

A PHILOSOPHY FOR THE MODELLING OF REALISTIC NONLINEAR SYSTEMS

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ABSTRACT. A nonlinear dynamical system is modelled as a nonlinear mapping from a set of input signals into a corresponding set of output signals. Each signal is specified by a set of real number parameters, but such sets may be uncountably infinite. For numerical simulation of the system each signal must be represented by a finite parameter set and the mapping must be defined by a finite arithmetical process. Nevertheless the numerical simulation should be a good approximation to the mathematical model. We discuss the representation of *realistic* dynamical systems and establish a stable approximation theorem for numerical simulation of such systems.

1. INTRODUCTION

To construct a mathematical model of a *realistic* dynamical system it is necessary to formalize definitions of such crucial physical properties as *causality*, *finite memory* and *stationarity*. The philosophy of realistic systems has been considered by many authors including Russell [1], Paley and Wiener [2], Faires and Segal [3], Falb and Freedman [4], Willems [5], Gohberg [6] and Sandberg and Xu [7]. We propose a generic topological structure to describe *realistic* nonlinear systems and extend the methods of Torokhti and Howlett [8], [9], [10] to prove *stable* approximation theorems for numerical simulation of these systems. We define a class of \mathcal{R} -operators and prove that an \mathcal{R} -continuous operator F can be approximated by an \mathcal{R} -continuous operator S constructed from an algebra of elementary functions by a finite arithmetic process. The approximation is *stable* to small disturbances. Our theorem is a generalization of the Stone-Weierstrass theorem. Theorems of this type were extended to operators on topological vector spaces by Prenter [11] and Bruno [12]. A Stone-Weierstrass theory for approximation of continuous functions by superpositions of a sigmoidal function was given by Cybenko [13]. Daugavet [14] considered nonlinear operator approximation by generalized causal operators. We provide a substantial extension of this work and show that our definition of the \mathcal{R} -continuous operator includes the accepted notions of causality [1] - [7], [14] and other fundamental realistic properties as special cases. Several key results on operator approximation [11], [12], [14] also follow from particular applications of our main

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theorems. In future work we intend to show that certain specific approximation problems [15], [16] can be formulated and solved for \mathcal{R} -operators.

2. REPRESENTATION OF REALISTIC DYNAMICAL SYSTEMS

We define a class of *realistic* systems. The fundamental idea is that each input or output is uniquely defined by a corresponding legend of historical information. We pay particular attention to systems in which the output history depends continuously on the input history.

2.1. \mathcal{R} -spaces.

Definition 2.1. [14] Let X and A be Banach spaces and let $\mathcal{L}(X, A)$ be the set of continuous linear operators from X into A . Let $T = (T, \rho)$ be a compact metric space and let $\mathcal{M} = \{M_t\}_{t \in T}$ be a family of operators $M_t \in \mathcal{L}(X, A)$ with norm $\|M_t\| \leq 1$ for each $t \in T$ and such that $M_s[u] \rightarrow M_t[u]$ as $\rho(s, t) \rightarrow 0$ for each $u \in X$. The space X equipped with the family of operators \mathcal{M} is called an \mathcal{R} -space and is denoted by $X_{\mathcal{R}} = (X, A, T, \mathcal{M})$.

The family \mathcal{M} provides a mechanism for storing and manipulating information about elements in X .

Definition 2.2. For each $x \in X$ the collection of elements $\mathcal{M}[x] = \{M_t[x] \mid t \in T\} \subseteq A$ is called the legend or the complete history of the element x . For each $t \in T$ the element $M_t[x] \in A$ represents the current history of x .

We assume that each element $x \in X$ is uniquely defined by specifying the legend $\mathcal{M}[x]$ of the element.¹

Lemma 2.3. $\mathcal{M}[x] = \{0\} \Leftrightarrow x = 0$.

If we define $\mathcal{M}[x] + \mathcal{M}[y] = \mathcal{M}[x + y]$ and $\alpha\mathcal{M}[x] = \mathcal{M}[\alpha x]$ for each $\alpha \in \mathbb{C}$, then the set $\mathcal{X} = \mathcal{M}[X] = \{\mathcal{M}[x] \mid x \in X\}$ of all legends is a linear space over \mathbb{C} with zero element $\mathcal{M}[0]$. If we further define $\|\mathcal{M}[x]\| = \sup_{t \in T} \|M_t[x]\|$, then \mathcal{X} is a normed linear space.

Definition 2.4. The archival function $\mathcal{H} : \mathcal{X} \rightarrow X$ is a well-defined linear function given by the formula $\mathcal{H}(\mathcal{M}[x]) = x$ for all $\mathcal{M}[x] \in \mathcal{X}$.

Definition 2.5. The family \mathcal{M} is said to be pointwise normally extreme on X if, for each $x \in X$, there exists $t = t_x \in T$ such that $\|M_t[x]\| = \|x\|$.

Lemma 2.6. *If the family \mathcal{M} is pointwise normally extreme on X , then the normed linear spaces \mathcal{X} and X are isometrically isomorphic under the archival mapping $\mathcal{H} : \mathcal{X} \rightarrow X$.*

Corollary 2.7. *If the family \mathcal{M} is pointwise normally extreme on X , then \mathcal{X} is a Banach space and $\mathcal{H} \in \mathcal{L}(\mathcal{X}, X)$ with $\|\mathcal{H}\| = 1$.*

¹This is an adaption of the idea that a function is defined by specifying the complete set of function values.

2.2. \mathcal{R} -operators and \mathcal{R} -continuous operators.

Definition 2.8. [14] Let $X_{\mathcal{R}} = (X, A, T, \mathcal{M})$ and $Y_{\mathcal{R}} = (Y, B, T, \mathcal{N})$ be \mathcal{R} -spaces, and let the closed set $E \subseteq T \times T$ be an equivalence relation. Let $K \subseteq X$ be a compact set, and let $x \in K$ and $t \in T$. The operator $F : K \rightarrow Y$ is an \mathcal{R} -operator at $M_t[x] \in A$ if $M_s[v] = M_t[x] \Rightarrow N_s[F(v)] = N_t[F(x)]$ whenever $(s, t) \in E$ and $v \in K$. If $F : K \mapsto Y$ is an \mathcal{R} -operator at $M_t[x] \in A$ for all $x \in K$ and $t \in T$, then we say that $F : K \mapsto Y$ is an \mathcal{R} -operator.

A dynamical system defined by an \mathcal{R} -operator $F : K \mapsto Y$ has the following interpretation. For each $x \in K$ and $t \in T$ the current history $N_t[F(x)]$ of the output depends only on the current history $M_t[x]$ of the input. For a theory of constructive approximation we require the dependence to be continuous.

Definition 2.9. Let $X_{\mathcal{R}} = (X, A, T, \mathcal{M})$ and $Y_{\mathcal{R}} = (Y, B, T, \mathcal{N})$ be \mathcal{R} -spaces, and let the closed set $E \subseteq T \times T$ be an equivalence relation. Let $K \subseteq X$ be compact, and let $x \in K$ and $t \in T$. The operator $F : K \rightarrow Y$ is \mathcal{R} -continuous at $M_t[x] \in A$ if, for each open neighbourhood of zero $H \subseteq B$, there is an open neighbourhood of zero $G = G(x, t, H) \subseteq A$ such that $M_s[v] \in M_t[x] + G \Rightarrow N_s[F(v)] \in N_t[F(x)] + H$ when $(s, t) \in E$ and $v \in K$. If $F : K \mapsto Y$ is \mathcal{R} -continuous at $M_t[x] \in A$ for all $x \in K$ and $t \in T$, then we say that F is \mathcal{R} -continuous.

Lemma 2.10. *If $F : K \mapsto Y$ is \mathcal{R} -continuous, then F is also an \mathcal{R} -operator.*

Lemma 2.11. *For each $t \in T$ the set $M_t[K] = \{M_t[x] \mid x \in K\} \subseteq A$ is compact.*

Proof. If $\{G_\gamma\}_{\gamma \in \Gamma}$ is a collection of open sets, then $M_t[K] \subseteq \bigcup_{\gamma \in \Gamma} G_\gamma \Rightarrow K \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$ where each $U_\gamma = M_t^{-1}[G_\gamma]$ is also open. Since K is compact there is a finite subcollection $U_{\gamma_1}, \dots, U_{\gamma_r}$ such that $K \subseteq \bigcup_{i=1}^r U_{\gamma_i} \Rightarrow M_t[K] \subseteq \bigcup_{i=1}^r G_{\gamma_i}$. \square

For each $t \in T$ let $E_t = \{s \mid (s, t) \in E\} \subseteq T$. Note that E_t is compact. We wish to show that the set $\mathcal{M}_t[K] = \{M_s[K] \mid s \in E_t\}$ is also compact.

Lemma 2.12. *Let $s \in T$. If $M_s[K] \subseteq G$ where G is an open set, then we can find $\delta = \delta(s, G) > 0$ such that $M_r[K] \subseteq G$ when $\rho(r, s) < \delta$.*

Proof. If not \exists sequences $\{r_i\} \subseteq T$ with $\rho(r_i, s) \rightarrow 0$ and $\{u_i\} \subseteq K$ such that $M_{r_i}[u_i] \notin G$ for each i . We can assume $u_i \rightarrow v$ for some $v \in K$. Choose $\alpha > 0$ and $G_\alpha = \{a \mid \|a\| < \alpha\} \subseteq A$ so that $M_s[v] + G_\alpha \subseteq G$. If $U_\alpha = \{u \mid \|u\| < \alpha\} \subseteq X$, then $u \in U_\alpha/2 \Rightarrow M_r[u] \in G_\alpha/2$. If i is so large that $u_i - v \in U_\alpha/2$ and $M_{r_i}[v] \in M_s[v] + G_\alpha/2$, then $M_{r_i}[u_i] \in M_s[v] + G_\alpha \subseteq G$. This is a contradiction. \square

Lemma 2.13. *For each $t \in T$ the set $\mathcal{M}_t[K]$ is a compact subset of A .*

Proof. Let $t \in T$ and $s \in E_t$ and suppose that $\{G_\gamma\}_{\gamma \in \Gamma}$ is a collection of open sets with $\mathcal{M}_t[K] \subseteq \bigcup_{\gamma \in \Gamma} G_\gamma$. Since $M_s[K]$ is compact and $M_s[K] \subseteq \mathcal{M}_t[K]$ for each $s \in E_t$ there is a finite subset $\Gamma(s) \subseteq \Gamma$ with $M_s[K] \subseteq \bigcup_{\gamma \in \Gamma(s)} G_\gamma = G(s)$. Choose $\delta(s) > 0$ such that $M_r[K] \subseteq G(s)$ whenever $\rho(r, s) < \delta(s)$, and define the open sets $R(s) = \{r \mid \rho(r, s) < \delta(s)\} \subseteq T$ for each $s \in T$. Since E_t is compact we know that $E_t \subseteq \bigcup_{s \in E_t} R(s) \Rightarrow E_t \subseteq \bigcup_{j=1}^q R(s_j)$ for some finite subcollection $\{R(s_j)\}_{j=1,2,\dots,q}$ and since $\bigcup_{r \in R(s_j)} M_r[K] \subseteq G(s_j)$ for each $j = 1, \dots, q$ we have

$$\mathcal{M}_t[K] = \bigcup_{r \in E_t} M_r[K] = \bigcup_{j=1}^q \left[\bigcup_{r \in R(s_j)} M_r[K] \right] \subseteq \bigcup_{j=1}^q G(s_j) = \bigcup_{j=1}^q \left[\bigcup_{\gamma \in \Gamma(s_j)} G_\gamma \right].$$

Therefore $\mathcal{M}_t[K]$ is compact. \square

Lemma 2.14. *Let $F : K \mapsto Y$ be continuous and \mathcal{R} -continuous. For each neighbourhood of zero, $H \subseteq B$ there exists a neighbourhood of zero $G = G(H)$ such that $M_r[u] - M_s[v] \in G \Rightarrow N_r[F(u)] - N_s[F(v)] \in H$ whenever $(r, s) \in E$ and $u, v \in K$. Hence F is uniformly \mathcal{R} -continuous.*

Proof. If not then for some $\beta > 0$ there exist neighbourhoods of zero $H_\beta = \{b \mid \|b\| < \beta\} \subseteq B$ and $G_{1/n} = \{a \mid \|a\| < 1/n\} \subseteq A$ for each $n = 1, 2, \dots$ and $u_n, v_n \in K$ and $r(n), s(n), t(n) \in T$ with $r(n), s(n) \in E_{t(n)}$ for each $n = 1, 2, \dots$ such that $M_{r(n)}[u_n] - M_{s(n)}[v_n] \in G_{1/n}$ and $N_{r(n)}[F(u_n)] - N_{s(n)}[F(v_n)] \notin H_\beta$. We suppose, without loss of generality, that there exist $u, v \in K$ with $u_n \rightarrow u$ and $v_n \rightarrow v$ as $n \rightarrow \infty$ and points $r, s, t \in T$ with $\rho(r(n), r) \rightarrow 0$, $\rho(s(n), s) \rightarrow 0$ and $\rho(t(n), t) \rightarrow 0$ as $n \rightarrow \infty$. Since $(r(n), t(n)) \in E$ and $(s(n), t(n)) \in E$ and since E is closed, it follows that $(r, t) \in E$ and $(s, t) \in E$. Hence $r, s \in E_t$. Choose $\alpha > 0$ and define $G_\alpha = \{a \mid \|a\| < \alpha\} \subseteq A$. We have $M_r[x] \in G_\alpha/5$, whenever $x \in U_\alpha/5$ where $U_\alpha = \{x \mid \|x\| < \alpha\} \subseteq X$. If we take n so large that $u - u_n, v - v_n \in U_\alpha/5$, $M_{r(n)}[u] - M_r[u], M_{s(n)}[v] - M_s[v] \in G_\alpha/5$ and $G_{1/n} \subseteq G_\alpha/5$ then $M_r[u] - M_s[v] \in G_\alpha$. Since α is arbitrary it follows that $M_r[u] - M_s[v] = 0$ and since $r, s \in E_t$ the \mathcal{R} -continuity of F implies that $N_r[F(u)] - N_s[F(v)] = 0$. Define $V_\beta = \{y \mid \|y\| < \beta\} \subseteq Y$. Note that $N_r[y] \in H_\beta/4$ whenever $y \in V_\beta/4$. Choose n so large that $F(u_n) - F(u), F(v_n) - F(v) \in V_\beta/4$ and $N_{r(n)}[F(u)] - N_r[F(u)], N_{s(n)}[F(v)] - N_s[F(v)] \in H_\beta/4$. Hence $N_{r(n)}[F(u_n)] - N_{s(n)}[F(v_n)] \in H_\beta$, which is a contradiction. \square

2.3. The collection of auxiliary mappings. To establish a constructive approximation for \mathcal{R} -continuous mappings we define a collection of auxiliary mappings.

Definition 2.15. Let $F : K \mapsto Y$ be \mathcal{R} -continuous. For each $t \in T$ define the auxiliary mapping $f_t : \mathcal{M}_t[K] \mapsto B$ by setting $f_t(M_s[v]) = N_s[F(v)]$ for each $s \in E_t$ and $v \in K$.

This is a good definition because $M_r[u] = M_s[v] \Rightarrow N_r[F(u)] = N_s[F(v)]$ for each $r, s \in E_t$ and each $u, v \in K$. The mapping $f_t : \mathcal{M}_t[K] \mapsto B$ is continuous at each point $M_s[v] \in \mathcal{M}_t[K]$ because, for each open neighbourhood of zero $H \subseteq B$, there is a corresponding open neighbourhood of zero $G = G_t(v, s, H) = G(v, s, H) \cap \mathcal{M}_t[K] \subseteq A$ such that $M_r[u] - M_s[v] \in G \Rightarrow f_t(M_r[u]) - f_t(M_s[v]) = N_r[F(u)] - N_s[F(v)] \in H$ whenever $r \in E_t$ and $u \in K$. Because $\mathcal{M}_t[K]$ is compact the mapping $f_t : \mathcal{M}_t[K] \mapsto B$ is uniformly continuous and for each neighbourhood of zero $H \subseteq B$, there is a neighbourhood of zero $G = G_t(H) \subseteq A$ such that $M_r[u] - M_s[v] \in G \Rightarrow f_t(M_r[u]) - f_t(M_s[v]) \in H$ whenever $r, s \in E_t$ and $u, v \in K$. Lemma 2.14 shows that when $F : K \mapsto Y$ is continuous the collection $\{f_t\}_{t \in T}$ is uniformly equi-continuous. That is, for each neighbourhood of zero $H \subseteq B$ there is a neighbourhood of zero $G = G(H) \subseteq A$ such that for all $t \in T$ we have $M_r[u] - M_s[v] \in G \Rightarrow f_t(M_r[u]) - f_t(M_s[v]) \in H$ whenever $r, s \in E_t$ and $u, v \in K$.

2.4. Some examples of \mathcal{R} -continuous operators. The following theorem of M. Riesz is used in the examples to justify compactness of the set K .

Theorem 2.16. *Let $K \subseteq L^p([0, 1])$ and write $\mathcal{T}_h x(r) = x(r+h) \forall x \in K; r, r+h \in [0, 1]$. The set K is compact if and only if $\exists M > 0$ with $\|x\|_p \leq M$ and $\delta = \delta(\epsilon)$ such that $\|\mathcal{T}_h x - x\|_p < \epsilon$ whenever $|h| < \delta$ for all $x \in K$.*

2.4.1. *A causal operator.* Let $X = L^1([0, 1])$, $K = \{x \mid |x(s) - x(t)| \leq |s - t| \forall s, t \in [0, 1]\} \subseteq X$ and $Y = C([0, 1])$. The operator $F : K \mapsto Y$ is a \mathcal{C} -operator on the time interval $T = [0, 1]$ if, for all $t \in T$ and $u, x \in K$, $\{u(s) = x(s) \forall s \in [0, t]\} \Rightarrow \{[F(u)](s) = [F(x)](s) \forall s \in [0, t]\}$. Note that the output at time t depends only on the input prior to time t . The operator F is uniformly \mathcal{C} -continuous if, for all $t \in T$ and all $u, x \in K$ and for each $\beta > 0$, we can find $\alpha = \alpha(\beta) > 0$ such that $\{|\int_{[0,s]} u(r)dr - \int_{[0,s]} x(r)dr| < \alpha \forall s \in [0, t]\} \Rightarrow \{|[F(u)](s) - [F(x)](s)| < \beta \forall s \in [0, t]\}$. To show that a uniformly \mathcal{C} -continuous operator is a special case of a uniformly \mathcal{R} -continuous operator, set $A = B = C(T)$ and $\tau = \min(s, t)$ and define $M_t[x](s) = \int_{[0,\tau]} x(r)dr$ and $N_t[y](s) = y(\tau)$ for $x \in X, y \in Y$ and $s, t \in T$. Let $E = \{(t, t) \mid t \in T\}$. In this notation F is a uniformly \mathcal{C} -continuous operator on T if and only if for all $t \in T$, and all $u, x \in K$ and for each $\beta > 0$, we can find $\alpha = \alpha(\beta)$ such that $\{\|M_t[u] - M_t[x]\| < \alpha\} \Rightarrow \{\|[F(u)] - [F(x)]\| < \beta\}$, which, in turn, is equivalent to saying that F is a uniformly \mathcal{R} -continuous operator on T . For a particular instance we note that the operator $F_C : K \mapsto Y$ defined by $[F_C(x)](t) = e^{-t} \int_{[0,t]} e^s x(s)ds$ for each $t \in T$ is a uniformly \mathcal{C} -continuous operator.

2.4.2. *A stationary operator with finite memory.* Let $X = L^\infty(\mathbb{R})$, $K = \{x \mid x(t) = 0 \text{ for } t \notin [0, 1] \text{ and } |x(s) - x(t)| \leq |s - t| \forall s, t \in \mathbb{R}\} \subseteq X$ and $Y = C(\mathbb{R})$. The operator $F : K \mapsto Y$ is a stationary operator with finite memory $\Delta > 0$ on the time interval $T = [0, 1 + \Delta]$ if, for all $u, x \in K$ and all $s, t \in T$, $\{u(s+r-\Delta) = x(s+r-\Delta) \forall r \in [0, \Delta]\} \Rightarrow \{[F(u)](s) = [F(x)](t)\}$. The output at time t depends only on the inputs at times $s \in [t - \Delta, t]$. We say that F is an \mathcal{S} -operator. The operator F is uniformly \mathcal{S} -continuous on T if, $\forall \{u, x \in K; s, t \in T; \beta > 0\}$, we can find $\alpha = \alpha(\beta) > 0$ such that $\{|u(s+r-\Delta) - x(s+r-\Delta)| < \alpha \forall r \in [0, \Delta]\} \Rightarrow \{|[F(u)](s) - [F(x)](t)| < \beta\}$. To show that a uniformly \mathcal{S} -continuous operator is a special case of a uniformly \mathcal{R} -continuous operator, set $A = L^\infty([0, \Delta])$ and $B = C([0, 1 + \Delta])$. For each $t \in T$ define $M_t : X \mapsto A$ by $M_t[x](r) = x(r + t - \Delta) \forall r \in [0, \Delta]$ and $N_t : Y \mapsto C([0, 1 + \Delta])$ by $N_t[y](r) = y(t) \forall r \in [0, 1 + \Delta]$. Let $E = T \times T$. The operator F is uniformly \mathcal{S} -continuous on T if and only if for all $u, x \in K$ and all $s, t \in T$ and for each $\beta > 0$, we can find $\alpha = \alpha(\beta) > 0$ such that $\{\|M_s[u] - M_t[x]\| < \alpha\} \Rightarrow \{\|N_s[F(u)] - N_t[F(x)]\| < \beta\}$, which is equivalent to saying that F is a uniformly \mathcal{R} -continuous operator on T . In particular, the mapping $F_\Delta : K \mapsto Y$ defined by $[F_\Delta(x)](t) = \frac{1}{\Delta} \int_{[t-\Delta,t]} x(r)dr$ for each $x \in X$ and $t \in \mathbb{R}$ is a uniformly \mathcal{S} -continuous operator.

3. THE MODULUS OF CONTINUITY

Definition 3.1. Let X and Y be separable Banach spaces. Let $K \subseteq X$ be a compact set, and let $F : K \rightarrow Y$ be a continuous map. The modulus of continuity $\omega = \omega[F] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is given by the formula

$$\omega(\delta) = \sup_{x_1, x_2 \in K, \|x_1 - x_2\| \leq \delta} \|F(x_1) - F(x_2)\|.$$

Note that $\omega(0) = 0$ and $\omega(\delta) \leq \omega(\delta')$ whenever $\delta \leq \delta'$. We will show that ω is a uniformly continuous function.

Lemma 3.2. Let X and Y be separable Banach spaces. Let $K \subseteq X$ be a compact set and $F : K \rightarrow Y$ a continuous map. Let $\omega = \omega[F] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the corresponding

modulus of continuity. Then for each $\tau > 0$ we can find $\sigma = \sigma(\tau) > 0$ such that $0 \leq \omega(\delta') - \omega(\delta) \leq \tau$ whenever $0 \leq \delta' - \delta \leq \sigma$.

Proof. Define $\Delta F : K \times K \mapsto Y$ by setting $\Delta F(x) = F(x_2) - F(x_1)$ for each $x = (x_1, x_2) \in K$. Clearly ΔF is continuous with respect to the norm $\|x\|_{K \times K} = \|x_1\| + \|x_2\|$ and hence, since $K \times K$ is compact, ΔF is uniformly continuous. If we define $D_\delta = \{x \mid \|x_2 - x_1\| \leq \delta\}$, then $D_\delta \subseteq K \times K$ is compact and $\omega(\delta) = \sup_{x \in D_\delta} \|\Delta F(x)\|$ for each $\delta \geq 0$. Fix $\tau > 0$ and choose $\sigma = \sigma(\tau) > 0$ such that $\|\Delta F(x') - \Delta F(x)\| < \tau$ whenever $\|x' - x\|_{K \times K} < \sigma$. Now suppose that $0 \leq \delta' - \delta \leq \sigma$. Find $x' \in D_{\delta'}$ with $x'_2 \neq x'_1$ and $\omega(\delta') = \|\Delta F(x')\|$, and define $\theta \in [0, 1]$ so that $\theta\|x'_2 - x'_1\| = \delta$. Let $x'' = (x'_2, x'_1)$ and define $x = \theta x' + (1 - \theta)(x' + x'')/2$. It is easy to see that $\|x_2 - x_1\| = \delta$ and that $\|x' - x\|_{K \times K} \leq \sigma$. It follows that $\omega(\delta') = \|\Delta F(x')\| \leq \|\Delta F(x)\| + \tau \leq \omega(\delta) + \tau$. Thus $0 \leq \omega(\delta') - \omega(\delta) \leq \tau$ when $0 \leq \delta' - \delta \leq \sigma$ and hence ω is uniformly continuous on \mathbb{R}_+ . \square

3.1. The \mathcal{R} -modulus of continuity. The \mathcal{R} -modulus of continuity will be used to characterize our constructive approximation theorems for \mathcal{R} -continuous operators.

Definition 3.3. [14] Let $X_{\mathcal{R}} = \{X, A, T, \mathcal{M}\}$ and $Y_{\mathcal{R}} = \{Y, B, T, \mathcal{N}\}$ be \mathcal{R} -spaces, and let $E \subseteq T \times T$ be the given equivalence relation. Let $K \subseteq X$ be a compact set, and suppose that the map $F : K \rightarrow Y$ is \mathcal{R} -continuous. The function $\omega_{\mathcal{R}} = \omega_{\mathcal{R}}[F] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by

$$\omega_{\mathcal{R}}(\delta) = \sup_{\substack{u, v \in K; (r, s) \in E: \\ \|M_r[u] - M_s[v]\| \leq \delta}} \|N_r[F(u)] - N_s[F(v)]\|$$

is called the \mathcal{R} -modulus of continuity of the operator F .

Definition 3.4. We say that $(X_{\mathcal{R}}, Y_{\mathcal{R}})$ is a complete \mathcal{R} -pair if $E = T \times T$ and an incomplete \mathcal{R} -pair if $E \neq T \times T$.

Lemma 3.5. Let $(X_{\mathcal{R}}, Y_{\mathcal{R}})$ be a complete \mathcal{R} -pair and suppose that $F : K \mapsto Y$ is \mathcal{R} -continuous. Then the \mathcal{R} -modulus of continuity $\omega_{\mathcal{R}} = \omega_{\mathcal{R}}[F] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is uniformly continuous with $\omega_{\mathcal{R}}(0) = 0$.

Proof. Since $E_t = T$ for all $t \in T$ it follows that $\mathcal{M}[K] = \mathcal{M}_t[K] = \{M_s[x] \mid x \in K \text{ and } s \in T\} \subseteq A$ for all $t \in T$. Define an auxiliary mapping $f : \mathcal{M}[K] \mapsto B$ by setting $f(M_t[x]) = N_t[Fx]$ for each $x \in K$ and $t \in T$. Recall from our earlier remarks that the mapping $f : \mathcal{M}[K] \mapsto B$ is uniformly continuous. The function $\omega_f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is the modulus of continuity of f . Lemma 3.2 shows that ω_f is uniformly continuous. Since $\omega_f(\delta) = \omega_{\mathcal{R}}(\delta)$ we obtain the desired result. \square

Lemma 3.6. Let $(X_{\mathcal{R}}, Y_{\mathcal{R}})$ be an incomplete \mathcal{R} -pair and suppose that $F : K \mapsto Y$ is both continuous and \mathcal{R} -continuous. Then the \mathcal{R} -modulus of continuity $\omega_{\mathcal{R}} = \omega_{\mathcal{R}}[F] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is uniformly continuous with $\omega_{\mathcal{R}}(0) = 0$.

Proof. Since $(X_{\mathcal{R}}, Y_{\mathcal{R}})$ is an incomplete \mathcal{R} -pair we consider the various equivalence classes E_t for each $t \in T$. We saw earlier that for each $t \in T$ there is an auxiliary mapping $f_t : M_t[K] \mapsto B$ defined by $f_t(M_t[x]) = N_t[F(x)]$ for all $x \in K$. Let $\omega[f_t] : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the modulus of continuity for the map f_t , and consider the argument used in Lemma 3.2. Define $\Delta f_t : M_t[K] \times M_t[K]$ by the formula $\Delta f_t(p, q) = \|f_t(p) - f_t(q)\|$ for each $(p, q) \in M_t[K] \times M_t[K]$. Choose $\tau > 0$. From our earlier remarks about the uniform equi-continuity of the family of auxiliary mappings $\{f_t\}_{t \in T}$, we can choose $\sigma = \sigma(\tau) > 0$ such that for all $t \in T$ we have $\|\Delta f_t(p', q') - \Delta f_t(p, q)\| < \tau$

whenever $\|(p', q') - (p, q)\| < \sigma$. Now it is clear from Lemma 3.2 that for all $t \in T$ we have $0 \leq \omega[f_t](\delta') - \omega[f_t](\delta) \leq \tau$ whenever $0 \leq \delta' - \delta \leq \sigma$. Thus the family $\{\omega[f_t]\}_{t \in T}$ is also uniformly equi-continuous. Since $\omega_{\mathcal{R}}(\delta) = \sup_{t \in T} \omega[f_t](\delta)$, it follows that $0 \leq \omega_{\mathcal{R}}(\delta') - \omega_{\mathcal{R}}(\delta) \leq \tau$ whenever $0 \leq \delta' - \delta \leq \sigma$. \square

4. APPROXIMATION OF NONLINEAR OPERATORS ON COMPACT SETS

We describe briefly the recent work by Torokhti and Howlett [8]. Let X, Y be locally convex topological vector spaces and let $K \subseteq X$ be a compact subset. Let $F : K \subseteq X \rightarrow Y$ be a continuous map. If F is known only on K , then for some suitable neighbourhood ϵ of zero in X the construction of an extended operator $S : K + \epsilon \subseteq X \rightarrow Y$ is an important ingredient in the approximation procedure. The extension of the domain allows consideration of a small disturbance in the input signal. Such disturbances are unavoidable in the modelling process. The main result is formulated as follows. Let X, Y be topological vector spaces with the Grothendieck property of approximation² and with approximating sequences $\{G_m\}_{m=1,2,\dots} \subseteq \mathcal{L}(X, X_m)$, $\{H_n\}_{n=1,2,\dots} \subseteq \mathcal{L}(Y, Y_n)$ of continuous linear operators, where $X_m \subseteq X, Y_n \subseteq Y$ are subspaces of dimension m, n . Write $X_m = \{x_m \in X \mid x_m = \sum_{j=1}^m a_j u_j\}$ and $Y_n = \{y_n \in Y \mid y_n = \sum_{k=1}^n b_k v_k\}$, where $a \in \mathbb{R}^m, b \in \mathbb{R}^n$ and $\{u_j\}_{j=1,2,\dots,m}, \{v_k\}_{k=1,2,\dots,n}$ are bases in X_m, Y_n respectively. Let $\mathcal{G} = \{g\}$ be an algebra of continuous functions $g : \mathbb{R}^m \rightarrow \mathbb{R}$ that satisfies the conditions of Stone's algebra. Define the operators $Q \in \mathcal{L}(X_m, \mathbb{R}^m)$, $Z : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $W \in \mathcal{L}(\mathbb{R}^n, Y_n)$ by $Q(x_m) = a$, $Z(a) = (g_1(a), g_2(a), \dots, g_n(a))$ and $W(z) = \sum_{k=1}^n z_k v_k$ where each $g_k \in \mathcal{G}$ and $z_k = g_k(a)$. Subject to an appropriate choice of the functions $\{g_k\} \in \mathcal{G}$, so that z_k provides a sufficiently good approximation to b_k , the following *stable* approximation theorems can be established.

Theorem 4.1. *Let X, Y be locally convex topological vector spaces as above, and let X be normal. Let $K \subseteq X$ be a compact set and $F : K \rightarrow Y$ a continuous map. For a given convex neighbourhood of zero $\tau \subseteq Y$ there exists a neighbourhood of zero $\sigma \subseteq X$, an associated continuous operator $S : X \rightarrow Y_n$ defined by finite arithmetic in the form $S = S_\sigma = WZQG_m$ and a neighbourhood of zero $\epsilon \subseteq X$ such that for all $x \in K$ and all $x' \in X$ with $x' - x \in \epsilon$ we have $F(x) - S(x') \in \tau$.*

Theorem 4.2. *Let X and Y be separable Banach spaces. Let $K \subseteq X$ be a compact set and $F : K \rightarrow Y$ a continuous map. For any given numbers $\delta > 0$ and $\tau > 0$ and for all $x \in K$ and all $x' \in X$ with $\|x' - x\| \leq \delta$, there exists an operator $S = WZQG_m : X \rightarrow Y$ defined by finite arithmetic such that $\|F(x) - S(x')\| \leq \frac{1}{2}\omega[F](2\delta) + \tau$.*

Proof. The proof of the latter result uses an argument proposed by Daugavet [14]. Since [14] is difficult to obtain, the proof is given in Appendix A.

4.1. A model for constructive approximation in the class of \mathcal{R} -continuous operators. When F is an \mathcal{R} -continuous operator we prove the existence of an approximating \mathcal{R} -continuous operator S that is *stable* to small disturbances. The operator S defines a model of the real system and is constructed from an algebra of elementary continuous functions by a process of finite arithmetic.

²The space X possesses the Grothendieck property of approximation if there is a sequence $\{G_m\}_{m \in \mathbb{N}} \subseteq \mathcal{L}(X, X_m)$ where $X_m \subseteq X$ is a subspace of dimension m and the operators G_m are equi-continuous on compacta and uniformly convergent to unit operators on those compacta.

Theorem 4.3. *Let A and B be Banach spaces with the Grothendieck property of approximation, and let $X_{\mathcal{R}} = (X, A, T, \mathcal{M})$ and $Y_{\mathcal{R}} = (Y, B, T, \mathcal{N})$ be \mathcal{R} -spaces. Suppose that $(X_{\mathcal{R}}, Y_{\mathcal{R}})$ is a complete \mathcal{R} -pair and that \mathcal{N} is pointwise normally extreme on Y . Let $K \subseteq X$ be a compact set, and let the map $F : K \mapsto Y$ be an \mathcal{R} -continuous operator. Then for any fixed real numbers $\delta > 0$ and $\tau > 0$ there exists an associated \mathcal{R} -continuous operator S defined by finite arithmetic in the form $S = WZQG : X \mapsto Y$ such that for all $x \in K$ and $x' \in X$ with $\|x - x'\| \leq \delta$ we have $\|F(x) - S(x')\| \leq \frac{1}{2}\omega_{\mathcal{R}}(2\delta) + \tau$.*

Proof. We recall from Lemma 3.5 that the auxiliary mapping $f : \mathcal{M}[K] \mapsto B$ is uniformly continuous. We will construct a mapping $\sigma : A \rightarrow B$ in the form $\sigma = \pi\nu\lambda\theta$ where $A_m \subseteq A$ is a subspace of dimension m and $B_n \subseteq B$ is a subspace of dimension n , and where $\theta \in \mathcal{L}(A, A_m)$ and $\lambda \in \mathcal{L}(A_m, \mathbb{R}^m)$, where $\nu : \mathbb{R}^m \mapsto \mathbb{R}^n$ is continuous and where $\pi \in \mathcal{L}(\mathbb{R}^n, B_n)$. By Theorem 4.2 there exists a continuous mapping $\sigma : A \mapsto B$ in the above form such that for all $w \in \mathcal{M}[K]$ and all w' with $\|w - w'\| < \delta$ we have $\|f(w) - \sigma(w')\| \leq \frac{1}{2}\omega_{\mathcal{R}}(2\delta) + \tau$, where we have used the fact that the modulus of continuity of f satisfies $\omega_f(\alpha) = \omega_{\mathcal{R}}(\alpha)$ for all $\alpha \in \mathbb{R}_+$. Now define $S : X \mapsto Y$ by setting $N_t[Sx] = \sigma(M_t[x])$ for each $x \in X$ and each $t \in T$. Our indirect definition assumes that if $N_t[y] \in B$ is known for all $t \in T$, then $y \in Y$ is also known. We will follow our earlier notation and write $y = \mathcal{K}(\mathcal{N}[y])$ where $\mathcal{K} : \mathcal{Y} \mapsto Y$ is the appropriate archival function. The mapping $\sigma : A \mapsto B$ is continuous and hence $S : X \mapsto Y$ is an \mathcal{R} -continuous operator. Since $\|M_t[x - x']\| \leq \|x - x'\|$, it follows that $\|N_t[Fx - Sx']\| = \|f(M_t[x]) - \sigma(M_t[x'])\| < \frac{1}{2}\omega_{\mathcal{R}}(2\delta) + \tau$ for all $t \in T$ whenever $x \in K$ and $\|x - x'\| < \delta$. But we can choose $t = t_{[F(x) - S(x')]} \in T$ such that $\|N_t[Fx - Sx']\| = \|F(x) - S(x')\|$ and so $\|F(x) - S(x')\| < \frac{1}{2}\omega_{\mathcal{R}}(2\delta) + \tau$ whenever $x \in K$ and $\|x - x'\| < \delta$. Since we defined $N_t[Sx] = \pi\nu\lambda\theta M_t[x]$ we can now write $\mathcal{N}[Sx] = \pi\nu\lambda\theta\mathcal{M}[x]$ or, equivalently, $S(x) = \mathcal{K}\pi\nu\lambda\theta\mathcal{H}^{-1}(x)$ for each $x \in X$. Note that $\|\mathcal{H}^{-1}\| \leq 1$ and that $\|\mathcal{K}\| = 1$. If we define $G = \theta\mathcal{H}^{-1}$, $Q = \lambda$, $Z = \nu$ and $W = \mathcal{K}\pi$, then we can see that S has the desired form. We assume that G and W can be defined by finite arithmetic or replaced by suitable approximations. \square

Lemma 4.4. *Let $K \subseteq X$ be a compact set. Then for each $\epsilon > 0$ we can find $\delta > 0$ such that $\|M_s[x] - M_t[x]\| < \epsilon$ for all $x \in K$ whenever $s, t \in T$ and $\rho(s, t) < \delta$.*

Theorem 4.5. *Let A and B be Banach spaces with the Grothendieck property of approximation. Let $X_{\mathcal{R}} = (X, A, T, \mathcal{M})$ and $Y_{\mathcal{R}} = (Y, B, T, \mathcal{N})$ be \mathcal{R} -spaces and suppose that $(X_{\mathcal{R}}, Y_{\mathcal{R}})$ is an incomplete \mathcal{R} -pair and that \mathcal{N} is pointwise normally extreme on Y . Let $K \subseteq X$ be a compact set and let the map $F : K \mapsto Y$ be continuous and \mathcal{R} -continuous. Then for any fixed real numbers $\delta > 0$ and $\tau > 0$ there exists an associated operator $S : X \mapsto Y$ defined by $N_t[Su] = \sum_{j=1}^N \psi_j(t)N_t[S_ju]$ where $\psi_j : T \mapsto \mathbb{R}$ for each $j = 1, 2, \dots, N$ and $\{\psi_1, \dots, \psi_N\}$ is a partition of unity and where $S_j = W_jZ_jQ_jG_j : X \mapsto Y$ for each $j = 1, 2, \dots, N$ and each $u \in K$ and $t \in T$. The mapping S is continuous and \mathcal{R} -continuous and is defined by a process of finite arithmetic in such a way that for all $x \in K$ and $x' \in X$ with $\|x - x'\| \leq \delta$ we have $\|F(x) - S(x')\| \leq \frac{1}{2}\omega_{\mathcal{R}}(2\delta) + \tau$.*

Proof. Let $t \in T$ and consider the auxiliary mappings $f_t : M_t[K] \mapsto B$ and the associated moduli of continuity $\omega[f_t] : \mathbb{R}_+ \mapsto \mathbb{R}_+$. We recall from Lemmas 2.14 and 3.6 that the families $\{f_t\}_{t \in T}$ and $\{\omega[f_t]\}_{t \in T}$ are each uniformly equi-continuous. Hence, for the given $\tau > 0$, it is possible to choose $\epsilon = \epsilon(\tau) > 0$ so small that

$\lambda \leq \delta + \epsilon \Rightarrow \omega_{\mathcal{R}}(2\lambda) \leq \omega_{\mathcal{R}}(2\delta) + \tau$ and $\|M_r[u] - M_s[v]\| < \epsilon \Rightarrow \|N_r[Fu] - N_s[Fv]\| < \tau/12$ whenever $(r, s) \in E$ and $u, v \in K$. Since K and $F(K)$ are both compact, we can use Lemma 4.4 to find $\gamma > 0$ so that both $\|M_s[x] - M_t[x]\| < \epsilon$ and $\|N_s[Fx] - N_t[Fx]\| < \tau/4$ for all $x \in K$ when $\rho(s, t) < \gamma$. Choose a γ -net $\{t_1, \dots, t_N\} \subseteq T$ such that whenever $t \in T$ we can always find some $j = j(t)$ with $\|t - t_j\| < \gamma$ and let $\{\psi_1(t), \dots, \psi_N(t)\}$, where $\psi_j : T \mapsto \mathbb{R}$ for each $j = 1, 2, \dots, N$, be a partition of unity on T such that $\psi_1, \dots, \psi_N \in C(T)$, $\psi_j(t) \geq 0$ for all $t \in T$, $\sum_{j=1}^N \psi_j(t) = 1$ for all $t \in T$, and $\psi_j(t) = 0$ whenever $\rho(t, t_j) \geq \gamma$. Let $x \in K$ and choose $u \in X$ with $\|u - x\| \leq \delta$. If $\rho(t, t_j) < \gamma$, then $\|M_t[u] - M_{t_j}[x]\| \leq \|M_t[u - x]\| + \|M_t[x] - M_{t_j}[x]\| \leq \|u - x\| + \epsilon = \lambda \leq \delta + \epsilon$. By applying Theorem 4.2 we can define a function $\sigma_j : A \rightarrow B$ in the form $\sigma_j = \pi_j \nu_j \lambda_j \theta_j$ such that for all $w \in M_{t_j}[K]$ and w' with $\|w' - w\| < \lambda$ we have $\|f_j(w) - \sigma_j(w')\| < \frac{1}{2}\omega[f_j](2\lambda) + \frac{\tau}{4}$. Define $S_j : X \rightarrow Y$ by setting $N_t[S_j u] = \sigma_j(M_t[u])$ and $S : X \mapsto Y$ by the formula $N_t[S u] = \sum_{j=1}^N \psi_j(t) \sigma_j(M_t[u])$ for all $u \in X$ and $t \in T$. Now for $x \in K$, $u \in X$ with $\|x - u\| < \delta$ and all $t \in T$ we have

$$\|N_t[Fx] - N_t[S u]\| = \left\| \sum_{\rho(t, t_j) < r} \psi_j(t) [N_t[Fx] - \sigma_j(M_t[u])] \right\|.$$

We make two observations. Firstly, for $\rho(t, t_j) < r$ we have

$$\begin{aligned} \|N_t[Fx] - \sigma_j(M_t[u])\| &\leq \|N_t[Fx] - N_{t_j}[Fx]\| + \|N_{t_j}[Fx] - \sigma_j(M_t[u])\| \\ &\leq \|f_j(M_{t_j}[x]) - \sigma_j(M_t u)\| + \frac{\tau}{4}. \end{aligned}$$

Secondly, since $\|M_{t_j}[x] - M_t[u]\| \leq \lambda$, it follows that $\|f_j(M_{t_j}[x]) - \sigma_j(M_t[u])\| \leq \frac{1}{2}\omega_{\mathcal{R}}(2\lambda) + \frac{\tau}{4}$. The desired result can now be established. \square

APPENDIX A. PROOF OF THEOREM 4.2

It is well known that any separable Banach space is isometric and isomorphic to a subspace of the space $C([0, 1])$ of continuous functions on the interval $[0, 1]$. Thus, without loss of generality, we assume $X = Y = C([0, 1])$. Define $\varphi : K \times [0, 1] \rightarrow \mathbb{R}$ by setting $\varphi(x, t) = F[x](t)$ for all $t \in [0, 1]$. Fix $\delta > 0$ and $t \in [0, 1]$. For each $u \in K_\delta = \{u \mid \|u - x\| \leq \delta \text{ for some } x \in K\}$ choose $x^+[u] = x_{\delta,t}^+[u]$, $x^-[u] = x_{\delta,t}^-[u] \in K$ so that

$$\varphi_\delta^+(u, t) = \varphi(x^+[u], t) = \max_{x \in K, \|x-u\| \leq \delta} \varphi(x, t)$$

and

$$\varphi_\delta^-(u, t) = \varphi(x^-[u], t) = \min_{x \in K, \|x-u\| \leq \delta} \varphi(x, t)$$

and set $\varphi_\delta(u, t) = \frac{1}{2}[\varphi_\delta^+(u, t) + \varphi_\delta^-(u, t)]$. Define $F_\delta : K_\delta \rightarrow C([0, 1])$ by setting $F_\delta[u](t) = \varphi_\delta(u, t)$ for all $\delta > 0$ and each $t \in [0, 1]$. If $u \in K_\delta$ and $x \in K$ with $\|u - x\| \leq \delta$, then $|\varphi(x, t) - \varphi_\delta(u, t)| \leq \omega(2\delta)/2$ for all $t \in [0, 1]$, and hence it follows that $\|F(x) - F_\delta(u)\| \leq \frac{1}{2}\omega(2\delta)$. However, F_δ may not be continuous. Therefore for fixed $t \in [0, 1]$ and each pair of positive real numbers λ and μ we define

$$\varphi_{\lambda,\mu}(u, t) = \frac{1}{2\mu} \int_{[\lambda, \lambda+\mu]} [\varphi_\xi^+(u, t) + \varphi_\xi^-(u, t)] d\xi$$

and $F_{\lambda,\mu} : K_\lambda \rightarrow C([0, 1])$ by setting $F_{\lambda,\mu}[u](t) = \varphi_{\lambda,\mu}(u, t)$ for all $t \in [0, 1]$. If $\|u - v\| < \rho$, then it can be shown that

$$\|F_{\lambda,\mu}[u] - F_{\lambda,\mu}[v]\| \leq \frac{2\rho F_K}{\mu}$$

where $F_K = \max_{x \in K} \|F(x)\|$. This shows that the operator $F_{\lambda,\mu}$ is continuous. If $x \in K$ and $\|x - u\| < \lambda$, then it follows that $\|F(x) - F_{\lambda,\mu}(u)\| \leq \frac{1}{2}\omega(2\nu)$ where $\nu = \lambda + \mu$. To prove the desired result we take $\tau > 0$ and choose $\epsilon > 0$ so that $\omega(2\delta + \epsilon) \leq \omega(2\delta) + \tau$ for all $\delta > 0$. Now we set $\lambda = \delta + \epsilon/2$ and $\mu = \epsilon/2$ and note that if $\|x - u\| \leq \lambda$, then $\|F(x) - F_{\lambda,\mu}(u)\| \leq \frac{1}{2}\omega(2\delta) + \frac{\tau}{2}$. Let $0 = t_0 < \dots < t_N = 1$ be a partition of the interval $[0, 1]$, and define the operator $P_N \in \mathcal{L}(C([0, 1]), PL([0, 1]))$, where $PL([0, 1]) \subseteq C([0, 1])$ is the subspace of piecewise linear functions defined by setting $P_N[x](t_k) = x(t_k)$ for each $k = 0, \dots, N$ with the partition sufficiently fine to ensure that $\|x - P_N(x)\| \leq \epsilon/4$ for all $x \in K$. Let L_δ denote the closure of the set $P_N(K_\delta)$. Since L_δ lies in an $(N + 1)$ -dimensional subspace and is bounded and closed, it follows that L_δ is compact. It can be shown that $L_\delta \subseteq K_\lambda$, and hence $F_{\lambda,\mu}$ is well defined on L_δ . By Theorem 4.1 for all $v \in L_\delta$ there exists an operator $S_{\lambda,\mu} : X \rightarrow C(T)$ in the form $S_{\lambda,\mu} = WZQG_m^*$ such that $\|F_{\lambda,\mu}(v) - S_{\lambda,\mu}(v)\| \leq \frac{\tau}{2}$. We can now define the operator $S : X \rightarrow C(T)$ in the form $S = WZQG_m$, where $G_m = G_m^*P_N$, by the equality $S(u) = S_{\lambda,\mu}(P_N[u])$ for each $u \in K_\delta$. \square

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