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OPERATORS WITH EIGENVALUES AND EXTREME CASES OF STABILITY

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Abstract. In the following, we consider some cases where the point spectrum of an operator is either very stable or very unstable with respect to small perturbations of the operator. The main result is about the shift operator on l_2, whose point spectrum is what we will call strongly stable. We also give some general perturbation results, including a result about the size of the set of operators that have an eigenvalue.

1. Introduction

For a Banach space X, let $\mathcal{L}(X)$ denote the space of bounded linear operators from X to itself. In [4], Kato gives an example of an operator that has the open unit disk as its point spectrum, but which can be perturbed in an arbitrarily small way so that the perturbed operator has no eigenvalues. In particular, this example shows that the set of operators in $\mathcal{L}(X)$ that have an eigenvalue is not always an open set. However, we offer the following proof that this set of operators is a large set in the Baire sense.

First, we need to recall the following definitions and well-known lemmas.

Definition 1.1. An operator T with closed range is called **semi-Fredholm** if either $\mathrm{nul}(T)$ is finite or $\mathrm{def}(T)$ is finite, where $\mathrm{nul}(T)$ is the dimension of the kernel of T and $\mathrm{def}(T)$ is the co-dimension of the range of T.

Definition 1.2. The **essential spectrum** $\sigma_e(T)$ of an operator T is the subset of the spectrum of T consisting of all λ so that $\lambda - T$ is not semi-Fredholm.

Lemma 1.3 ([4])**.** *The essential spectrum is preserved under a compact perturbation.*

Lemma 1.4 ([4])**.** *Any boundary point of the resolvent of T belongs to $\sigma_e(T)$ unless it is an isolated point of the spectrum of T.*

Theorem 1.5. *Let X be a Banach space. The set of operators in $\mathcal{L}(X)$ with an eigenvalue contains an open dense set.*

Proof. It is well known, [4], that isolated eigenvalues having finite-dimensional eigenspaces change continuously with the operator. Thus, the set of operators with

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such eigenvalues is open, and our task is to show that this set is dense in $\mathcal{L}(X)$. Let T be an arbitrary operator and $\varepsilon > 0$. Because $\sigma(T)$ is compact, there exists $\lambda' \in \delta(\sigma(T))$ (the boundary of $\sigma(T)$) with $|\lambda'| \geq |\beta|$ for all $\beta \in \sigma(T)$. Let $\lambda \in \rho(T)$ with $|\lambda| > |\lambda'|$ and $|\lambda - \lambda'| < \varepsilon/2$. Now, being in $\delta(\sigma(T))$, λ' is an almost eigenvalue of T, [3]. Let (x_n) be a sequence of almost eigenvectors for λ' (so $\|x_n\| = 1$ for all n and $\|Tx_n - \lambda' x_n\| \to 0$). Let k be such that $\|Tx_k - \lambda' x_k\| < \varepsilon/2$, and let $v = \lambda x_k - Tx_k$ (note that $v \neq 0$ as $\lambda \in \rho(T)$). We now define $U : [x_k] \to [v]$ by $U(x_k) = v$ (where $[x_k]$ is the one-dimensional subspace containing x_k, etc.). Then

$$\begin{aligned}\|U\| = \|v\| = \|\lambda x_k - Tx_k\| &= \|\lambda x_k - \lambda' x_k + \lambda' x_k - Tx_k\| \\ &\leq |\lambda - \lambda'| + \|\lambda' x_k - Tx_k\| < \varepsilon.\end{aligned}$$

Because $[x_k]$ is one dimensional, there is a norm-one projection $P : X \longrightarrow [x_k]$. We now define $S = UP$, and note that $\|S\| < \varepsilon$. Then $(T+S)(x_k) = Tx_k + \lambda x_k - Tx_k = \lambda x_k$, so that $T + S$ has λ as an eigenvalue.

We now show that λ is isolated. Since S is of finite rank, it is a compact operator, and since the essential spectrum, $\sigma_e(T)$, is unchanged by a compact perturbation (Lemma 1.3), $\sigma_e(T) = \sigma_e(T + S)$. Now, suppose λ is not isolated. Then by Lemma 1.4, λ cannot be in $\delta(\rho(T+S))$, or else λ would be in $\sigma_e(T+S)$, which cannot be since $\sigma_e(T + S) = \sigma_e(T)$ and $\lambda \in \rho(T)$. Because λ is not in $\delta(\rho(T + S))$, there is a disk of positive radius with center λ and contained in $\sigma(T + S)$. Let $r = \inf\{t > 1 \mid t\lambda \in \delta(\rho(T + S))\}$. Then $r > 1$ and $r\lambda$ is a non-isolated point of $\sigma(T+S)$ which is also in $\delta(\rho(T+S))$. Therefore, again by Lemma 1.4, $r\lambda \in \sigma_e(T + S)$. This means then that $r\lambda \in \sigma_e(T)$, and since $|r\lambda| > |\lambda| > |\lambda'|$, we have a contradiction because λ' is of maximal modulus in $\sigma(T)$. Hence, λ is isolated.

Finally, it is easily seen that λ has a one-dimensional eigenspace, namely $[x_k]$. Indeed, if $y = w + x_k$, where $Pw = 0$, then $(T + S)(y) = Tw + \lambda x_k$. However, $\lambda y = \lambda w + \lambda x_k$, so $(T + S)y = \lambda y$ implies that $Tw = \lambda w$. Because $\lambda \in \rho(T)$, it must be that $w = 0$. Therefore, the set of operators that have an isolated eigenvalue with finite eigenspace is an open dense subset of $\mathcal{L}(X)$. □

Our investigation will now focus on a condition under which the stability of eigenvalues is realized. Although a simple condition, we will see that for an operator T and $\lambda \in \sigma(T)$, the surjectivity or non-surjectivity of the operator $\lambda - T$ will provide us with fairly strong results. We first give two well-known results and the relevant definitions.

Lemma 1.6 ([1]). *Let X and Y be Banach spaces. The surjective linear operators are an open set in $\mathcal{L}(X, Y)$.*

Definition 1.7. The **index** of a semi-Fredholm operator T is defined as $\mathbf{ind}(T) = \mathrm{nul}(T) - \mathrm{def}(T)$.

Theorem 1.8 ([2]). *Let T be a semi-Fredholm operator. Then there is an $\varepsilon > 0$ so that $\|T - S\| < \varepsilon \Rightarrow \mathrm{ind}(T) = \mathrm{ind}(S)$.*

Proposition 1.9. *Let T be a semi-Fredholm operator from a Banach space to itself and $\lambda \in \sigma_p(T)$ with $\lambda - T$ surjective. Then there is an $\varepsilon > 0$ so that*

$$\|T - S\| < \varepsilon \Rightarrow B_\varepsilon(\lambda) \subseteq \sigma_p(S).$$

Proof. Let T and λ be as stated above. By Lemma 1.6, there is an $\varepsilon_1 > 0$ such that $\|\lambda - T - S'\| < \varepsilon_1 \Rightarrow S'$ is onto. By Theorem 1.8, there is an $\varepsilon_2 > 0$ such that

$$\|\lambda - T - S'\| < \varepsilon_2 \Rightarrow \operatorname{ind}(S') = \operatorname{ind}(\lambda - T).$$

Now, with $\varepsilon_3 = \min(\varepsilon_1, \varepsilon_2)$, we choose $\varepsilon = \varepsilon_3/2$. Suppose that we have an operator S with $\|T - S\| < \varepsilon$, and suppose further that β satisfies $|\beta - \lambda| < \varepsilon$. Then

$$\|\lambda - T - (\beta - S)\| \leq |\lambda - \beta| + \|T - S\| < \frac{\varepsilon_3}{2} + \frac{\varepsilon_3}{2} = \varepsilon_3.$$

Hence, $\beta - S$ is onto and $\operatorname{ind}(\beta - S) = \operatorname{ind}(\lambda - T)$. Since $\beta - S$ and $\lambda - T$ are onto, we have

$$1 \leq \operatorname{nul}(\lambda - T) = \operatorname{ind}(\lambda - T) = \operatorname{ind}(\beta - S) = \operatorname{nul}(\beta - S).$$

Therefore, β is an eigenvalue of S. □

By taking $S = T$ in Proposition 1.9, we obtain the following two corollaries:

Corollary 1.10. *Let T be a semi-Fredholm operator and λ an eigenvalue of T with $\lambda - T$ surjective. Then, in fact, T has an entire disk (of positive radius) of eigenvalues centered at λ.*

Corollary 1.11. *Let T be any operator and λ be an isolated eigenvalue. Then either T is not semi-Fredholm or $\lambda - T$ is not surjective.*

In the Introduction, we mentioned an example by Kato of an operator whose entire point spectrum (which is the open unit disk) could be destroyed by an arbitrarily small perturbation. The operator in this example is a left shift T from $l_p(\mathbb{Z})$ to $l_p(\mathbb{Z})$ and given by $T(\ldots, x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \ldots) = (\ldots, x_{-}1, 0, \underline{x_1}, x_2, x_3, \ldots)$, where we have underlined the 0th coordinate. If the one-dimensional operator A is defined by $A(..., x_{-2}, x_{-1}, \underline{x_0}, x_1, x_2, \ldots) = (\ldots, 0, \varepsilon x_0, \underline{0}, 0, 0, \ldots)$, then it can be shown that $T + A$ has an empty point spectrum. The following theorem is simply a generalization of that example. For convenience, we introduce the following terminology:

Definition 1.12. For a collection of eigenvalues A of an operator T, we define $E_A \equiv \{x \mid Tx = \lambda x \text{ for some } \lambda \in A\}$.

Theorem 1.13. *Let T be an operator from a Banach space X to itself, and let A be a collection of eigenvalues of T so that $X/\bigcup_{\lambda \in A} \mathcal{R}(\lambda - T)$ is not empty. If there exists a vector $y \in X$ such that $P_y(x) \neq 0$ for each $x \in E_A$, where P_y is a norm-one projection onto $[y]$, then there exists a one-dimensional operator S of arbitrarily small norm so that $T + S$ has no eigenvalues in A.*

Proof. Let $\varepsilon > 0$. Let y be the vector from the hypothesis, and we assume without loss of generality that $\|y\| = 1$. Let $\|v\| = 1$ with $v \in X/\bigcup_{\lambda \in A} \mathcal{R}(\lambda - T)$, and define

$$P : [y] \longrightarrow [v]$$

by $P(y) = \varepsilon v$. We now claim that $S = PP_y$ is the operator we desire. Suppose that $Tx + Sx = \lambda x$ for some $\lambda \in A$ and some nonzero x in X. Then $Sx = \lambda x - Tx$. Now, it cannot be that $Sx \neq 0$. Otherwise, v would be in the range of $\lambda - T$. So we assume that $Sx = 0$. Hence, x is an eigenvector for λ. But then $P_y(x) = \alpha y$ for some $\alpha \neq 0$. Therefore, $Sx = P(\alpha y) = \varepsilon \alpha v \neq 0$, again a contradiction. Noting that $\|S\| = \varepsilon$, the proof is complete. □

2. The shift operator

We now turn our attention to the study of a particular operator, namely the left shift $L : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ (from now on we will simply write l_2 for $l_2(\mathbb{N})$). It is easily seen that L has the open unit disk of eigenvalues, and we will see that this operator enjoys a great deal of stability with regard to its eigenvalues. However, the main result, Theorem 2.3, is a stronger result than would be obtained from the previous general results. We introduce some terminology below to describe this kind of stability, and give a preliminary lemma.

Definition 2.1. A subset M of $\sigma(T)$ is **strongly stable** if, given any r with $0 < r < 1$, there is an $\varepsilon > 0$ so that $\|T - S\| < \varepsilon$ implies that $rM \subset \sigma(S)$.

Lemma 2.2. *Let $y = (y_n) \in l_2$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Define the sequence $x = (x_n)$ by $x_n = \lambda^n y_1 + \lambda^{n-1} y_2 + \cdots + \lambda y_n$. Then the linear map satisfying $T_\lambda y = x$ is in $\mathcal{L}(l_2)$ and $\|T_\lambda\| \le \sqrt{\sum_{m=2}^{\infty}(m-1)|\lambda|^m}$.*

Proof. For a given n, we have

$$\begin{aligned} |x_n|^2 \le (|\lambda|^n|y_1| + \cdots + |\lambda||y_n|)^2 &= \sum_{i,j=1}^{n} |\lambda|^{i+j}|y_{n-i+1}||y_{n-j+1}| \\ &\le \sum_{i,j=1}^{n} |\lambda|^{i+j}\frac{|y_{n-i+1}|^2 + |y_{n-j+1}|^2}{2} = \sum_{i,j=1}^{n} |\lambda|^{i+j}|y_{n-i+1}|^2. \end{aligned}$$

Calling this last sum a_n, we see that $\sum_{n=1}^{\infty}|x_n|^2 \le \sum_{n=1}^{\infty} a_n$, and we now show that $\sum_{n=1}^{\infty} a_n < \infty$. Letting k be fixed, we consider the coefficients of $|y_k|^2$ appearing in $\sum_{n=1}^{\infty} a_n$. First, if $n < k$, then $|y_k|^2$ does not appear in a_n. For $n \ge k$, the coefficients of $|y_k|^2$ in a_n are $|\lambda|^{i+1}, |\lambda|^{i+2}, \ldots, |\lambda|^{i+n}$ where $n - i + 1 = k$. Now consider now any positive integer m. If $n > m+k-2$, then $i+1 > m$, so that $|\lambda|^m$ is not among the coefficients of $|y_k|^2$ in a_n. Thus, $|\lambda|^m$ is a coefficient of $|y_k|^2$ in $\sum_{n=1}^{\infty} a_n$ exactly $m-1$ times, corresponding to $n = k, k+1, \ldots, m+k-2$. Hence, in fact,

$$\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty}\Big(\sum_{m=2}^{\infty}(m-1)|\lambda|^m\Big)|y_k|^2 = \Big(\sum_{m=2}^{\infty}(m-1)|\lambda|^m\Big)\|y\|^2.$$

□

We now state the main result concerning the shift operator.

Theorem 2.3. *The point spectrum of the left shift operator $L : l_2 \longrightarrow l_2$ is strongly stable.*

Proof. Let L be the left shift operator. We show that for $0 < r < 1$, there exists an $\varepsilon > 0$ so that $\|T\| < \varepsilon \Rightarrow L - T$ has the open disk of radius r of eigenvalues. Let us now fix r and let $|\lambda| < r$. For an arbitrary operator T, the perturbation of L by T, namely $L - T$, has λ as an eigenvalue provided that for some nonzero

$x = (x_n) \in l_2$,

$$
\begin{aligned}
x_2 &= \lambda x_1 + T_1 x, \\
x_3 &= \lambda^2 x_1 + \lambda T_1 x + T_2 x \\
&\vdots \\
x_j &= \lambda^{j-1} x_1 + \lambda^{j-2} T_1 x + \cdots + \lambda T_{j-2} x + T_{j-1} x \\
&\vdots
\end{aligned} \tag{1}
$$

where $T_i x$ is the ith coordinate of Tx. Now, we must first show that a sequence as defined in (1) would be in l_2, and secondly we must show the existence of an $x = (x_n)$ in l_2 that satisfies the equations in (1). But the sequence

$$
\begin{aligned}
&(x_1, \lambda x_1 + T_1 x, \lambda^2 x_1 + \lambda T_1 x + T_2 x, \dots) \\
&= x_1 (\lambda^n)_{n=0}^{\infty} + (0, 0, \lambda T_1 x, \lambda^2 T_1 x + \lambda T_2 x, \dots) + (0, T_1 x, T_2 x, \dots)
\end{aligned}
$$

is a sum of three sequences in l_2. The first is in l_2 since it is geometric, the third since $T \in \mathcal{L}(l_2)$, and the second by Lemma 2.2. We now show that such a sequence exists as long as $\|T\|$ is sufficiently small. Consider the map $l_2 \xrightarrow{S} l_2 \xrightarrow{T} l_2$ where $S(z) \equiv (x_1, \lambda x_1 + z_1, \lambda^2 x_1 + \lambda z_1 + z_2, \cdots)$. Here, $S(z)$ is formed by placing z_i where there is a $T_i x$ in (1) and where x_1 is chosen to be nonzero. To show the existence of a sequence of the desired form, we show that $T \circ S$ has a fixed point z (then $x = S(z)$ works, since then $T_i x = z_i$). But S is just an affine linear map given by $z \longrightarrow x_1 (\lambda^n)_{n=0}^{\infty} + R(z) + R^2 \circ T_\lambda(z)$ where R is the right shift and T_λ is the operator mentioned in Lemma 2.2. Thus, if $\|T\| < \frac{1}{\|R + R^2 \circ T_\lambda\|}$, then $T \circ S$ is a contraction and therefore has a fixed point, so that λ is an eigenvalue of $L - T$. Now, by Lemma 2.2, one readily obtains the inequality

$$
\frac{1}{\|R + R^2 \circ T_\lambda\|} \geq \frac{1}{1 + \sqrt{\sum_{m=2}^{\infty} (m-1)|r|^m}},
$$

for every $|\lambda| < r$. Thus, letting $\varepsilon = \frac{1}{1 + \sqrt{\sum_{m=2}^{\infty} (m-1)|r|^m}}$, we have that $\|T\| < \varepsilon \implies L - T$ has the full open disk of radius r of eigenvalues, or equivalently, that every operator in the ε-ball around L has the open r-disk of eigenvalues. $\square$

Remarks. We feel that the notion of strong stability can be investigated further, both for other operators and for other parts of the spectrum. Also, the result for the shift operator should be true for l_p, but the proof would require some involved combinatorics (for integer p) and the use of interpolation.

References

[1] Dieudonné, J., *Sur les homomorphismes d'espaces normés*, Bull. Sci. Math., Vol. 67, (1943), 72–84. MR **7:**124c

[2] Atkinson, F. V., *The Normal Solubility of Linear Equations in Normed Spaces*, Mat. Sbornik, Vol. 28, (1951), 3–14. MR **13:**46d

[3] Beauzamy, B., *Introduction to Operator Theory and Invariant Subspaces*, North-Holland Publishing Co., Amsterdam, 1988. MR **90d:**47001

[4] Kato, T., *Perturbation Theory for Linear Operators*, Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag, New York, 1966. MR **34:**3324

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