

POLAR DECOMPOSITION OF ORDER BOUNDED DISJOINTNESS PRESERVING OPERATORS

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ABSTRACT. We constructively prove (i.e., in ZF set theory) a decomposition theorem for certain order bounded disjointness preserving operators between any two Riesz spaces, real or complex, in terms of the absolute value of another order bounded disjointness preserving operator. In this way, we constructively generalize results by Abramovich, Arensen and Kitover (1992), Grobler and Huijsmans (1997), Hart (1985), Kutateladze, and Meyer-Nieberg (1991).

1. INTRODUCTION

Polar decomposition theorems are a classical staple in analysis. However, for the theory of Riesz spaces (also called vector lattices) polar decomposition theorems for operators are a relative novelty. Indeed, from its inception the theory of Riesz spaces has mainly been focused on the real numbers rather than on the complex numbers. From that perspective it is surprising that in 1997 Grobler and Huijsmans proved the following polar decomposition theorem for order bounded disjointness preserving operators on complex Riesz spaces (Theorem 8 in [8]). For convenience we have slightly changed their notation: in our terminology we call E a complex Riesz space if it is the complexification of a uniformly complete Riesz space.

Theorem 1 (Grobler-Huijsmans). *Let E and F be complex Riesz spaces and suppose that F has the Z -extension property for principal ideals. Let $T : E \rightarrow F$ be an order bounded disjointness preserving operator. Then there exists an injective orthomorphism S on the ideal generated by the range of T in F such that $T = U|T|$. Moreover, $|U| = I$.*

Several years before, in 1992, Abramovich, Arensen and Kitover proved (see Theorem 3.3 in [2]) the following decomposition result for order bounded disjointness preserving operators, preceding their theorem with the comment that it is valid for complex Riesz spaces as well. (In their theorem the Dedekind completion of an Archimedean Riesz space F is denoted by \hat{F} , and we have slightly paraphrased the last part of the conclusion for $|U|$.)

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Theorem 2 (Abramovich-Arenson-Kitover). *Let E and F be Archimedean Riesz spaces, and let $T : E \rightarrow F$ be an order bounded disjointness preserving operator. Then T admits a unique decomposition $T = UT_1$, where $T_1 : E \rightarrow F$ is a lattice homomorphism, $U \in Z(\hat{F})$ and $|U|$ equals the projection onto the band generated by $T(E)$ in \hat{F} .*

Whereas, in contrast with the first theorem above, the theorem by Abramovich, Arenson and Kitover has no restrictions on the range space F and offers a larger domain for the operator U ; it has the disadvantage that its proof in [2] depends on the representation of Riesz spaces as vector spaces of extended continuous functions. In fact, Theorem 3.3 in [2] uses the Stone space of the Boolean algebra of bands in F in the formulation of the less involved part of its result and such framing cannot be proved in Zermelo-Fraenkel set theory, which is exactly our reason for paraphrasing the last part of its conclusion for $|U|$ in the way we did. However, Theorem 2 above is, if one accepts the Axiom of Choice, trivially equivalent to the last part of Theorem 3.3 in [2]. It is exactly representation theorems that use the Axiom of Choice which Grobler and Huijsmans in their paper [8] explicitly and successfully avoid. However, in turn, [8] does not entertain the possibility of a constructively valid Theorem 2, which is a trivial reformulation of the Abramovich-Arenson-Kitover result. In this paper we will combine the best of both theorems above: we will remove the extra condition on F from the result by Grobler and Huijsmans, following their path in avoiding representation theorems that involve the Axiom of Choice, as well as prove Theorem 2 constructively. To explain how we will go about this, a few words about the constructive aspects of [8] are in order. It refers to Zaanen's program, i.e., the desire to not unnecessarily use representation theorems for Riesz spaces in order to not unnecessarily use the Axiom of Choice. The paper [8] falls into that Zaanen program. An alternative approach to Zaanen's program was outlined in [5] and is based on the idea that sufficiently elementary results only involve countably many elements of the Riesz spaces under consideration. In this alternative way, one bypasses the very representation theorems that Grobler and Huijsmans wish to avoid, while one reasons in so-called small Riesz spaces only. Thus we derive Theorem 2 above in a constructive way, underscoring once again that if the existence of a mathematical object is at stake (polar decomposition in this case) no Axiom of Choice is needed when uniqueness is assured. Amongst the results in [5], all of which are valid in Zermelo-Fraenkel set theory, we find Theorem 4.1, an extension theorem for orthomorphisms, which is one of the main ingredients of our approach here to polar decomposition theorems for order bounded disjointness preserving operators. Where the two theorems above prove a relationship between an order bounded disjointness preserving operator $T : E \rightarrow F$ and its absolute value $|T|$, we show a similar relation for certain operators S for which $S(E)$ is somewhat related to $|T|(E)$ and we study the related question of which operators S can be decomposed as a product WT , where W is an orthomorphism. Thus, the innovations in this paper lie in the novelties of the latter (though already Theorem 6.9 in [3], again by Abramovich, Arenson and Kitover, but then without a constructive proof, precluded our results) but foremost in the constructively valid generalization of the Grobler-Huijsmans polar decomposition theorem, which constructively implies the generality of the Abramovich-Arenson-Kitover theorem above. In addition, our constructive proof of the polar decomposition theorem is considerably shorter than the proof of Theorem 1 above in [8]. Our approach also links an array of previously

unrelated results from the literature. Our main results also include generalizations of Kutateladze's theorem (Theorem 8.16 in [3]), Hart's Theorem 2.1 and Corollary 2.2 in [9], as well as the parts of Meyer-Nieberg's Corollary 3.1.19 and Theorem 3.1.20 in [13], that assume the range space to be Dedekind complete. The latter two results seem to have been at the origin of the circle of ideas surrounding the theorem by Grobler and Huijsmans, though Meyer-Nieberg employs a nonconstructive but very effective Hahn-Banach argument. For all unexplained terminology in the theory of Riesz spaces we refer to [3], [13] and [15]. All Riesz spaces in this paper are Archimedean. We refer to a result as elementary if it admits a proof in Zermelo-Fraenkel set theory. For recent results and more bibliography on disjointness preserving operators, we refer to the remarkable memoir [1] by Abramovich and Kitover. At the end of this introduction the authors thank the anonymous referee for many comments that improved the paper significantly.

2. THE POLAR DECOMPOSITION

In this section E and F are Riesz spaces, either real or complex. One has to keep in mind that for complex Riesz spaces E , this includes the assumption that E is the complexification of a uniformly complete real Riesz space. The words *uniformly complete Riesz space* are then understood to mean either a uniformly complete Riesz space over the reals or simply a Riesz space over the complex numbers. We refer the reader to the section on complex Riesz spaces and complex operators in [13] or the corresponding section in [15].

For a subset A of F we denote by $\mathcal{B}(A)$ the band generated by A in F , by $\mathcal{I}(A)$ the ideal generated by A in F , and by $\mathcal{R}(A)$ the Riesz subspace generated by A in F . Instead of $\mathcal{I}(\{z\})$ we write $\mathcal{I}(z)$ and instead of $\mathcal{B}(\{z\})$ we write $\mathcal{B}(z)$. We need the following definitions. Notice that though the first one is not the standard definition, it is the more convenient for us in this paper and it is trivially equivalent to the standard definition when $E = F$. The notation $z \perp x$ in the Riesz space E means $|z| \wedge |x| = 0$.

Definition 1. Suppose E is a Riesz subspace of F . A map $U : E \rightarrow F$ is called an orthomorphism if U is order bounded and

$$z \perp x \Rightarrow Uz \perp x$$

for all $z \in E$ and all $x \in F$.

Definition 2. For Riesz spaces E and F , a linear map $T : E \rightarrow F$ is called disjointness preserving if

$$z \perp x \Rightarrow Tz \perp Tx$$

for all $z, x \in E$.

Observe that every orthomorphism is, in particular, disjointness preserving.

It is the next result (Theorem 4.1 in [5]) that enables us to get around the nonconstructive extension argument used in Corollaries 3.1.19 and 3.1.20 of [13].

Theorem 3 (Buskes-van Rooij). *Let G be a majorizing Riesz subspace of a uniformly complete Riesz space F . An orthomorphism $G \rightarrow F$ can be extended uniquely to an orthomorphism $F \rightarrow F$.*

The previous theorem was proved for real Riesz spaces in [5] with an elementary proof, and the extension to the complex case is straightforward. An elementary

proof of the next proposition in the complex case follows immediately from its real version which is contained in Corollary 3 of [6] (for the case of orthomorphisms see Lemma 3.3 (ii) in [5]). It can also be found as the first part of Theorem 3.3 in [2].

Proposition 1. *If E is a Riesz subspace of F and $T : E \rightarrow F$ is an order bounded disjointness preserving operator, then*

$$|z| \leq |x| \Rightarrow |Tz| \leq |Tx|$$

for all $z, x \in E$.

The following fundamental result, for the real case as well as the complex case, was first proved by Meyer (see [11] and [12]). An elementary proof for the complex case was provided in [8]. In the real case elementary proofs can also be found in [4] and [6]. Meyer's theorem is in many ways the starting point of this paper.

Theorem 4 (Meyer). *If $T : E \rightarrow F$ is an order bounded disjointness preserving operator, then $|T|$ exists and*

$$|T||z| = ||T|z| = |Tz|$$

for all $z \in E$. In particular, $|T|$ is a Riesz homomorphism.

For a disjointness preserving operator $T \in \mathcal{L}_b(E, F)$, we define

$$\mathcal{Z}_T = \{S \in \mathcal{L}_b(E, F) : S = U|T| \text{ for some } U \in \text{Orth}(\mathcal{I}(TE))\}.$$

An immediate consequence to Meyer's Theorem is the following lemma.

From here on E and F are Riesz spaces, either real or complex, and $T : E \rightarrow F$ is an order bounded disjointness preserving operator.

Lemma 1. *The following assertions hold:*

- (i) $|T|(E)$ is a Riesz subspace of F and $|T|(E) \subset \mathcal{R}(T(E))$. Furthermore, $\mathcal{I}(T(E)) = \mathcal{I}(|T|(E))$.
- (ii) \mathcal{Z}_T is a Riesz space under the ordering of $\mathcal{L}_b(E, F)$ and for every $S = U|T|$ in \mathcal{Z}_T we have $|S| = |U||T|$.

Easy examples show that the Riesz subspace generated by $T(E)$ in F may not be equal to $|T|(E)$ in spite of (i) above. We now present our central result.

Theorem 5. *Let $S \in \mathcal{L}_b(E, F)$ such that $Sz \in \mathcal{B}(Tz)$ for all $z \in E$. If G is a Riesz subspace of F and G contains $S(E)$, then there exists a unique orthomorphism $U : |T|(E) \rightarrow G$ for which $S = U|T|$.*

Proof. Let $S \in \mathcal{L}_b(E, F)$ for which $Sz \in \mathcal{B}(Tz) \subset F$ for all $z \in E$. We first prove that

$$(*) \quad |T|z = |T|x \Rightarrow Sz = Sx.$$

Indeed, if $|T|z = |T|x$, then, by Meyer's theorem,

$$0 = ||T|(z - x)| = |T|(z - x)|$$

and hence $T(z - x) = 0$. But then $Sz = Sx$. Thus we can define a map $U : |T|(E) \rightarrow F$ by

$$U|T|z = Sz \text{ for all } z \in E.$$

The remainder of the proof is divided into 3 steps.

Step 1. Obviously, U is linear.

Step 2. Suppose that $\|T|z| \wedge |x| = 0$ for $z \in E$ and $x \in F$. Then (once more by Meyer's theorem) since $S(z) \in \mathcal{B}(Tz)$, we find that $U|T|z \perp |x|$. Thus U is an orthomorphism $|T|(E) \rightarrow F$ if we can prove that U is order bounded.

Step 3. U is order bounded. Let $a \in E$. If $\|T|z| \leq \|T|a|$, then (because $|T|$ is a Riesz homomorphism) $|T||z| \leq |T||a|$, thus $|T||z| = |T|(|z| \wedge |a|)$. Now choose $b \in F$ such that $|Sy| \leq b$ for all $|y| \leq |a|$. Consequently, from (*) it follows that

$$|S|z|| = |S(|z| \wedge |a|)| \leq b.$$

From the conditions in the theorem, S is disjointness preserving as well. We then have that $|Sz| \leq b$. Hence $|U(|T|z)| \leq b$ and thus $U : |T|(E) \rightarrow F$ is order bounded. Therefore, $U : |T|(E) \rightarrow F$ is an orthomorphism, and we find from Proposition 5 that

$$\|T|z| \leq \|T|a| \text{ implies } |U|T|z| \leq |U|T|a| = |Sa| \in G.$$

The conclusion follows. \square

As a corollary we obtain Theorem 2 of the introduction.

Strengthening the condition on S and additionally assuming that F is uniformly complete yields the following.

Theorem 6. *Let $S \in \mathcal{L}_b(E, F)$ such that $S(E) \subset \mathcal{I}(T(E))$ for all $z \in E$. If, in addition, F is uniformly complete, then there exists a unique orthomorphism $U \in \text{Orth}(\mathcal{I}(T(E)))$ for which $S = U|T|$. Consequently, \mathcal{Z}_T is Riesz isomorphic to $\text{Orth}(\mathcal{I}(T(E)))$.*

Proof. Since $S(E) \subset \mathcal{I}(T(E))$, it follows from Theorem 5 that there exists a unique orthomorphism $U : |T|(E) \rightarrow \mathcal{I}(T(E))$ for which $S = U|T|$. Then by (i) of Lemma 1 and Theorem 3, U can be extended uniquely to an element of $\text{Orth}(\mathcal{I}(T(E)))$.

It is easy to show that \mathcal{Z}_T is Riesz isomorphic to $\text{Orth}(\mathcal{I}(T(E)))$. \square

The argument in the proof above, based on the extension Theorem 3, can be replaced with a Hahn-Banach argument if one does not care about avoiding the Axiom of Choice. There is a special case of the preceding theorem that warrants a separate statement (see Theorem 6.9 in [3]).

Proposition 2 (Abramovich-Arenson-Kitover). *Let $S \in \mathcal{L}_b(E, F)$ such that $|Sz| \leq |Tz|$ for all $z \in E$. If F is uniformly complete, then there exists a unique orthomorphism $U \in \text{Orth}(\mathcal{I}(T(E)))$ with $|U| \leq I$ for which $S = U|T|$.*

Proof. From the preceding theorem, $S = U|T|$ for a unique $U \in \text{Orth}(\mathcal{I}(T(E)))$. From Lemma 1, $|S| = |U||T|$ and hence $|U| \leq I$ (first on $|T|(E)$, but then on all of $\mathcal{I}(T(E))$). \square

A corollary is the polar decomposition theorem for T itself.

Theorem 7 (Polar decomposition theorem). *If F is uniformly complete, then there exists a unique bijective $U \in \text{Orth}(\mathcal{I}(T(E)))$ for which $T = U|T|$. Moreover, $|U| = I$.*

Proof. From Theorem 5, $T = U|T|$ for a unique $U \in \text{Orth}(\mathcal{I}(T(E)))$. Hence, $|U|(|T|z) = |U(|T|z)| = |Tz| = |T|z$ for all $z \in E^+$ and consequently

$$|U||T|z = |T|z$$

for all $z \in |T|(E)$. But then $|U| = I$ on all of $\mathcal{I}(|T|(E))$ from Theorem 3 by the uniqueness of the extension of $|U|$ from $|T|(E)$ to $\mathcal{I}(T(E))$. The bijectivity of U follows. \square

As a corollary we now obtain Theorem 2 of the introduction (recall that \hat{F} denotes the Dedekind completion of F).

Theorem 8 (Abramovich, Arenson, Kitover). *T admits a unique decomposition $T = UT_1$, where $T_1 : E \rightarrow F$ is a lattice homomorphism, $U \in Z(\hat{F})$ and $|U|$ equals the projection onto the band generated by $T(E)$ in \hat{F} .*

Proof. $T : E \rightarrow F$ can and in this proof will be considered as an order bounded disjointness preserving operator $E \rightarrow \hat{F}$. By applying the previous theorem we find a unique $V \in \text{Orth}(\mathcal{I}(T(E)))$ for which $S = V|T|$ and $|V| = I$. Since $\mathcal{I}(T(E))$ is an ideal in F , the formula

$$g \rightarrow \sup\{V^+(f) : 0 \leq f \leq g, f \in \mathcal{I}(T(E))\}$$

($g \in E^+$) defines an additive and positively homogeneous map, hence extends uniquely to a linear map, namely, that linear map in $Z(\hat{F})$, named U^+ . Extending V^- similarly, the map $U := U^+ - U^-$ is the desired element of $Z(\hat{F})$. Alternatively, one can use the Hahn-Banach Theorem 8.15 in [3]. In fact, Wickstead in [14] proved that any orthomorphism in the centre of any ideal of a Riesz space E can be extended to an orthomorphism on all of E if and only if E is Dedekind complete. \square

Two other corollaries of Theorem 7 follow.

Corollary 1. *If F is uniformly complete, then there exists a unique bijective $V \in \text{Orth}(\mathcal{I}(T(E)))$ for which $|T| = VT$, where V is the inverse of U in $\text{Orth}(\mathcal{I}(T(E)))$ given by Theorem 6 and $V = \text{Re}U - i \text{Im}U$.*

Proof. In the real case one observes that $U^2 = I$, hence $U^{-1} = U$. The complex case is equally easy. \square

Corollary 2. *If F is uniformly complete and $S \in \mathcal{L}_b(E, F)$ such that $S(E) \subset \mathcal{I}(T(E))$, then there exists a unique $W \in \text{Orth}(\mathcal{I}(T(E)))$ for which $S = WT$.*

Proof. By Theorem 5 there exists $U \in \text{Orth}(\mathcal{I}(T(E)))$ for which $S = U|T|$. Now take V as in the preceding corollary, and one obtains $S = UV|T|$. Define $W := UV$ in $\text{Orth}(\mathcal{I}(T(E)))$. Then $S = WT$. \square

As an immediate application of the corollary above, we obtain the following generalization of Kutateladze's theorem (see Theorem 8.16 in [3]). That generalization also follows from Theorem 4.2 in [5] (with an elementary proof) and Theorems 5.4 and 6.9 in [3] with proofs that use the Axiom of Choice. It is theorems like the latter three that are very close in spirit and content to the main results of the present paper.

Corollary 3 (Kutateladze). *Suppose F is uniformly complete, $T : E \rightarrow F$ is a Riesz homomorphism, and $S \in \mathcal{L}_b(E, F)$ such that $0 \leq S \leq T$. Then there exists a unique $W \in \text{Orth}(\mathcal{I}(T(E)))$ for which $S = WT$.*

For an application to a result by Hart, we need the following variation of Theorem 5.

Theorem 9. *Let $S \in \mathcal{L}_b(E, F)$ for which $Sz \in \mathcal{B}(Tz)$ for all $z \in E$ and $S(E) \subset \mathcal{R}(T(E))$. Then there exists a unique $U \in \text{Orth}(\mathcal{R}(T(E)))$ for which $S = U|T|$. If one takes $S = T$, then, in addition, $|U| = I$.*

Proof. Consider S and T as operators from E to the uniform completion F^{ru} of F . Of course, $Sz \in \mathcal{B}(Tz)$ for all $z \in E$ and $S(E) \subset \mathcal{R}(T(E)) \subset \mathcal{I}(T(E))$, where— with a slight abuse of notation—the calligraphic letters represent a band, a Riesz space, and an ideal, respectively, in F^{ru} . Then we can find an orthomorphism $U \in \text{Orth}(\mathcal{I}(T(E)))$ (i.e., the space of orthomorphisms on the ideal generated by $T(E)$ in F^{ru}) such that $S = U|T|$. But it is easy to see, from the fact that every element in $\mathcal{R}(T(E))$ is of the form $\bigwedge_{i \in I} \bigvee_{j \in J} x_{ij}$, where I and J are finite and $x_{ij} \in T(E)$ (see 2.2.11 in [10]), that U maps $\mathcal{R}(T(E))$ into $\mathcal{R}(T(E))$. \square

As an application, we turn to [9] with a proof of Hart's Theorem 2.1 therein.

Theorem 10 (Hart). *For every $U \in \text{Orth}(E)$ there exists $\tilde{U} \in \text{Orth}(\mathcal{R}(T(E)))$ for which $\tilde{U}T = TU$.*

Proof. First we observe from Meyer's theorem that $T(\mathcal{B}(z)) \subset \mathcal{B}(Tz)$ and then $TUz \in \mathcal{B}(Tz)$ for all $z \in E$. Define $S = TU$. Hence $Sz \in \mathcal{B}(Tz)$ for all $z \in E$ and $S(E) \subset \mathcal{R}(T(E))$. Thus by Theorem 5, we can find an orthomorphism $W \in \text{Orth}(\mathcal{R}(T(E)))$ such that $W|T| = TU$. By applying Theorem 5 again, we find an invertible $V \in \text{Orth}(\mathcal{R}(T(E)))$ for which $T = V|T|$. Then $WV^{-1}T = TU$. Take $\tilde{U} = WV^{-1}$. \square

Notice that our proof circumvented the extension Lemma 1.5 in [9] via Theorem 3.

If, on the other hand, one prefers to consider orthomorphisms on larger Riesz subspaces, then one can go beyond Theorem 6 (and if one so wishes in a similar way in Proposition 2) by applying our next theorem, without requiring more than uniform completeness of F . The details follow from Lemma 20.1 in [7].

Theorem 11. *If F is uniformly complete, then there exists a unique bijective $U \in \text{Orth}(\mathcal{I}(T(E))^-)$ for which $T = U|T|$, where $\mathcal{I}(T(E))^-$ is the uniform closure of $\mathcal{I}(T(E))$ in F . In addition, $|U| = I$.*

REFERENCES

1. Y. A. Abramovich and A. K. Kitover, *Inverses of Disjointness Preserving Operators*, *Memoirs Amer. Math. Soc.*, 143 (2000), no. 679. MR **2000k**:47047
2. Y. A. Abramovich, E. L. Arensen and A. K. Kitover, *Banach $C(K)$ -modules and Operators Preserving Disjointness*, *Pitman Research Notes in Mathematics Series*, 277, Longman Scientific & Technical, Harlow, 1992. MR **94d**:47027
3. C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, *Pure and Applied Math.*, Vol. 119, Academic Press, Orlando, FL, 1985. MR **87h**:47086
4. S. J. Bernau, *Orthomorphisms of Archimedean vector lattices*, *Math. Proc. Cambridge Philos. Soc.*, 89 (1981), 119-128. MR **82b**:47043
5. G. Buskes and A. van Rooij, *Small Riesz spaces*, *Math. Proc. Cambridge Philos. Soc.*, 105, (1989), 523-536. MR **90e**:46009
6. B. de Pagter, *A note on disjointness preserving operators*, *Proc. Amer. Math. Soc.*, 90 (1984), 543-549. MR **85e**:47057
7. B. de Pagter, *f-Algebras and Orthomorphisms*, Ph.D. Dissertation, Leiden, 1981.
8. J. J. Grobler and C. B. Huijsmans, *Disjointness preserving operators on complex Riesz spaces*, *Positivity*, 1 (1997), 155-164. MR **99k**:47079

9. D. R. Hart, *Some properties of disjointness preserving operators*, Nederl. Akad. Wetensch. Indag. Math., 47 (1985), 183-197. MR **87e**:47045
10. G. Jameson, *Ordered Linear Spaces*, Lecture Notes in Math 141, Springer-Verlag, Berlin-Heidelberg-New York, 1970. MR **55**:10996
11. M. Meyer, *Les homomorphismes d'espaces vectoriels réticulés complexes*, C. R. Acad. Sci. Paris, Sér. I Math., t. 292, (1981), 793-796. MR **82f**:47051
12. M. Meyer, *Le stabilisateur d'un espace vectoriel réticulé*, C. R. Acad. Sci. Paris, Sér. I Math., t. 283, (1976), 249-250. MR **55**:6170
13. P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, Heidelberg, New York, 1991. MR **93f**:46025
14. A. W. Wickstead, *Extensions of orthomorphisms*, J. Austral. Math. Soc. Ser. A 29 (1980), 87-98. MR **81b**:47049
15. A. C. Zaanen, *Riesz Spaces II*, North Holland, Amsterdam-New York-Oxford, 1983. MR **86b**:46001

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