if all hyperbolic groups are embeddable into
$U(R^\omega)$, then so is $\Gamma_\alpha$. We remark that whether all hyperbolic groups are residually
finite (and thus embeddable into $U(R^\omega)$) is one of the major open problems in
geometric group theory.

Let us consider the category of unital $C^*$-algebras and unital completely positive
maps. A $C^*$-algebra $A$ is said to be complementary universal for a class $C$ of $C^*$-
algebras if for every member $B$ of $C$, there are unital completely positive maps
$\psi : B \to A$ and $\varphi : A \to B$ such that $\varphi\psi = \text{id}_B$. It follows from Kirchberg’s theorem
[K2] that any separable $C^*$-algebra not of type I is complementary universal for the
class of separable nuclear $C^*$-algebras. The full group $C^*$-algebra $C^*F_\infty$ of the
free group $F_\infty$ on countably many generators is complementary universal for the
class of separable $C^*$-algebras with the lifting property (abbreviated to LP). See
[K1] for information on the LP. It is not known whether there exists a separable
complementary universal $C^*$-algebra for the class of separable exact $C^*$-algebras.

Theorem 4. Let $\Gamma$ and $\{\Gamma_\alpha\}_{\alpha \in I}$ be as in Theorem 1 and let $\mathcal{C} = \{C^*\Gamma_\alpha : \alpha \in I\}$
or $\mathcal{C} = \{C^*_{\text{red}}\Gamma_\alpha : \alpha \in I\}$. Then, there is no separable complementary universal
$C^*$-algebra for $\mathcal{C}$.

From the above discussion, we immediately obtain the following corollary.

Corollary 5. The full group $C^*$-algebra $C^*\Gamma_\alpha$ of $\Gamma_\alpha$ fails the LP for some $\alpha \in I$.

2. Proofs

Proof of Theorem 1. Since there exists a torsion-free noncyclic hyperbolic group
with Kazhdan’s property (T) (e.g., a co-compact lattice in $Sp(n, 1)$ or in $F_4(-20)$),
the second part is a straight consequence of the first. We just indicate how to
modify the proof of Corollary 3 in [O] to obtain the first part of our Theorem.
So, we stick by the notation used in [O]. Let $G = \{g_1, g_2, \ldots\}$ be a torsion-free noncyclic hyperbolic group. It follows that $G$ is ICC since the set $E(G)$ of all $x \in G$ whose conjugacy class is finite is a finite subgroup in $G$ (cf. Proposition 1 in [O]). Recall that the quasifinite quotient group $G'$ was the inductive limit of a sequence of epimorphisms $G = G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots$. By the construction, every $E(G_i)$ is trivial, or equivalently every $G_i$ is ICC. Hence, manipulating the construction, for every $i$ and $j \leq i$, we can carry at least $i$ mutually distinct elements from the conjugacy class of $g_j$ in $G_i$ injectively into $G'$ unless $g_j = 1$ in $G_i$. This ensures the ICC property of $G'$. In the construction, there has to be infinitely many $i$’s such that $g_i$ is torsion-free in $G_{2i-2}$. For such $i$, we may choose an arbitrarily large number for the order of $g_i$ in $G_{2i-1}$ which will be equal to that in $G'$. Combined with a diagonal argument, this implies that there are uncountably many normal subgroups in $G$ all of whose corresponding quotient groups are quasifinite and ICC. Theorem 4 now follows from this result and Lemma III.42 in [H]. \hfill \Box

**Proof of Theorem 5** To prove the theorem by reductio ad absurdum, suppose that \begin{eqnarray*}
I_0 = \{\alpha \in I : U(M) \text{ contains a subgroup isomorphic to } \Gamma_{\alpha}\}
\end{eqnarray*}
is uncountable. For each $\alpha \in I_0$, let $u_\alpha : \Gamma \to U(M)$ be a nontrivial homomorphism which factors through $\Gamma_{\alpha}$. We fix a standard representation of $M$ on $H$ with a unit cyclic separating trace vector $\xi$ in $H$. It follows that there are $\delta > 0$ and an uncountable subset $I_1$ of $I_0$ such that $\max_{s \in \Gamma} \|u_{\alpha}(s)\xi - \xi\| > \delta$ for all $\alpha \in I_1$.

Let a finite subset $E$ of generators in $\Gamma$ and a function $f$ be as in the above definition of Kazhdan’s property (T). Take $\varepsilon > 0$ small enough so that $2f(\varepsilon) < \delta$. Since $H$ is separable and $I_1$ is uncountable, there are distinct $\alpha$ and $\beta$ in $I_1$ such that $\max_{s \in E} \|u_{\alpha}(s)\xi - u_{\beta}(s)\xi\| < \varepsilon$. We consider the unitary representation $\pi : \Gamma \ni s \mapsto u_{\alpha}(s)J u_{\beta}(s)J \in B(H)$, where $J$ is the canonical conjugation on $H$ associated with $M$ and $\xi$. Then, we have $\max_{s \in \Gamma} \|\pi(s)\xi - \xi\| < \varepsilon$. It follows from Kazhdan’s property (T) of $\Gamma$ that there is a vector $\eta \in H$ with $\|\xi - \eta\| < f(\varepsilon)$ such that $\pi(s)\eta = \eta$ for all $s \in \Gamma$. Let $\Delta = \{s \in \Gamma : u_{\alpha}(s)\eta = \eta\}$. It is easy to see that $\Delta$ is a subgroup of $\Gamma$ and that $\Delta$ contains the normal subgroups $\ker u_{\alpha}$ and $\ker u_{\beta}$. Since $\Gamma_{\alpha}$ and $\Gamma_{\beta}$ are simple and $\ker u_{\alpha}$ and $\ker u_{\beta}$ are distinct, we actually have $\Delta = \Gamma$. It follows that $\max_{s \in \Gamma} \|u_{\alpha}(s)\xi - \xi\| < 2f(\varepsilon) < \delta$, which is absurd. \hfill \Box

**Proof of Corollary 6** It is not difficult to see that if $L\Gamma_{\alpha}$ is isomorphic to a (not necessarily unital) subfactor of $M$, then $\Gamma_{\alpha}$ is isomorphic to a subgroup of $U(M)$. Therefore, it follows from Theorem 5 that \begin{eqnarray*}
\{\alpha \in I : M_n(M) \text{ contains a (not necessarily unital) subfactor isomorphic to } L\Gamma_{\alpha}\}
\end{eqnarray*}
is at most countable for every $n \in \mathbb{N}$, and the conclusion follows. \hfill \Box

**Proof of Theorem 4** We only deal with the case where $C = \{C^*\Gamma_{\alpha} : \alpha \in I\}$. To prove the theorem by reductio ad absurdum, suppose that there is a separable $C^*$-algebra $A$ which is complementary universal for $C$. We fix unital completely positive maps $\psi_\alpha : C^*\Gamma_{\alpha} \to A$ and $\varphi_\alpha : A \to C^*\Gamma_{\alpha}$ such that $\varphi_\alpha \psi_\alpha = \text{id}_{C^*\Gamma_{\alpha}}$. Let $E \subset \Gamma$ be a finite set of generators of $\Gamma$ containing 1 and let $u_{\alpha}(s)$ be the unitary element in $C^*\Gamma_{\alpha}$ corresponding to $s \in \Gamma$. Let $\varepsilon > 0$ be arbitrary. Since $A$ is separable and $I$ is uncountable, there are distinct $\alpha$ and $\beta$ in $I$ such that $\max_{s \in E} \|\psi_\alpha(u_{\alpha}(s)) - \psi_\beta(u_{\beta}(s))\| < \varepsilon$. It follows that denoting the left regular
representation of $\Gamma_\alpha$ by $\lambda_\alpha$, we have
\[
\left\| \frac{1}{|E|} \sum_{s \in E} \lambda_\alpha(s) \otimes u_\beta(s) \right\|_{C^*_{\text{red}} \Gamma_\alpha \otimes_{\text{max}} C^* \Gamma_\beta} \\
\geq \left\| \frac{1}{|E|} \sum_{s \in E} \lambda_\alpha(s) \otimes \varphi_\alpha \psi_\beta(s) \right\|_{C^*_{\text{red}} \Gamma_\alpha \otimes_{\text{max}} C^* \Gamma_\alpha} \\
\geq \left\| \frac{1}{|E|} \sum_{s \in E} \lambda_\alpha(s) \otimes u_\alpha(s) \right\|_{C^*_{\text{red}} \Gamma_\alpha \otimes_{\text{max}} C^* \Gamma_\alpha} - \varepsilon \\
= 1 - \varepsilon.
\]
Since $\Gamma$ has Kazhdan’s property (T), if we choose $\varepsilon > 0$ sufficiently small, this implies that the trivial representation of $\Gamma$ is weakly contained in $C^*_{\text{red}} \Gamma_\alpha \otimes_{\text{max}} C^* \Gamma_\beta$ (cf. Proposition 4.9 in [V]). Reasoning in the same way as the proof of Theorem 2, one can show that the trivial representation is weakly contained in $C^*_{\text{red}} \Gamma_\alpha$. This is absurd.

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\section*{References}

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