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UNIQUENESS OF EXCEPTIONAL SINGULAR QUARTICS

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Abstract. We prove that given a general collection Γ of 14 points of $\mathbb{P}^4 = \mathbb{P}^4_{\mathcal{K}}$ ($\mathcal{K}$ an infinite field) there is a *unique* quartic hypersurface that is singular on Γ.

This completes the solution to the open problem of the dimension of a linear system of hypersurfaces of $\mathbb{P}^n$ that are singular on a collection of general points.

1. Introduction

Let $\mathcal{K}$ be an infinite field and $\mathbb{P}^n = \mathbb{P}^n_{\mathcal{K}}$.

The following problem has aroused a good deal of interest over the last few centuries:

Question 1. *Let Γ be a general set of d points in $\mathbb{P}^n$. Given a degree $m \geq 3$, does the vector space of sections in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ that are singular on Γ have the expected dimension of* $\max(0, \dim H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) - (n+1)d)$*?*

The answer is that the only exceptions are the following 4 cases: $(n, m, d) = (2, 4, 5)$, $(3, 4, 9)$, $(4, 4, 14)$, and $(4, 3, 7)$. This was proved by J. Alexander and A. Hirschowitz ([H], [A], [AH1], [AH2], and [AH3]). (A simpler proof was later given in [Ch2] and [Ch3].)

A correspondence between the question on singularities and the Waring problem for general linear forms was (for $\operatorname{char} \mathcal{K} = 0$) described by Lasker [L]. Terracini [T2] applied the duality of Macaulay to make this precise. Terracini [T1], as well as Palatini [P], gave a further relation to the study of a secant variety to a Veronese. (See [EI] for an extension to $\operatorname{char} \mathcal{K} \neq 0$.) The Waring problem asks: given n, m, what is the minimal $d = (n, m)$ for which the general form of degree m in $n+1$ variables may be written as a sum of d mth powers of linear forms? The expectation is that $(n+1)d \geq \binom{n+m}{m}$ should suffice (since there are d choices from the $(n+1)$-dimensional space of linear forms). The exceptional case of $(n, m, d) = (2, 4, 5)$ was discovered by Clebsch [C], followed by those of $(3, 4, 9)$, $(4, 4, 14)$ due to Sylvester [S], and the more subtle case of $(4, 3, 7)$ presented by Palatini [P].

In each of the exceptional cases we have $(n+1)d \geq \dim H^0(\mathbb{P}^n, \mathcal{O}(\mathbb{P}^n(m))$ hence no m-ic form is "numerically" expected to be singular at a general collection of d points. However, one may easily find such an m-ic in each of these cases. We consider, therefore, the question of the "next best" possibility:

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Question 2. *In the exceptional cases, is there a* ***unique*** *m-ic singular along d general points?*

The affirmative answer for the case of 7 points in $\mathbb{P}^4$ and degree 3 was given by C. Ciliberto and Hirschowitz [CH]. This is discussed, e.g., in [Ch2].

We consider the exceptional cases in degree 4, namely, 5 points in $\mathbb{P}^2$, 9 in $\mathbb{P}^3$, and 14 in $\mathbb{P}^4$. In each, $d = \dim H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)) - 1$, so that there is a quadric Q vanishing on d general points, hence Q^2 is singular at each point. Thus we show that Q^2 is the only such quartic. J. Alexander proves this in the cases of $\mathbb{P}^2$ and $\mathbb{P}^3$ in [A]. To obtain uniqueness in $\mathbb{P}^4$ we use *both* of these cases together with a Horace differential argument. This is unlike the usual application of the "méthode d'Horace" in which a codimension 1 result suffices in carrying out the induction. The result is:

Theorem 3. *If* $(n, d) = (2, 5)$, $(3, 9)$, *or* $(4, 14)$, *there is a unique quartic of* $\mathbb{P}^n$ *that is singular on* d *general points of* $\mathbb{P}^n$.

Corollary 4. *Suppose that* $\operatorname{char} \mathcal{K} \neq 2$. *Take* $(n, d) = (2, 5), (3, 9)$, *or* $(4, 14)$. *In the space of homogeneous forms of degree* 4 *in* $n+1$ *variables, the closure of the set of those expressible as a sum of* d *fourth powers of linear forms has codimension* 1.

Corollary 5. *Given* n, m, d, *let* $N = \binom{n+m}{m} - 1$. *Let* $\nu_m : \mathbb{P}^n \to \mathbb{P}^N$ *be the mth Veronese embedding of* $\mathbb{P}^n$. *Call* $S_{n,m,d}$ *the variety of secant* $(d-1)$*-planes to* $\nu_m(\mathbb{P}^n)$ *in* $\mathbb{P}^N$. *Then for* $(n, d) = (2, 5), (3, 9)$, *or* $(4, 14)$, $S_{n,m,d}$ *is a hypersurface of* $\mathbb{P}^N$.

Let us recall standard definitions in the study of such objects:

Definition 1. Let $p \in \mathbb{P}^n$. The **double point** at p in $\mathbb{P}^n$ is the subscheme of $\mathbb{P}^n$ defined by the square of the ideal sheaf of p.

If $\Phi \subset \mathbb{P}^n$, we denote by Φ^2 the union of the double points supported on Φ.

Hence a homogeneous form in the coordinate ring of $\mathbb{P}^n$ is singular on a set Φ precisely if it vanishes on Φ^2.

Definition 2. Given a scheme $X \subset \mathbb{P}^n$ and a hyperplane H of $\mathbb{P}^n$, the **Castelnuovo exact sequence** is given by

$$0 \to \mathcal{I}_{\tilde{X}}(-1) \to \mathcal{I}_X \to \mathcal{I}_{X \cap H, H} \to 0, \tag{1}$$

where $\tilde{X}$ (called the **residual** scheme to X with respect to H) is given by the ideal sheaf $\mathcal{I}_{\tilde{X}} = \mathcal{I}_X : \mathcal{O}_{\mathbb{P}^n}(-H)$.

From this, it is straightforward to prove the uniqueness in $\mathbb{P}^2$ and $\mathbb{P}^3$ using specialisation, as is done in [A]. But in $\mathbb{P}^4$ the exact sequence reveals only that there is at most a pencil of quartics through 14 double points. This is because the case of $\mathbb{P}^3$ is extra-exceptional: although $4 \cdot 9 > \dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$, there is a quartic singular on 9 general points. Hence the base locus of the system of quartics singular on 8 double points and a point q meets the double point at q in a scheme ρ of degree 3. Applying Castelnuovo to a suitable collection $Z \subset \mathbb{P}^4$ of 13 double points, of which 8 lie on a $\mathbb{P}^3$ containing a point q, shows that the base locus of quartics through $Z \cup \{q\}$ meets $\{q\}^2$ in the scheme ρ determined by those 8 points on $\mathbb{P}^3$. So $Z \cup \{q\}^2$ lies on a pencil of quartics.

To conquer this obstacle, we apply the lemme d'Horace différentielle (Lemma 6) of Alexander and Hirschowitz ([AH1]). The statement extracted from the lemma is that from such a scheme $Z \cup \{q\}^2$ lying on a pencil of quartics together with base locus scheme ρ, one may find a point p for which $Z \cup \{p\}^2$ is on a unique quartic provided that $\tilde{Z} \cup \rho$ does not lie on a cubic. The idea is to degenerate a point $p \in \mathbb{P}^4 - \mathbb{P}^3$ to q along with a subscheme $\rho' \subset \{p\}^2$ degenerating to ρ. Hence the base locus of quartics through $Z \cup \{p\}$ meets $\{p\}^2$ in a subscheme of ρ'. But then, Castelnuovo's exact sequence may be applied directly to $Z \cup \rho'$, to see that if $\tilde{Z} \cup \rho'$ does not lie on a cubic, then the base locus of quartics through $Z \cup \{p\}$ cannot contain all of ρ'. Hence by upper semicontinuity it suffices that $\tilde{Z} \cup \rho$ does not lie on a cubic.

The uniqueness in $\mathbb{P}^4$ is therefore accomplished by producing such a scheme $Z \cup \{q\}$ along with base locus scheme ρ determined by $Z \cap \mathbb{P}^3$ for which $\tilde{Z} \cup \rho$ does not lie on a cubic. Just as well, we arrange that ρ has a subscheme ρ_0 of degree 2 whose union with $\tilde{Z}$ does not lie on a cubic. Hence it is desired to have some control over the base locus scheme ρ at q. For this we arrange by further specialisation (analogous to [Ch2], in the initial case of 12 points in $\mathbb{P}^5$) that ρ has a *recognizable* such subscheme ρ_0 that does not depend on all the points. Namely, 4 of the points of $Z \cap \mathbb{P}^3$ are put onto a plane containing q, so that the base locus scheme ρ must contain the degree 2 scheme ρ_0 on q given by the conic through the 5 planar points.

Hence the problem is reduced to a matter of studying cubics on the union of 5 general double points, 4 simple points on $\mathbb{P}^3$ (and otherwise set free), with a degree 6 curvilinear subscheme of $\mathbb{P}^2$ (in linearly general position). Now the four simple points may be further specialised to $\mathbb{P}^2$, yielding $\mathbb{P}^2$ in the base locus. Then it is easy to see that no cubic of $\mathbb{P}^4$ vanishes on the general union of $\mathbb{P}^2$ with five double points, which completes the proof.

Notation. For a subscheme $X \subset \mathbb{P}^n$, we write $h_{\mathbb{P}^n}(X, m)$ for the **Hilbert function** of X in degree m: the number of conditions that X imposes on the linear system of hypersurfaces of degree m.

Taking global sections on the Castelnuovo exact sequence (1) then provides the inequality:

$$h_{\mathbb{P}^n}(X, m) \geq h_{\mathbb{P}^n}(\tilde{X}, m-1) + h_H(X \cap H, m)$$

where H is a hyperplane and $\tilde{X}$ the residual of X with respect to H.

2. Proof of Theorem 3

Fix a flag $\mathbb{P}^2 \subset \mathbb{P}^3 \subset \mathbb{P}^4$.

We show that there is a unique quartic hypersurface of $\mathbb{P}^4$ through the union of 14 general double points. To do this, we construct a scheme from the ground up, collecting subschemes with support on $\mathbb{P}^2$ and on $\mathbb{P}^3$ and thereby observing uniqueness in dimensions 2 and 3 along the way.

Dimension 2. Suppose that $\Psi \cup \{q\} \subset \mathbb{P}^2$ is a set of 5 points, no three of which are collinear. So $\Psi \cup \{q\}$ lies on a unique conic C (nonsingular and irreducible) defined by a quadric form Q. Suppose F is a quartic form vanishing on Φ^2. Then F vanishes on a subscheme of C of degree 10, hence $Q|F$, say $F = G \cdot Q$, $\deg G = 2$. Then G also vanishes on $\Psi \cup \{q\}$ (since C is nonsingular); so, up to constants, we have $G = Q$ and $F = Q^2$. Hence we have uniqueness.

Notice, in particular, that the base locus of quartics through $\Psi^2 \cup \{q\}$ meets $\{q\}^2$ in precisely $\{q\}^2 \cap C$.

Dimension 3. Let $\Phi \subset \mathbb{P}^3 - \mathbb{P}^2$ be a set of 4 points in linearly general position. Then it is easy to see (e.g. straight from the ideal) that

$$h_{\mathbb{P}^3}(\Phi^2, 3) = 16$$

and

$$h_{\mathbb{P}^3}(\Phi^2 \cup \mathbb{P}^2, 3) = 20$$

(i.e., Φ^2 does not lie on a quadric). So we may find a (general) set $\Psi \subset \mathbb{P}^2$ of 4 points so that $\Phi^2 \cup \Psi$ does not lie on a cubic. Now choose $q \in \mathbb{P}^2$ so that $\Psi \cup \{q\}$ is in linearly general position (with respect to $\mathbb{P}^2$). Then $(\Psi^2 \cup \{q\}^2) \cap \mathbb{P}^2$ lies on a unique quartic of $\mathbb{P}^2$. Hence by (1) there is a unique quartic that is singular on the collection $\Phi \cup \Psi \cup \{q\}$ of 9 points of $\mathbb{P}^3$.

Further,

$$h_{\mathbb{P}^2}(\Psi^2 \cup \{q\}, 4) = 13$$

so

$$h_{\mathbb{P}^3}(\Phi^2 \cup \Psi^2 \cup \{q\}, 3) \geq 20 + 14 = 4 \cdot 8 + 1,$$

so that equality holds here. Therefore the system of quartics through $\Phi^2 \cup \Psi^2 \cup \{q\}$ has base locus meeting $\{q\}^2$ in precisely a scheme ρ of degree 3.

Let C be the conic through $\Psi \cup \{q\}$ in $\mathbb{P}^2$ and $\rho_0 = \{q\}^2 \cap C$. As we have seen, $\rho_0 \subset \rho$.

Dimension 4. Take $\Phi \subset \mathbb{P}^3$, $\Psi \cup \{q\} \subset \mathbb{P}^2$, ρ_0, ρ just as in the case of dimension 3.

Consider a set $\Sigma \subset \mathbb{P}^4 - \mathbb{P}^3$ of 5 points in linearly general position and $Z = \Sigma^2 \cup \Phi^2 \cup \Psi^2$.

We apply the following:

Lemma 6 ([AH1])**.** *Choose a hyperplane $H \subset \mathbb{P}^n$. Let $X \subset \mathbb{P}^n$ be a union of double and simple points of $\mathbb{P}^n$ and $\tilde{X}$ its residual with respect to $\mathbb{P}^{n-1}$. Let Υ be a subscheme of a double point supported at a point $q \in H$.*

Assume that:

- $\deg X \cup \Upsilon = \binom{n+m}{m}$,
- $(X \cup \Upsilon) \cap H$ *does not lie on an m-ic of H, and*
- *if ρ is the intersection of $\Upsilon \cap H$ with the base locus of m-ics through $(X \cup \{q\}) \cap H$, then $\tilde{X} \cup \rho$ does not lie on an $(m-1)$-ic of $\mathbb{P}^n$.*

Then there is a translation Υ' of Υ so that $X \cup \Upsilon'$ does not lie on an m-ic hypersurface of $\mathbb{P}^n$.

To use the lemma, let us start by taking a general point $r \in \mathbb{P}^3$ so that $(Z \cup \{r\} \cup \{q\}^2) \cap \mathbb{P}^3$ does not lie on a quartic (by the uniqueness in $\mathbb{P}^3$) and set $X = Z \cup \{r\}$.

Next, let us choose $\Upsilon \subset \{q\}^2 \subset \mathbb{P}^4$ of degree 4 and satisfying $\Upsilon \cap \rho = \rho_0$ (so $\deg X \cup \Upsilon = 70$). Then $(X \cup \Upsilon) \cap \mathbb{P}^3$ does not lie on a quartic of $\mathbb{P}^3$ (by virtue of the choice $\rho \not\subset \Upsilon$). The base locus of quartics through $(X \cup \{q\}) \cap \mathbb{P}^3$ then meets Υ in precisely ρ_0. Hence in order to apply Lemma 6 to X and Υ, we see that the scheme $\tilde{X} \cup \rho_0 = \Sigma^2 \cup \Phi \cup \Psi \cup \rho_0$ does not lie on a cubic.

We have $\Psi \cup \rho_0 \subset \mathbb{P}^2$ and $h_{\mathbb{P}^2}(\Psi \cup \rho_0, 3) = 6$ (since $h_{\mathbb{P}^2}(C, 3) = 7 > 6$).

Since ρ_0 does not depend on Φ we may degenerate Φ to a set $\Phi_0 \subset \mathbb{P}^2$, so that

$$h_{\mathbb{P}^2}(\Phi_0 \cup \Psi \cup \rho_0, 3) = 10;$$

that is, no cubic of $\mathbb{P}^2$ vanishes on $\Phi_0 \cup \Psi \cup \rho_0$.

Consider, then, a set of 5 points $\Sigma \subset \mathbb{P}^4$. If $\Sigma^2 \cup \mathbb{P}^2$ does not lie on a cubic of $\mathbb{P}^4$, then neither does $\Sigma^2 \cup \Phi_0 \cup \Psi \cup \rho_0$, and hence by upper semicontinuity $\Sigma^2 \cup \Phi \cup \Psi \cup \rho_0$ is not on a cubic, as desired.

Thus we are left with finding Σ, so that $h_{\mathbb{P}^4}(\Sigma^2 \cup \mathbb{P}^2, 3) = 35$.

Let us take $\Sigma \subset \mathbb{P}^4 - \mathbb{P}^3$ to be a set of 5 points in linearly general position.

Then Σ^2 does not lie on a quadric of $\mathbb{P}^4$, that is,

$$h_{\mathbb{P}^4}(\Sigma^2, 2) = 15.$$

Further, any cubic through Σ^2 must vanish on the union $\operatorname{Sec}\Sigma$ of lines between pairs of points of Σ. We have (see, e.g., [Ch1])

$$h_{\mathbb{P}^3}(\operatorname{Sec}\Sigma \cap \mathbb{P}^3, 3) = 10.$$

Hence

$$\begin{aligned} h_{\mathbb{P}^n}(\Sigma^2 \cup \mathbb{P}^2, 3) &= h_{\mathbb{P}^n}(\Sigma^2 \cup \operatorname{Sec}\Sigma \cup \mathbb{P}^2, 3) \\ &\geq h_{\mathbb{P}^4}(\Sigma^2, 2) + h_{\mathbb{P}^3}((\operatorname{Sec}\Sigma \cap \mathbb{P}^3) \cup \mathbb{P}^2, 3) \\ &\geq h_{\mathbb{P}^4}(\Sigma^2, 2) + h_{\mathbb{P}^3}(\operatorname{Sec}\Sigma \cap \mathbb{P}^3, 3) + h_{\mathbb{P}^2}(\mathbb{P}^2, 3) \\ &= 15 + 10 + 10 = 35. \end{aligned}$$

By Lemma 6 there is a point $p \in \mathbb{P}^4$ for which

$$h_{\mathbb{P}^4}(\Sigma^2 \cup \Phi^2 \cup \Psi^2 \cup \{p\}^2 \cup \{r\}, 3) = 70.$$

Thus, there is a unique quartic of $\mathbb{P}^4$ that is singular on the collection $\Sigma \cup \Phi \cup \Psi \cup \{p\}$ of 14 points. □

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