ON AN EXAMPLE OF ASPINWALL AND MORRISON

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(Communicated by Michael Stillman)

Abstract. In this paper, a family of smooth multiply-connected Calabi–Yau threefolds is investigated. The family presents a counterexample to global Torelli as conjectured by Aspinwall and Morrison.

Introduction

The aim of this paper is to prove

**Theorem 0.1.** The one-parameter family of smooth, multiply-connected Calabi–Yau threefolds \( \mathcal{Y} \to B \) over the base \( B = \mathbb{P}^1 \setminus \{1, \xi, \ldots, \xi^4, \infty\} \), constructed by Aspinwall–Morrison in \([1]\) (cf. Section \( \ref{section:construction} \), with \( \xi \) a primitive fifth root of unity, has the following properties:

- For any \( t \in B \), there exists an isomorphism
  \[ H^3(Y_t, \mathbb{Q}) \cong H^3(Y_{\xi t}, \mathbb{Q}) \]
  preserving rational polarized Hodge structures (for a stronger statement, see Theorem \( \ref{thm:isomorphism} \)).
- There is a Zariski-open set \( U \subset B \) such that for \( t \in U \), \( i = 0, \ldots, 4 \), the fibres \( Y_{\xi^i t} \) are pairwise nonisomorphic as algebraic varieties.

The family \( \mathcal{Y} \to B \) is a quotient of a family of quintics, manufactured in such a way that a certain symmetry of a cover \( \mathcal{Z} \to B \) of \( \mathcal{Y} \to B \) fails to descend in any obvious way to a symmetry of \( \mathcal{Y} \to B \). The existence of this symmetry on the cover implies the statement about Hodge structures (Theorem \( \ref{thm:isomorphism} \)). On the other hand, an isomorphism between \( Y_t \) and \( Y_{\xi t} \) for general \( t \) would force, via a specialization argument (Theorem \( \ref{thm:isomorphism} \)), the existence of an automorphism \( \sigma \) on the fibre \( Y_0 \) over \( 0 \) of a special kind. However, the automorphism group of \( Y_0 \) can be computed explicitly (Theorem \( \ref{thm:automorphism} \)), and such a \( \sigma \) does not exist. For technical reasons, the argument runs on a family of singular models \( \mathcal{Y} \to B \) of \( \mathcal{Y} \to B \). (See Section \( \ref{section:technical} \.)

Theorem \( \ref{thm:counterexample} \) establishes the fact, conjectured by Aspinwall and Morrison, that the family \( \mathcal{Y} \to B \) provides a counterexample to global Torelli for Calabi–Yau threefolds. Previous counterexamples to Torelli were given in \([13]\); there families of birationally equivalent Calabi–Yau threefolds were considered. By \( \ref{thm:counterexample} \) Theorem...
4.12], birational equivalence implies isomorphism between (rational) Hodge structures. However, in the present case the situation should be entirely different.

**Conjecture 0.2.** For general $t \in B$, the threefolds $Y_{t+i}$ for $i = 0, \ldots, 4$ are not birationally equivalent to one another.

One obvious direct approach to this conjecture is to aim to understand the various birational models of a fixed fibre $Y_t$. Birational models of minimal threefolds can be studied via their cones of nef divisors in the Picard group; so this method requires an explicit understanding of the nef cone of $Y_t$. An étale cover $Z_t$ of $Y_t$ is a toric hypersurface. A recent conjecture [3, Conjecture 6.2.8] of Cox and Katz aims at giving a complete understanding of the nef cone of toric Calabi–Yau hypersurfaces. However, it is proved in [13] that in fact the conjecture of Cox and Katz fails for $Z_t$. At this point the computation of the nef cone of $Y_t$ seems rather hopeless. A different approach to Conjecture 0.2 is required.

To conclude the Introduction, let me point out that the varieties $Y_t$ are multiply connected with fundamental group $\mathbb{Z}/5\mathbb{Z}$ (Proposition 1.5 and Proposition 1.7). This is a curious fact. The construction of Aspinwall and Morrison requires in an essential way that members of the mirror Calabi–Yau family should have a nontrivial (and in fact non-cyclic) fundamental group. Computations of Gross [7, Section 4.12], birational equivalence implies isomorphism between (rational) Hodge structures. However, in the present case the situation should be entirely different.

**Proposition 1.1.** In the contravariant description, $\mathbb{P}^4/G \cong \mathbb{P}_{M, \Delta}$, where $M \cong \mathbb{Z}^4$ and $\Delta \subset M_\mathbb{R}$ is the polyhedron

$$\Delta = \text{span}\{(1, 0, 0, 0), (-3, 5, -4, -2), (0, 0, 1, 0), (0, 0, 0, 1), (2, -5, 3, 1)\}.$$
With $N = \text{Hom}(M, \mathbb{Z})$, the dual polyhedron $\Delta^* \subset N_\mathbb{R}$ of $\Delta$ is

$$\Delta^* = \text{span}\{(−1, −2, −1, −1), (4, 1, −1, −1), (−1, −1, −1, −1),$$

$$(−1, 2, 4, −1), (−1, 0, −1, 4)\}.$$

The polyhedron $\Delta^*$ has no interior lattice points apart from the origin, has no lattice points in the interiors of its three- or one-dimensional faces, and has precisely two lattice points, $P_{2i−1}, P_{2i}$, $i = 1, \ldots, 10$, in the interiors of each of its ten two-dimensional faces.

Proof. This is a standard toric calculation; for details see [14, Proposition 1.1]. □

Let $\Sigma$ be the fan consisting of cones over faces of $\Delta^*$ in $N_\mathbb{R}$. This fan defines the toric variety $X_{N, \Sigma} = P_{M, \Delta}$.

**Proposition 1.2.** $P_{M, \Delta}$ is a $\mathbb{Q}$-factorial Gorenstein variety, with ten curves of canonical singularities. Every permutation $\eta$ of the lattice points $\{P_i\}$ gives rise to a partial resolution $X_{\Sigma, \eta} \to P_{M, \Delta}$. The varieties $X_{\Sigma, \eta}$ have isolated singularities only.

Proof. This is basic toric geometry. The curves of singularities correspond to the ten two-dimensional faces of $\Delta^*$. The singularities can be partially resolved by subdividing the fan $\Sigma$ using the lattice points $\{P_i\}$ in any order. Any permutation $\eta$ of these points gives a fan $\Sigma_{\eta}$ in the space $N_\mathbb{R}$ and a corresponding toric partial resolution $X_{\Sigma, \eta}$ with isolated singularities. □

The family of hypersurfaces of interest in this paper is constructed from

$$Q = \left\{ \sum_{i=0}^{4} z_i^5 - 5t \prod_{i=0}^{4} z_i = 0 \right\} \subset \mathbb{P}^4 \times B,$$

where $B = \mathbb{C} \setminus \{1, \xi, \ldots, \xi^4\}$. The second projection gives a smooth family $p : Q \to B$ of Calabi–Yau quintics $Q_t$. The groups $G$ and $H$ act on $\mathbb{P}^4 \times B$ by acting trivially on $B$, and hence on $Q$; these actions preserve holomorphic three-forms in the fibres. Let

$$\tilde{Z} = Q/H,$$

$$\tilde{Y} = Q/G = \tilde{Z}/K.$$

Here $K \cong \mathbb{Z}/5\mathbb{Z}$ is the group generated by the image of $g_3$ in $\text{Aut}(\tilde{Z})$. Both $\tilde{Z}$ and $\tilde{Y}$ are naturally families over $B$ with fibres $\tilde{Z}_t$ and $\tilde{Y}_t$, respectively.

**Proposition 1.3.** For $t \in B$, $\tilde{Z}_t$ is a canonical Calabi–Yau threefold with ten isolated $\frac{1}{5}(1, 1, 3)$ quotient singularities. The group $K$ acts freely on $\tilde{Z}_t$. The variety $\tilde{Y}_t$ is a canonical Calabi–Yau threefold with two isolated $\frac{1}{5}(1, 1, 3)$ quotient singularities.

Proof. Easy explicit check. □

The family $\tilde{Z} \to B$ is a family of nondegenerate anti-canonical hypersurfaces in the toric variety $P_{\Delta}$. The partial resolutions $X_{\Sigma, \eta} \to P_{M, \Delta}$ give rise to morphisms $Z_\eta \to \tilde{Z}$ over $B$, with $Z_\eta \to B$ a family of nonsingular threefolds as $X_{\Sigma, \eta}$ is nonsingular in codimension three.
Proposition 1.4. The families $Z_\eta$ are all canonically isomorphic to a unique toric resolution $Z \rightarrow \tilde{Z}$ over $B$. For $t \in B$, the fibre $Z_t$ is a smooth Calabi–Yau threefold with Hodge numbers $h^{1,1}(Z_t) = 21$, $h^{2,1}(Z_t) = 1$. In the resolution $Z_t \rightarrow \tilde{Z}_t$ there are two exceptional divisors over every singular point $S_t$, a Hirzebruch surface $E_t \cong \mathbb{P}_3$ and a projective plane $F_t \cong \mathbb{P}^2$ intersecting in a $\mathbb{P}^1$ which is the negative section in the Hirzebruch surface and a line in $\mathbb{P}^2$.

Proof. Let $\eta_1$, $\eta_2$ be two permutations of the interior lattice points. There is a corresponding birational map $X_{\Sigma_{\eta_1}} \rightarrow X_{\Sigma_{\eta_2}}$ whose exceptional sets are disjoint from the families $Z_{\eta_i}$. This implies the first part. The other statements follow from easy toric calculations. \qed

Proposition 1.5. The action of the group $K \cong \mathbb{Z}/5\mathbb{Z}$ on $\tilde{Z}$ extends to a free action on the resolution $Z \rightarrow B$. Thus there is an étale cover $Z \rightarrow Y = \tilde{Z}/K$ over $B$. The fibre $Y_t$ for $t \in B$ is a Calabi–Yau resolution of $Y_t$ with Hodge numbers $h^{1,1}(Y_t) = 5$, $h^{2,1}(Y_t) = 1$.

Proof. The action of $K$ is generated by the symmetry $g_3$ of $\mathbb{P}^4$. This symmetry descends to the toric variety $\mathbb{P}_\Delta$ as a toric symmetry induced by a lattice isomorphism $\alpha_3 : M \rightarrow M$ fixing the polyhedron $\Delta$ and permuting the lattice points $\{P_i\}$. Composition with the permutation induced by $\alpha_3$ gives a correspondence $\eta \rightarrow \eta'$ between permutations of the set $\{P_i\}$, and $\alpha_3$ gives rise to an isomorphism $\tilde{g}_3 : X_{\Sigma_{\eta}} \rightarrow X_{\Sigma_{\eta'}}$. This isomorphism restricts to anti-canonical families as an isomorphism $Z_{\eta} \rightarrow Z_{\eta'}$, or, by Proposition 1.4 as an automorphism $Z \rightarrow Z$. By construction, this automorphism is the required extension of $g_3$ and it clearly generates a free group action on $Z$ over $B$.

We conclude this section by proving two auxiliary statements.

Proposition 1.6. The family $\tilde{Y} \rightarrow B$ restricted to a neighbourhood of $0 \in B$ is the universal deformation space of its central fibre $\tilde{Y}_0$ in the analytic category.

Proof. By general theory, the projective variety $\tilde{Y}_0$ has a versal deformation space $\mathcal{X} \rightarrow S$ in the analytic category. $\tilde{Y}_0$ is a canonical Calabi–Yau threefold. Thus $H^0(\tilde{Y}_0, T_{\tilde{Y}_0}) = 0$ and this implies that $\mathcal{X} \rightarrow S$ is in fact universal. By Ran’s extension [12] of the Bogomolov–Tian–Todorov theorem, unobstructedness holds for $\tilde{Y}_0$. Thus $S$ is smooth. Further, the codimension of the singularities of $\tilde{Y}_0$ is three. By the argument of [3 A.4.2], it follows that the first-order tangent space of $S$ at the base point is isomorphic to $H^1(\tilde{Y}_0, T_{\tilde{Y}_0})$, a one-dimensional complex vector space.

In order to prove that $\tilde{Y} \rightarrow B$ is the universal deformation space, all we need to show is that its Kodaira–Spencer map is injective. Recall the family $Q \rightarrow B$, a deformation of the Fermat quintic $Q_0$ over $B$. Choosing a ($G$-invariant) three-form on $Q_0$ gives rise to a commutative diagram:

\[
\begin{array}{ccc}
T_0(B) & \xrightarrow{k} & H^1(Q_0, T_{Q_0}) \\
\| & & \sim \\
T_0(B) & \xrightarrow{l} & H^1(\tilde{Y}_0, T_{\tilde{Y}_0}) \sim H^1(\tilde{Y}_0, \Omega^2_{\tilde{Y}_0})
\end{array}
\]

Here $k$ and $l$ are the Kodaira–Spencer maps, whereas the map $j$ is given by pullback of (orbifold) two-forms (the sheaf of orbifold two-forms $\Omega^2_{\tilde{Y}_0}$ is defined carefully in [3]
A.3]). The map \(k\) is injective, as \(Q\) is a nontrivial first-order deformation of \(Q_0\). By commutativity, \(l\) is also injective. This proves the proposition. \(\Box\)

**Proposition 1.7.** For \(t \in B\), the Calabi–Yau manifold \(Z_t\) is simply connected.

**Proof.** The variety \(Z_t\) is a resolution of the threefold \(\tilde{Z}_t = Q_t/H\). Let \(Q_t^0\) be the open set of \(Q_t\) on which the action of \(H\) is free; it is the complement of a finite set of points and hence is simply connected. Let \(Z_t^0 = Q_t^0/H; \pi_1(Z_t^0) \cong H\).

The fundamental group of \(Z_t\) is a quotient group of \(H\). Let \(T_t\) be the universal cover of \(Z_t\); by the generalized Riemann existence theorem, \(T_t\) is an algebraic variety and it clearly has trivial canonical bundle. Let \(T_t^0\) be the preimage of \(Z_t^0\) under the covering map. Then \(T_t^0\) has finite fundamental group; let \(\tilde{T}_t^0\) be its universal cover. \(\tilde{T}_t^0\) is an algebraic variety again. Notice, however, that \(Q_t^0, T_t^0\) are both universal covers of the variety \(Z_t^0\), and thus by the uniqueness part of the generalized Riemann existence theorem they must be isomorphic. Thus there exists a diagram:

\[
\begin{array}{ccc}
Q_t & \supset & Q_t^0 \\
\downarrow & & \downarrow \\
T_t^0 & \subset & T_t \\
\downarrow & & \downarrow \\
\tilde{Z}_t & \supset & \tilde{Z}_t^0 \subset Z_t
\end{array}
\]

The covering \(Q_t^0 \rightarrow T_t^0\) corresponds to a group \(L\) of holomorphic automorphisms of \(Q_t^0\). An automorphism of \(Q_t^0\) can be thought of as a birational self-map of \(Q_t\). However, as \(Q_t\) is a minimal Calabi–Yau threefold with Picard number one, it has no birational self-maps with a nontrivial exceptional locus. So \(L\) consists of automorphisms of \(Q_t\). The fact that the map \(Q_t^0 \rightarrow T_t^0\) factors the map \(Q_t^0 \rightarrow Z_t^0\) implies that \(L\) must be a subgroup of \(H\).

Thus we conclude that \(T_t\) is birational to a quotient \(Q_t/L\) for a subgroup \(L\) of \(H\). Moreover, \(\chi(Z_t) = 40\), so \(\chi(T_t)\) equals either 40, 200 or 1000. On the other hand, for every subgroup \(L\) of \(H\), the quotient \(Q_t/L\) has a Calabi–Yau desingularization. As the Euler number is a birational invariant of smooth Calabi–Yau threefolds, the Euler number of this desingularization must be equal to that of \(T_t\). Finally, it is easy to check that \(H\) has no subgroup \(L\) such that a Calabi–Yau desingularization of \(Q_t/L\) has Euler number 200 or 1000. Thus \(L = H\) and so \(T_t = Z_t\) is its own universal cover.\(\Box\)

2. Hodge structures

Let \(Z, Y\) denote the differentiable manifolds underlying the fibres \(Z_t, Y_t\). Let \(V_Z = H^3(Z, \mathbb{Z})_{\text{free}}, V_Y = H^3(Y, \mathbb{Z})_{\text{free}}\), with antisymmetric pairings \(Q_Z, Q_Y\) given by cup product.

**Lemma 2.1.** Pullback by the map \(\pi: Z \rightarrow Y\) induces an injection

\[
\pi^*: V_Y \hookrightarrow V_Z
\]

with image of index at most 25. Under this embedding,

\[
Q_Z(\pi^* \alpha_1, \pi^* \alpha_2) = 5 Q_Y(\alpha_1, \alpha_2).
\]

Consequently, there is an embedding of groups

\[
\text{Aut}_Z(V_Z, Q_Z) \subset \text{Aut}_Q(V_Y \otimes \mathbb{Q}, Q_Y).
\]
Proof. The group \(K \cong \mathbb{Z}/5\mathbb{Z}\) acts without fixed points on \(\mathbb{Z}\), so the map \(\pi\) induces a spectral sequence

\[ E_2^{p,q} = H^p(K; H^q(\mathbb{Z}, \mathbb{Z})) \Rightarrow H^{p+q}(Y, \mathbb{Z}). \]

The terms \(E_2^{p,q}\) for \(p > 0\) are torsion, so \(V_Y = (E_\infty^{0,3})_{\text{free}}\). On the other hand, \((E_2^{0,3})_{\text{free}} = H^0(K, H^3(\mathbb{Z}, \mathbb{Z}))_{\text{free}} = (V_Z)^K\). There are two differentials from \((E_2^{0,3})\), both having image \(\mathbb{Z} = 5\mathbb{Z}\). So there is an injection

\[ \pi^* : V_Y \hookrightarrow (V_Z)^K \]

with image of index at most 25. This map is an isomorphism when tensored by \(\mathbb{Q}\). As both \(V_Z\) and \(V_Y\) have rank four, \(K\) must act trivially on \(V_Z\) and this proves the first part. The other two statements are immediate. \(\square\)

Let \(\mathcal{D}_Y\) be the period domain parameterizing weight 3 polarized Hodge structures on \((V_Y, Q_Y)\). Fixing a point \(t \in B\), a marking \(H^3(Y_t, \mathbb{Z})_{\text{free}} \cong V_Y\) and a universal cover \(\tilde{B}\) of \(B\) leads to holomorphic period maps

\[ \tilde{B} \xrightarrow{\psi} \mathcal{D}_Y \]
\[ \downarrow \quad \downarrow \]
\[ B \xrightarrow{\psi} \mathcal{D}_Y / \Gamma \]

where \(\Gamma\) is any subgroup of \(\text{Aut}_\mathbb{Q}(V_Y \otimes \mathbb{Q}, Q_Y)\) containing all geometric monodromies and acting properly discontinuously on \(\mathcal{D}\). Choose \(\Gamma = j(\text{Aut}_\mathbb{Z}(V_Z, Q_Z)) \subset \text{Aut}_\mathbb{Q}(V_Y \otimes \mathbb{Q}, Q_Y)\) under the embedding \(j\) of Lemma 2.1.

Lemma 2.2. \(\Gamma\) acts properly discontinuously on \(\mathcal{D}_Y\), so \(\mathcal{D}_Y / \Gamma\) is an analytic space.
Proof. See [6, Section I.2]. \(\square\)

After all these preparations, we can state

Theorem 2.3. For \(\Gamma\) chosen as above, the period map \(\psi : B \rightarrow \mathcal{D}_Y / \Gamma\) is of degree at least five. More precisely, if \(t_1, t_2 \in B\) satisfy \(t_1^5 = t_2^5\), then \(\psi(t_1) = \psi(t_2)\). In particular, \(Y_{t_1}\) and \(Y_{t_2}\) have isomorphic rational Hodge structure.

Proof. The symmetry

\[ g : [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [\xi^{-1} z_0 : z_1 : z_2 : z_3 : z_4] \]

descends to a symmetry of \(\mathbb{P}^4 / H\) and maps \(Z_t\) isomorphically to \(Z_{t^5}\). By an argument analogous to the proof of Proposition 1.5, this isomorphism extends to an isomorphism \(Z_t \rightarrow Z_{t^5}\). This gives a diagram of polarized Hodge structures:

\[ H^3(Y_t, \mathbb{Z})_{\text{free}} \xrightarrow{\pi^*} H^3(Z_t, \mathbb{Z})_{\text{free}} \]
\[ \cong \]
\[ H^3(Y_{t^5}, \mathbb{Z})_{\text{free}} \xrightarrow{\pi^*} H^3(Z_{t^5}, \mathbb{Z})_{\text{free}} \]

Comparing this with the action of \(\Gamma\) on \(\mathcal{D}_Y\) defined above gives the first statement. The second statement is immediate. \(\square\)
Remark 2.4. The proof of Lemma 2.1 implies that the spectral sequence

$$E_2^{p,q} = H^p(K; H^q(Z, A)) \Rightarrow H^{p+q}(Y, A)$$

degenerates at $E_2$ whenever 5 is invertible in $A$. In particular, there is an isomorphism of polarized Hodge structures

$$H^3(Y_t, \mathbb{Z}[1/5]) \cong H^3(Y_{t_0}, \mathbb{Z}[1/5]).$$

The problem is that $\text{Aut}(V \otimes \mathbb{Z}[1/5], Q_Y)$ does not act properly discontinuously on $D_Y$, so such a statement is weaker than the one proved above. On the other hand, it seems difficult to determine the precise behavior of the spectral sequence with $\mathbb{Z}$ coefficients, i.e. to compute the torsion in the cohomology of $Y$.

Remark 2.5. The isomorphism of $\mathbb{Q}$-Hodge structures is due to Aspinwall and Morrison. They give a different proof coming from mirror symmetry which goes as follows. The mirror family $\mathcal{X}$ of $\mathcal{Y}$ is the quotient of a suitable family of quintic hypersurfaces by the group $\langle g_1, g_3 \rangle$. In particular, the antichiral ring of the central fibre $X_0$ of $\mathcal{X}$ with a choice of (complexified) Kähler class is isomorphic to the chiral ring of $Y_t$. On the other hand, the antichiral ring of $X_0$ can be shown to depend, via the mirror map, on $t^5$ only and not on $t$. Thus the varieties $Y_{t^i}$ for $i = 0, \ldots, 4$ have the same chiral ring, i.e. isomorphic rational Hodge structure.

Remark 2.6. Suppose that $Y_0$ is an $n$-fold, $G$ (a nontrivial quotient of) the fundamental group $\pi_1(Y_0)$. Then there is an étale cover $Z \to Y$; in fact there is a cover $Z_t \to Y_t$ for every deformation $Y_t$ of $Y_0$. The (primitive) cohomology $H^n_t(Z_t)$ becomes a $G$-representation, and in some cases one can recover information about $Y_t$ from the pair

$$(H^n_t(Z_t), \text{ action of } G).$$

A particular example of this construction is the theorem of Horikawa [3], giving a Torelli-type result for Enriques surfaces using global Torelli for K3s. However, by Proposition 2.1, the threefold $Z_t$ under investigation is simply connected. On the other hand, as the proofs above show, the Hodge structure on the middle-dimensional rational cohomology of the universal cover $Z_t$ contains no extra information, and it carries the trivial action of the fundamental group $\pi_1(Y_t)$.

3. THE AUTOMORPHISM GROUP OF THE CENTRAL FIBRE

Theorem 3.1. The automorphism groups of the varieties $Y_0$, $\tilde{Y}_0$ are

$$\text{Aut}(Y_0) \cong \text{Aut}(\tilde{Y}_0) \cong \langle G, g_4, g_5 \rangle / G,$$

where

$$g_4 : \left[ z_0 : z_1 : z_2 : z_3 : z_4 \right] \mapsto \left[ z_0 : z_1 : \xi z_2 : \xi z_3 : z_4 \right],$$

$$g_5 : \left[ z_0 : z_1 : z_2 : z_3 : z_4 \right] \mapsto \left[ z_0 : z_2 : z_4 : z_1 : z_3 \right].$$

In particular, every automorphism of $\tilde{Y}_0$ extends to an automorphism on all (small) deformations $\tilde{Y}_t$ of $\tilde{Y}_0$.

Proof. The proof of Theorem 3.1 uses three lemmas. The first one should certainly be well known, but a suitable reference could not be found so a proof is included.

Lemma 3.2. Let

$$X = \left\{ \sum_{i=0}^{n} x_i^d = 0 \right\} \subset \mathbb{P}_k^n$$

be the Fermat hypersurface. Assume that $d \geq 3$, $n \geq 2$ and that $(n,d) \neq (2,3)$ or $(3,4)$. Then

$$\text{Aut}(X) \cong G_{n,d},$$

where $G_{n,d}$ is the semi-direct product $\Sigma_{n+1} \rtimes (\mu_d)^n$ of a symmetric group and a power of the group of $d$-th roots of unity.

**Proof.** For $n = 2$, the result is proved in [15]. If $n \geq 3$ and $(n,d) \neq (3,4)$, then we first claim that every automorphism comes from a projective automorphism in the given embedding. If $n \geq 4$, Lefschetz implies $\text{Pic}(X) \cong \mathbb{Z}$ and then the claim is clear. If $n = 3$ and $d \neq 4$, then the canonical class is (anti-)ample and this easily implies the claim again; see [10].

Take an element $\sigma \in \text{Aut}(X)$ represented by an invertible matrix $A = (a_{ij})$. Apply $A$ to the equation of $X$ and consider the coefficients of $x_0^{d-1}x_1$, $x_0^{d-2}x_1^2$, and $x_0^d x_1^3$ for $i > 1$. Their vanishing shows that the set of numbers

$$\{a_0^{d-2}a_{01}, a_1^{d-2}a_{11}, \ldots, a_n^{d-2}a_{n1}\}$$

solves the homogeneous system of equations given by the invertible matrix $A^T$. So all these quantities are zero. By symmetry, $a_{ij}a_{ik} = 0$ whenever $j \neq k$. Hence $A$ has at most one nonzero entry in each row. Multiplying by a suitable element in $\Sigma_{n+1}$, $A$ can be brought into diagonal form, and then all its entries are $d$-th roots of unity. \[ \square \]

**Lemma 3.3.** Let $\hat{X}$ be a canonical Calabi–Yau threefold with a finite number $m \geq 2$ of isolated $\frac{\nu}{2}(1,1,3)$ quotient singularities and Picard number one. Let $\pi : X \to \hat{X}$ be the Calabi–Yau resolution. Then $\text{Aut}(X) \cong \text{Aut}(\hat{X})$.

**Proof.** The Picard group of the resolution $X$ is

$$\text{Pic}_Q(X) \cong \mathbb{Q}H \oplus \mathbb{Q}E_1 \oplus \mathbb{Q}F_1 \oplus \ldots \oplus \mathbb{Q}F_m \oplus \mathbb{Q}F_m,$$

where $H = \pi^*(\mathcal{O}_X(1))$ and $E_i$, $F_i$ are the classes of the exceptional divisors as described in Proposition [14]. The intersection numbers are as follows:

\[
\begin{align*}
H^3 &= d > 0 & \text{the degree of } \hat{X}, \\
H \cdot E_i &= H \cdot F_i = 0 & \text{as } H \text{ is a pullback}, \\
E_i \cdot E_j &= F_i \cdot F_j &= F_i \cdot F_j = 0 & \text{unless } i = j, \\
E_i^3 &= (K_{E_i})^2 = 8 & \text{as } E_i \cong \mathbb{F}_3, \\
F_i^3 &= (K_{F_i})^2 = 9 & \text{as } F_i \cong \mathbb{P}^2, \\
E_i^2 F_i &= 1, & \\
F_i^2 E_i &= -3.
\end{align*}
\]

Introducing the basis $H_0 = H$, $H_{2i-1} = E_i + \frac{1}{3}F_i$, $H_{2i} = F_i$ of $\text{Pic}_Q(X)$, the cubic form takes the shape

$$\left( \sum_{i=0}^{2m} \alpha_i H_i \right)^3 = d\alpha_0^3 + 8 \frac{1}{3} \sum_{i=1}^{m} \alpha_{2i-1}^3 + 9 \sum_{i=1}^{m} \alpha_{2i}^3.$$

Finally, the values of the second Chern class are

$$c_2(X) \cdot E_i = -4, \quad c_2(X) \cdot F_i = -6, \quad c_2(X) \cdot H = c \geq 0,$$

where the last inequality follows from a result of Miyaoka [11, Theorem 1.1].

Let $\sigma \in \text{Aut}(X)$ be an automorphism. It acts via pullback on $\text{Pic}_Q(X)$, fixing the cubic form together with the linear form given by cup product with $c_2(X)$. We claim
that the element $H_0 = H$ of $\text{Pic}_Q(X)$ must be fixed under the action. To see this, note that the cubic form has been manufactured to take the shape of the Fermat cubic. Every automorphism of $\text{Pic}_Q(X)$ must fix the associated (projectivized) hypersurface. The possible automorphisms are known from Lemma 3.2. Moreover, in the present case, the multiplications by roots of unity are excluded since $c_2$ must fix a rational vector space. The possible permutations are constrained by the fact that $c_2$ is negative on the $H_{i}$ for $i > 0$ and nonnegative on $H = H_0$, the latter is fixed and this proves the claim.

For large and divisible $m$, the divisor class $mH$ is base-point free and, since the torsion in $\text{Pic}(X)$ is finite, is the unique representative of its numerical equivalence class. As $H \in \text{Pic}_Q(X)$ is fixed by the induced action of $\sigma$, for large and divisible $m$ the space of sections of the linear system $|mH|$ is also acted on by $\sigma$. In other words, the automorphism $\sigma$ descends to the image of the associated morphism which is exactly $\tilde{X}$.

For the converse, note that the quotient singularity $\frac{1}{G}(1, 1, 3)$ has a unique crepant resolution. Hence every automorphism $\tilde{\sigma} \in \text{Aut}(\tilde{X})$ extends to a birational automorphism $\sigma \in \text{Aut}(X)$ of the resolution. The lemma follows.

Lemma 3.4. Let $X$ be a smooth algebraic variety with finite fundamental group $F$. Let $Y$ be the universal cover of $X$, a smooth algebraic variety with an action of $F$ by automorphisms. Then

$$ \text{Aut}(X) \cong N_{\text{Aut}(Y)}(F)/F. $$

Proof. Obvious.

To finish the proof of Theorem 3.1 let $Q_0$ be the open set of the Fermat quintic $Q_0$ on which the action of $G$ is free. Let $Y_0 = Q_0/G$. There is a sequence of maps

$$ \text{Aut}(\tilde{Y}_0) \hookrightarrow \text{Aut}(Y_0) \cong N_{\text{Aut}(Q_0)}(G)/G \cong N_{\text{Aut}(Q_0)}(G)/G. $$

The first isomorphism follows from Lemma 3.4. The second isomorphism uses $\text{Aut}(Q_0) \cong \text{Aut}(Q_0)$; here $\text{Aut}(Q_0) \subset \text{Aut}(Q_0)$ is proved by the argument used already in Proposition 1.7 and the other direction is clear by Lemma 3.2.

On the other hand, by Lemma 3.2 the automorphism group of $Q_0$ is the semi-direct product $G_{4,5}$ of the permutation and diagonal symmetries. Finding the normalizer of $G$ in $G_{4,5}$ is a finite search best done using a computer; a short Mathematica routine computes this normalizer to be

$$ N_{\text{Aut}(Q_0)}(G)/G \cong \langle G, g_4, g_5 \rangle /G $$

with $g_4$, $g_5$ as in the statement of Theorem 3.1. So we obtain

$$ \text{Aut}(\tilde{Y}_0) \hookrightarrow \langle G, g_4, g_5 \rangle /G $$

and it is easy to see that this is in fact an isomorphism. Finally, by Lemma 3.3

$$ \text{Aut}(\tilde{Y}_0) \cong \text{Aut}(Y_0). $$

This proves the first statement. The second statement follows by inspection: every generator of the normalizer fixes $Q_1$. \qed
4. The proof of Theorem 4.1

The proof is based on the following rather standard result, a version of which was already used in [13]:

**Theorem 4.1.** Let $X_i \to B$, $i = 1, 2$, be families of canonical Calabi–Yau varieties over a base scheme $B$, having simultaneous resolutions $\mathcal{Y}_i \to X_i$ over $B$. Let $\mathcal{L}_i$ be relatively ample relative Cartier divisors on $X_i$. Let $\text{Isom}_B(X_i, \mathcal{L}_i)$ be the functor

\[ \text{Isom}_B(X_i, \mathcal{L}_i) : \text{Schemes} \to \text{Sets} \]

defined by

\[ \text{Isom}_B(X_i, \mathcal{L}_i)(S) = \{ \text{polarized } S\text{-isomorphisms } (X_1)_S \to (X_2)_S \}, \]

where the pullback families $(X_i)_S$ are polarized by the relatively ample line bundles $(\mathcal{L}_i)_S$. This functor is represented by a scheme $\text{Isom}_B(X_i, \mathcal{L}_i)$, proper and unramified over $B$.

**Proof.** By Grothendieck’s theory of the representability of Hilbert schemes and related functors, the above functor is represented by a scheme $\text{Isom}_B(X_i, \mathcal{L}_i)$, separated and of finite type over $B$. The fact that the fibres have no infinitesimal automorphisms implies that $\text{Isom}_B(X_i, \mathcal{L}_i)$ is unramified over $B$. Properness follows from the valuative criterion along the lines of [4, Proposition 4.4]; the existence of a simultaneous resolution is needed for this final step. \(\square\)

**Theorem 4.2.** Let $\mathcal{Y} \to B$ be the family constructed in Section 1, a primitive fifth root of unity. Then there is a Zariski dense subset $U \subset B$, such that the fibres $\mathcal{Y}_t$ and $\mathcal{Y}_{t^5}$ are not isomorphic as algebraic varieties for $t \not\in U$.

**Proof.** First we work with the singular family $\mathcal{Y}$; for ease of notation, let $\mathcal{Y}_1 = \mathcal{Y}$. Fixing an ample divisor $L$ on $\mathbb{P}_\Delta/K$ gives by restriction a relatively ample divisor $\mathcal{L}$ on $\mathcal{Y}_1$. Let $\mathcal{L}_1 = \mathcal{L}^{\otimes 5}$.

Let $\gamma : B \to B$ be the map of the base which is multiplication by $\xi^{-1}$. Let $\mathcal{Y}_2 \to B$ denote the pullback of $\mathcal{Y}_1 \to B$ by $\gamma$. The family $\mathcal{Y}_2 \to B$ is equipped with the relatively ample line bundle $\mathcal{L}_2 = \gamma^*(\mathcal{L}_1)$ and its fibre over $t \in B$ is $\mathcal{Y}_{t^5}$.

**Lemma 4.3.** Let $t \in B$, and let $\mathcal{Y}_{1,t}$ and $\mathcal{Y}_{2,t}$ be the fibres of the two families polarized by the ample divisors $L_{1,t}$ and $L_{2,t}$. Then every isomorphism

\[ \varphi : \mathcal{Y}_{1,t} \to \mathcal{Y}_{2,t} \]

satisfies $\varphi^*(L_{2,t}) \sim L_{1,t}$.

**Proof.** The fibres have Picard number one, and multiplication by five annihilates every torsion element in their Picard groups. So the divisors $L_{1,t}$ are canonical elements of the respective Picard groups. The lemma follows. \(\square\)

Continuing the proof of Theorem 4.2, consider the relative isomorphism scheme

\[ \text{Isom} = \text{Isom}_B(\mathcal{Y}, \mathcal{L}) \]

together with the natural map $\text{Isom} \to B$. By Theorem 4.1 this map is proper, so its image $V$ is a closed subvariety of the quasi-projective variety $B$.

Assume first that $V = B$. Then $\text{Isom}$ has a component $\mathbf{I}$ with a surjective unramified map onto a Zariski neighbourhood of $0 \in B$. Now switch to the complex topology; let $\Delta$ be a disc in $\mathbf{I}$ mapping isomorphically onto a neighbourhood of
Consider the pullback families $\tilde{Y}_i, \to \Delta$. By the definition of $I$, these families are isomorphic under an isomorphism $\varphi$ over $\Delta$.

Consider the composition

$$\tilde{Y}_1, \to \beta \tilde{Y}_2, \to (\gamma^{-1})^* \tilde{Y}_1, \Delta.$$

Its restriction to the central fibre $\tilde{Y}_0$ is a polarized automorphism $\sigma$.

By Proposition 1.6 $\tilde{Y}_1 \to \Delta$ is the universal deformation space of $\tilde{Y}_0$ in the analytic category. The automorphism $\sigma$ acts on the base of the deformation space by universality. This action equals the composite of the actions of $\varphi$ and $$(\gamma^{-1})^*$$ on the base $\Delta$. However, $\varphi$ is an isomorphism over $\Delta$, so the action of $\sigma$ on $\Delta$ is multiplication by a primitive fifth root of unity, i.e. a rotation of the disc.

On the other hand, by Theorem 3.1, the action of every automorphism of $\tilde{Y}_0$ on the base of the universal deformation space is trivial. Thus $\sigma$ cannot exist. So the assumption $V = B$ leads to a contradiction.

Thus $V$ is a proper closed subset of $B$. Let $U = B \setminus V$, a Zariski open subset of $B$. Over $t \in U$ the scheme $\text{Isom}$ has no points. Using Lemma 4.3 this implies that for $t \in U$ there cannot exist any isomorphism between $\tilde{Y}_1$ and $\tilde{Y}_\xi$.

Finally, if $\tilde{Y}_i \cong \tilde{Y}_\xi$ for some $t \in B$, then an argument analogous to the proof of Lemma 3.3 shows that the singular Calabi–Yau models $\tilde{Y}_i, \tilde{Y}_\xi$ are also isomorphic. This concludes the proof of Theorem 4.2.

Applying this theorem for $\xi^i, i = 1, \ldots, 4$, and taking the intersection of the resulting open sets concludes the proof of Theorem 0.1 announced in the Introduction.

Remark 4.4. Theorem 0.1 is also argued for in the paper [1]. Aspinwall and Morrison write down a power series in the coordinate $t$ of the base $B$, following [2], related to higher genus Gromov–Witten invariants of the family mirror family $X'$. This series is a function of $t$ rather than $t^5$, and this is a strong indication of the validity of Theorem 0.1. As a matter of fact, I believe that this is also an indication of the validity of Conjecture 0.2. However, a solid mathematical definition, let alone computation, of this power series has not been given to date.

Acknowledgments

I thank Pelham Wilson, Mark Gross and Peter Newstead for comments and help.

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