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ON AN EXAMPLE OF ASPINWALL AND MORRISON

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ABSTRACT. In this paper, a family of smooth multiply-connected Calabi–Yau threefolds is investigated. The family presents a counterexample to global Torelli as conjectured by Aspinwall and Morrison.

INTRODUCTION

The aim of this paper is to prove

Theorem 0.1. *The one-parameter family of smooth, multiply-connected Calabi–Yau threefolds $\mathcal{Y} \to B$ over the base $B = \mathbb{P}^1 \setminus \{1, \xi, \dots, \xi^4, \infty\}$, constructed by Aspinwall–Morrison in* [1] *(cf. Section 1), with ξ a primitive fifth root of unity, has the following properties:*

- *For any $t \in B$, there exists an isomorphism*
$$H^3(Y_t, \mathbb{Q}) \cong H^3(Y_{\xi t}, \mathbb{Q})$$
preserving rational polarized Hodge structures (for a stronger statement, see Theorem 2.3).
- *There is a Zariski-open set $U \subset B$ such that for $t \in U$, $i = 0, \dots, 4$, the fibres $Y_{\xi^i t}$ are pairwise nonisomorphic as algebraic varieties.*

The family $\mathcal{Y} \to B$ is a quotient of a family of quintics, manufactured in such a way that a certain symmetry of a cover $\mathcal{Z} \to B$ of $\mathcal{Y} \to B$ fails to descend in any obvious way to a symmetry of $\mathcal{Y} \to B$. The existence of this symmetry on the cover implies the statement about Hodge structures (Theorem 2.3). On the other hand, an isomorphism between Y_t and $Y_{\xi t}$ for general t would force, via a specialization argument (Theorem 4.2), the existence of an automorphism σ on the fibre Y_0 over 0 of a special kind. However, the automorphism group of Y_0 can be computed explicitly (Theorem 3.1), and such a σ does not exist. For technical reasons, the argument runs on a family of singular models $\bar{\mathcal{Y}} \to B$ of $\mathcal{Y} \to B$. (See Section 4.)

Theorem 0.1 establishes the fact, conjectured by Aspinwall and Morrison, that the family $\mathcal{Y} \to B$ provides a counterexample to global Torelli for Calabi–Yau threefolds. Previous counterexamples to Torelli were given in [13]; there families of birationally equivalent Calabi–Yau threefolds were considered. By [9, Theorem

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4.12], birational equivalence implies isomorphism between (rational) Hodge structures. However, in the present case the situation should be entirely different.

Conjecture 0.2. *For general $t \in B$, the threefolds $Y_{\xi^i t}$ for $i = 0, \dots, 4$ are not birationally equivalent to one another.*

One obvious direct approach to this conjecture is to aim to understand the various birational models of a fixed fibre Y_t. Birational models of minimal threefolds can be studied via their cones of nef divisors in the Picard group; so this method requires an explicit understanding of the nef cone of Y_t. An étale cover Z_t of Y_t is a toric hypersurface. A recent conjecture [3, Conjecture 6.2.8] of Cox and Katz aims at giving a complete understanding of the nef cone of toric Calabi–Yau hypersurfaces. However, it is proved in [14] that in fact the conjecture of Cox and Katz fails for Z_t. At this point the computation of the nef cone of Y_t seems rather hopeless. A different approach to Conjecture 0.2 is required.

To conclude the Introduction, let me point out that the varieties Y_t are multiply connected with fundamental group $\mathbb{Z}/5\mathbb{Z}$ (Proposition 1.5 and Proposition 1.7). This is a curious fact. The construction of Aspinwall and Morrison requires in an essential way that members of the mirror Calabi–Yau family should have a nontrivial (and in fact non-cyclic) fundamental group. Computations of Gross [7, Section 3] connect torsion in the integral cohomologies of mirror Calabi–Yau threefolds, and these computations imply that the cohomology (and hence homology) of Y_t should have torsion of some kind. However, the direct relationship between failure of Torelli and the fundamental group seems rather mysterious; compare also Remark 2.6.

Notation and conventions. All schemes and varieties are defined over $\mathbb{C}$. A *Calabi–Yau threefold* is a normal projective threefold X with canonical Gorenstein singularities satisfying $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$. Some statements use the language of toric geometry; my notation follows Fulton [5] and Cox–Katz [3, Chapter 3]. If A is a $\mathbb{Z}$-module, then A_{free} denotes the torsion-free part.

1. The construction

Following [1], define maps $g_i : \mathbb{P}^4 \to \mathbb{P}^4$ by

$$\begin{aligned} g_1 &: [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_0 : \xi z_1 : \xi^2 z_2 : \xi^3 z_3 : \xi^4 z_4], \\ g_2 &: [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_0 : \xi z_1 : \xi^3 z_2 : \xi z_3 : z_4], \\ g_3 &: [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [z_1 : z_2 : z_3 : z_4 : z_0] \end{aligned}$$

where ξ is a fixed primitive fifth root of unity. Let

$$G = \langle g_1, g_2, g_3 \rangle, \ H = \langle g_1, g_2 \rangle$$

be subgroups of $PGL(5, \mathbb{C})$ generated by the transformations g_i. As abstract groups $H \cong (\mathbb{Z}/5\mathbb{Z})^2$, $G \cong \mathbb{Z}/5\mathbb{Z} \ltimes (\mathbb{Z}/5\mathbb{Z})^2$.

We will be interested in hypersurfaces in the varieties $\mathbb{P}^4/G$ and $\mathbb{P}^4/H$; the latter is a toric variety and its toric description will be useful in the sequel.

Proposition 1.1. *In the contravariant description, $\mathbb{P}^4/H \cong \mathbb{P}_{M,\Delta}$, where $M \cong \mathbb{Z}^4$ and $\Delta \subset M_{\mathbb{R}}$ is the polyhedron*

$$\Delta = \text{span}\{(1,0,0,0), (-3,5,-4,-2), (0,0,1,0), (0,0,0,1), (2,-5,3,1)\}.$$

With $N = \mathrm{Hom}(M, \mathbb{Z})$, the dual polyhedron $\Delta^* \subset N_{\mathbb{R}}$ of Δ is

$$\Delta^* = \mathrm{span}\{(-1,-2,-1,-1),(4,1,-1,-1),(-1,-1,-1,-1),\\ (-1,2,4,-1),(-1,0,-1,4)\}.$$

The polyhedron Δ^* has no interior lattice points apart from the origin, has no lattice points in the interiors of its three- or one-dimensional faces, and has precisely two lattice points, P_{2i-1}, P_{2i}, $i = 1, \ldots, 10$, in the interiors of each of its ten two-dimensional faces.

Proof. This is a standard toric calculation; for details see [14, Proposition 1.1]. □

Let Σ be the fan consisting of cones over faces of Δ^* in $N_{\mathbb{R}}$. This fan defines the toric variety $\mathbb{X}_{N,\Sigma} \cong \mathbb{P}_{M,\Delta}$.

Proposition 1.2. *$\mathbb{P}_{M,\Delta}$ is a $\mathbb{Q}$-factorial Gorenstein variety, with ten curves of canonical singularities. Every permutation η of the lattice points $\{P_i\}$ gives rise to a partial resolution $\mathbb{X}_{\Sigma_\eta} \to \mathbb{P}_{M,\Delta}$. The varieties $\mathbb{X}_{\Sigma_\eta}$ have isolated singularities only.*

Proof. This is basic toric geometry. The curves of singularities correspond to the ten two-dimensional faces of Δ^*. The singularities can be partially resolved by subdividing the fan Σ using the lattice points $\{P_i\}$ in any order. Any permutation η of these points gives a fan Σ_η in the space $N_{\mathbb{R}}$ and a corresponding toric partial resolution $\mathbb{X}_{\Sigma_\eta}$ with isolated singularities. □

The family of hypersurfaces of interest in this paper is constructed from

$$\mathcal{Q} = \left\{ \sum_{i=0}^{4} z_i^5 - 5t \prod_{i=0}^{4} z_i = 0 \right\} \subset \mathbb{P}^4 \times B,$$

where $B = \mathbb{C} \setminus \{1, \xi, \ldots, \xi^4\}$. The second projection gives a smooth family $p : \mathcal{Q} \to B$ of Calabi–Yau quintics Q_t. The groups G and H act on $\mathbb{P}^4 \times B$ by acting trivially on B, and hence on $\mathcal{Q}$; these actions preserve holomorphic three-forms in the fibres. Let

$$\begin{array}{rcccl} \bar{\mathcal{Z}} & = & \mathcal{Q}/H, & & \\ \bar{\mathcal{Y}} & = & \mathcal{Q}/G & = & \bar{\mathcal{Z}}/K. \end{array}$$

Here $K \cong \mathbb{Z}/5\mathbb{Z}$ is the group generated by the image of g_3 in $\mathrm{Aut}(\bar{\mathcal{Z}})$. Both $\bar{\mathcal{Z}}$ and $\bar{\mathcal{Y}}$ are naturally families over B with fibres $\bar{Z}_t$ and $\bar{Y}_t$, respectively.

Proposition 1.3. *For $t \in B$, $\bar{Z}_t$ is a canonical Calabi–Yau threefold with ten isolated $\frac{1}{5}(1,1,3)$ quotient singularities. The group K acts freely on $\bar{Z}_t$. The variety $\bar{Y}_t$ is a canonical Calabi–Yau threefold with two isolated $\frac{1}{5}(1,1,3)$ quotient singularities.*

Proof. Easy explicit check. □

The family $\bar{\mathcal{Z}} \to B$ is a family of nondegenerate anti-canonical hypersurfaces in the toric variety $\mathbb{P}_\Delta$. The partial resolutions $\mathbb{X}_{\Sigma_\eta} \to \mathbb{P}_{M,\Delta}$ give rise to morphisms $\mathcal{Z}_\eta \to \bar{\mathcal{Z}}$ over B, with $\mathcal{Z}_\eta \to B$ a family of nonsingular threefolds as $\mathbb{X}_{\Sigma_\eta}$ is nonsingular in codimension three.

Proposition 1.4. *The families $\mathcal{Z}_\eta$ are all canonically isomorphic to a unique toric resolution $\mathcal{Z} \to \bar{\mathcal{Z}}$ over B. For $t \in B$, the fibre Z_t is a smooth Calabi–Yau threefold with Hodge numbers $h^{1,1}(Z_t) = 21$, $h^{2,1}(Z_t) = 1$. In the resolution $Z_t \to \bar{Z}_t$ there are two exceptional divisors over every singular point S_i, a Hirzebruch surface $E_i \cong \mathbb{F}_3$ and a projective plane $F_i \cong \mathbb{P}^2$ intersecting in a $\mathbb{P}^1$ which is the negative section in the Hirzebruch surface and a line in $\mathbb{P}^2$.*

Proof. Let η_1, η_2 be two permutations of the interior lattice points. There is a corresponding birational map $\mathbb{X}_{\Sigma_{\eta_1}} \dashrightarrow \mathbb{X}_{\Sigma_{\eta_2}}$ whose exceptional sets are disjoint from the families $\mathcal{Z}_{\eta_i}$. This implies the first part. The other statements follow from easy toric calculations. □

Proposition 1.5. *The action of the group $K \cong \mathbb{Z}/5\mathbb{Z}$ on $\bar{\mathcal{Z}}$ extends to a free action on the resolution $\mathcal{Z}$ over B. Thus there is an étale cover $\mathcal{Z} \to \mathcal{Y} = \mathcal{Z}/K$ over B. The fibre Y_t for $t \in B$ is a Calabi–Yau resolution of $\bar{Y}_t$ with Hodge numbers $h^{1,1}(Y_t) = 5$, $h^{2,1}(Y_t) = 1$.*

Proof. The action of K is generated by the symmetry g_3 of $\mathbb{P}^4$. This symmetry descends to the toric variety $\mathbb{P}_\Delta$ as a toric symmetry induced by a lattice isomorphism $\alpha_3 : M \to M$ fixing the polyhedron Δ and permuting the lattice points $\{P_i\}$. Composition with the permutation induced by α_3 gives a correspondence $\eta \to \eta'$ between permutations of the set $\{P_i\}$, and α_3 gives rise to an isomorphism $\tilde{g}_3 : \mathbb{X}_{\Sigma_\eta} \to \mathbb{X}_{\Sigma_{\eta'}}$. This isomorphism restricts to anti-canonical families as an isomorphism $\mathcal{Z}_\eta \to \mathcal{Z}_{\eta'}$, or, by Proposition 1.4, as an automorphism $\mathcal{Z} \to \mathcal{Z}$. By construction, this automorphism is the required extension of g_3 and it clearly generates a free group action on $\mathcal{Z}$ over B. □

We conclude this section by proving two auxiliary statements.

Proposition 1.6. *The family $\bar{\mathcal{Y}} \to B$ restricted to a neighbourhood of $0 \in B$ is the universal deformation space of its central fibre $\bar{Y}_0$ in the analytic category.*

Proof. By general theory, the projective variety $\bar{Y}_0$ has a versal deformation space $\mathcal{X} \to S$ in the analytic category. $\bar{Y}_0$ is a canonical Calabi–Yau threefold. Thus $H^0(\bar{Y}_0, T_{\bar{Y}_0}) = 0$ and this implies that $\mathcal{X} \to S$ is in fact universal. By Ran's extension [12] of the Bogomolov–Tian–Todorov theorem, unobstructedness holds for $\bar{Y}_0$. Thus S is smooth. Further, the codimension of the singularities of $\bar{Y}_0$ is three. By the argument of [3, A.4.2], it follows that the first-order tangent space of S at the base point is isomorphic to $H^1(\bar{Y}_0, T_{\bar{Y}_0})$, a one-dimensional complex vector space.

In order to prove that $\bar{\mathcal{Y}} \to B$ is the universal deformation space, all we need to show is that its Kodaira-Spencer map is injective. Recall the family $\mathcal{Q} \to B$, a deformation of the Fermat quintic Q_0 over B. Choosing a (G-invariant) three-form on Q_0 gives rise to a commutative diagram:

$$\begin{array}{ccccc} T_0(B) & \xrightarrow{k} & H^1(Q_0, T_{Q_0}) & \xrightarrow{\sim} & H^1(Q_0, \Omega^2_{Q_0}) \\ \| & & & & \uparrow j \\ T_0(B) & \xrightarrow{l} & H^1(\bar{Y}_0, T_{\bar{Y}_0}) & \xrightarrow{\sim} & H^1(\bar{Y}_0, \hat{\Omega}^2_{\bar{Y}_0}) \end{array}$$

Here k and l are the Kodaira–Spencer maps, whereas the map j is given by pullback of (orbifold) two-forms (the sheaf of orbifold two-forms $\hat{\Omega}^2_{\bar{Y}_0}$ is defined carefully in [3, A.3]). The map k is injective, as $\mathcal{Q}$ is a nontrivial first-order deformation of Q_0. By commutativity, l is also injective. This proves the proposition. □

Proposition 1.7. *For $t \in B$, the Calabi–Yau manifold Z_t is simply connected.*

Proof. The variety Z_t is a resolution of the threefold $\bar{Z}_t = Q_t/H$. Let Q_t^0 be the open set of Q_t on which the action of H is free; it is the complement of a finite set of points and hence is simply connected. Let $Z_t^0 = Q_t^0/H$; $\pi_1(Z_t^0) \cong H$.

The fundamental group of Z_t is a quotient group of H. Let T_t be the universal cover of Z_t; by the generalized Riemann existence theorem, T_t is an algebraic variety and it clearly has trivial canonical bundle. Let T_t^0 be the preimage of Z_t^0 under the covering map. Then T_t^0 has finite fundamental group; let $\tilde{T}_t^0$ be its universal cover. $\tilde{T}_t^0$ is an algebraic variety again. Notice, however, that Q_t^0, $\tilde{T}_t^0$ are both universal covers of the variety Z_t^0, and thus by the uniqueness part of the generalized Riemann existence theorem they must be isomorphic. Thus there exists a diagram:

$$\begin{array}{ccccc}
Q_t & \supset & Q_t^0 & & \\
 & & \downarrow & & \\
\Big\downarrow & & T_t^0 & \subset & T_t \\
 & & \downarrow & & \downarrow \\
\bar{Z}_t & \supset & Z_t^0 & \subset & Z_t
\end{array}$$

The covering $Q_t^0 \to T_t^0$ corresponds to a group L of holomorphic automorphisms of Q_t^0. An automorphism of Q_t^0 can be thought of as a birational self-map of Q_t. However, as Q_t is a minimal Calabi–Yau threefold with Picard number one, it has no birational self-maps with a nontrivial exceptional locus. So L consists of automorphisms of Q_t. The fact that the map $Q_t^0 \to T_t^0$ factors the map $Q_t^0 \to Z_t^0$ implies that L must be a subgroup of H.

Thus we conclude that T_t is birational to a quotient Q_t/L for a subgroup L of H. Moreover, $\chi(Z_t) = 40$, so $\chi(T_t)$ equals either 40, 200 or 1000. On the other hand, for every subgroup L of H, the quotient Q_t/L has a Calabi–Yau desingularization. As the Euler number is a birational invariant of smooth Calabi–Yau threefolds, the Euler number of this desingularization must be equal to that of T_t. Finally, it is easy to check that H has no subgroup L such that a Calabi–Yau desingularization of Q_t/L has Euler number 200 or 1000. Thus $L = H$ and so $T_t = Z_t$ is its own universal cover. □

2. Hodge structures

Let Z, Y denote the differentiable manifolds underlying the fibres Z_t, Y_t. Let $V_Z = H^3(Z, \mathbb{Z})_{\text{free}}$, $V_Y = H^3(Y, \mathbb{Z})_{\text{free}}$, with antisymmetric pairings Q_Z, Q_Y given by cup product.

Lemma 2.1. *Pullback by the map $\pi : Z \to Y$ induces an injection*

$$\pi^* : V_Y \hookrightarrow V_Z$$

with image of index at most 25. *Under this embedding,*

$$Q_Z(\pi^*\alpha_1, \pi^*\alpha_2) = 5\, Q_Y(\alpha_1, \alpha_2).$$

Consequently, there is an embedding of groups

$$\mathrm{Aut}_{\mathbb{Z}}(V_Z, Q_Z) \xrightarrow{\ j\ } \mathrm{Aut}_{\mathbb{Q}}(V_Y \otimes \mathbb{Q}, Q_Y).$$

Proof. The group $K \cong \mathbb{Z}/5\mathbb{Z}$ acts without fixed points on Z, so the map π induces a spectral sequence

$$E_2^{p,q} = H^p(K; H^q(Z, \mathbb{Z})) \Rightarrow H^{p+q}(Y, \mathbb{Z}).$$

The terms $E_2^{p,q}$ for $p > 0$ are torsion, so $V_Y = (E_\infty^{0,3})_{\text{free}}$. On the other hand, $(E_2^{0,3})_{\text{free}} = H^0(K, H^3(Z, \mathbb{Z})_{\text{free}}) = (V_Z)^K$. There are two differentials from $(E_2^{0,3})$, both having image $\mathbb{Z}/5\mathbb{Z}$. So there is an injection

$$\pi^* : V_Y \hookrightarrow (V_Z)^K$$

with image of index at most 25. This map is an isomorphism when tensored by $\mathbb{Q}$. As both V_Z and V_Y have rank four, K must act trivially on V_Z and this proves the first part. The other two statements are immediate. □

Let $\mathcal{D}_Y$ be the period domain parameterizing weight 3 polarized Hodge structures on (V_Y, Q_Y). Fixing a point $t \in B$, a marking $H^3(Y_t, \mathbb{Z})_{\text{free}} \cong V_Y$ and a universal cover $\tilde{B}$ of B leads to holomorphic period maps

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\tilde{\psi}} & \mathcal{D}_Y \\ \downarrow & & \downarrow \\ B & \xrightarrow{\psi} & \mathcal{D}_Y/\Gamma \end{array}$$

where Γ is any subgroup of $\mathrm{Aut}_{\mathbb{Q}}(V_Y \otimes \mathbb{Q}, Q_Y)$ containing all geometric monodromies and acting properly discontinuously on $\mathcal{D}$. Choose

$$\Gamma = j(\mathrm{Aut}_{\mathbb{Z}}(V_Z, Q_Z)) \subset \mathrm{Aut}_{\mathbb{Q}}(V_Y \otimes \mathbb{Q}, Q_Y)$$

under the embedding j of Lemma 2.1.

Lemma 2.2. *Γ acts properly discontinuously on $\mathcal{D}_Y$, so $\mathcal{D}_Y/\Gamma$ is an analytic space.*

Proof. See [6, Section I.2]. □

After all these preparations, we can state

Theorem 2.3. *For Γ chosen as above, the period map $\psi : B \to \mathcal{D}_Y/\Gamma$ is of degree at least five. More precisely, if $t_1, t_2 \in B$ satisfy $t_1^5 = t_2^5$, then $\psi(t_1) = \psi(t_2)$. In particular, Y_{t_1} and Y_{t_2} have isomorphic rational Hodge structure.*

Proof. The symmetry

$$g : [z_0 : z_1 : z_2 : z_3 : z_4] \mapsto [\xi^{-1} z_0 : z_1 : z_2 : z_3 : z_4]$$

descends to a symmetry of $\mathbb{P}^4/H$ and maps $\bar{Z}_t$ isomorphically to $\bar{Z}_{\xi t}$. By an argument analogous to the proof of Proposition 1.5, this isomorphism extends to an isomorphism $Z_t \to Z_{\xi t}$. This gives a diagram of polarized Hodge structures:

$$\begin{array}{ccc} H^3(Y_t, \mathbb{Z})_{\text{free}} & \xrightarrow{\pi^*} & H^3(Z_t, \mathbb{Z})_{\text{free}} \\ & & \big\downarrow \cong \\ H^3(Y_{\xi t}, \mathbb{Z})_{\text{free}} & \xrightarrow{\pi^*} & H^3(Z_{\xi t}, \mathbb{Z})_{\text{free}} \end{array}$$

Comparing this with the action of Γ on $\mathcal{D}_Y$ defined above gives the first statement. The second statement is immediate. □

Remark 2.4. The proof of Lemma 2.1 implies that the spectral sequence

$$E_2^{p,q} = H^p(K; H^q(Z, A)) \Rightarrow H^{p+q}(Y, A)$$

degenerates at E_2 whenever 5 is invertible in A. In particular, there is an isomorphism of polarized Hodge structures

$$H^3(Y_t, \mathbb{Z}[1/5]) \cong H^3(Y_{\xi t}, \mathbb{Z}[1/5]).$$

The problem is that $\mathrm{Aut}(V_Y \otimes \mathbb{Z}[1/5], Q_Y)$ does not act properly discontinuously on $\mathcal{D}_Y$, so such a statement is weaker than the one proved above. On the other hand, it seems difficult to determine the precise behavior of the spectral sequence with $\mathbb{Z}$ coefficients, i.e. to compute the torsion in the cohomology of Y.

Remark 2.5. The isomorphism of $\mathbb{Q}$-Hodge structures is due to Aspinwall and Morrison. They give a different proof coming from mirror symmetry which goes as follows. The mirror family $\mathcal{X}$ of $\mathcal{Y}$ is the quotient of a suitable family of quintic hypersurfaces by the group $\langle g_1, g_3 \rangle$. In particular, the antichiral ring of the central fibre X_0 of $\mathcal{X}$ with a choice of (complexified) Kähler class is isomorphic to the chiral ring of Y_t. On the other hand, the antichiral ring of X_0 can be shown to depend, via the mirror map, on t^5 only and not on t. Thus the varieties $Y_{\xi^i t}$ for $i = 0, \dots, 4$ have the same chiral ring, i.e. isomorphic rational Hodge structure.

Remark 2.6. Suppose that Y_0 is an n-fold, G (a nontrivial quotient of) the fundamental group $\pi_1(Y_0)$. Then there is an étale cover $Z \to Y$; in fact there is a cover $Z_t \to Y_t$ for every deformation Y_t of Y_0. The (primitive) cohomology $H^n_0(Z_t)$ becomes a G-representation, and in some cases one can recover information about Y_t from the pair

$$(H^n_0(Z_t), \text{ action of } G).$$

A particular example of this construction is the theorem of Horikawa [8], giving a Torelli-type result for Enriques surfaces using global Torelli for K3s. However, by Proposition 1.7, the threefold Z_t under investigation is simply connected. On the other hand, as the proofs above show, the Hodge structure on the middle-dimensional rational cohomology of the universal cover Z_t contains no extra information, and it carries the trivial action of the fundamental group $\pi_1(Y_t)$.

3. The automorphism group of the central fibre

Theorem 3.1. *The automorphism groups of the varieties Y_0, $\bar{Y}_0$ are*

$$\mathrm{Aut}(Y_0) \cong \mathrm{Aut}(\bar{Y}_0) \cong \langle G, g_4, g_5 \rangle / G,$$

where

$$\begin{array}{lccc} g_4: & [z_0 : z_1 : z_2 : z_3 : z_4] & \mapsto & [z_0 : z_1 : z_2 : \xi^4 z_3 : \xi z_4], \\ g_5: & [z_0 : z_1 : z_2 : z_3 : z_4] & \mapsto & [z_0 : z_2 : z_4 : z_1 : z_3]. \end{array}$$

In particular, every automorphism of $\bar{Y}_0$ extends to an automorphism on all (small) deformations $\bar{Y}_t$ of $\bar{Y}_0$.

Proof. The proof of Theorem 3.1 uses three lemmas. The first one should certainly be well known, but a suitable reference could not be found so a proof is included.

Lemma 3.2. *Let*

$$X = \left\{ \sum_{i=0}^{n} x_i^d = 0 \right\} \subset \mathbb{P}^n_k$$

be the Fermat hypersurface. Assume that $d \geq 3$, $n \geq 2$ and that $(n,d) \neq (2,3)$ or $(3,4)$. Then

$$\mathrm{Aut}(X) \cong G_{n,d},$$

where $G_{n,d}$ is the semi-direct product $\Sigma_{n+1} \ltimes (\mu_d)^n$ of a symmetric group and a power of the group of d-th roots of unity.

Proof. For $n = 2$, the result is proved in [15]. If $n \geq 3$ and $(n,d) \neq (3,4)$, then we first claim that every automorphism comes from a projective automorphism in the given embedding. If $n \geq 4$, Lefschetz implies $\mathrm{Pic}(X) \cong \mathbb{Z}$ and then the claim is clear. If $n = 3$ and $d \neq 4$, then the canonical class is (anti-)ample and this easily implies the claim again; see [10].

Take an element $\sigma \in \mathrm{Aut}(X)$ represented by an invertible matrix $A = (a_{ij})$. Apply A to the equation of X and consider the coefficients of $x_0^{d-1}x_1$, $x_0^{d-2}x_1^2$, and $x_0^{d-2}x_1x_i$ for $i > 1$. Their vanishing shows that the set of numbers

$$\{a_{00}^{d-2}a_{01}, a_{10}^{d-2}a_{11}, \ldots, a_{n0}^{d-2}a_{n1}\}$$

solves the homogeneous system of equations given by the invertible matrix A^T. So all these quantities are zero. By symmetry, $a_{ij}a_{ik} = 0$ whenever $j \neq k$. Hence A has at most one nonzero entry in each row. Multiplying by a suitable element in Σ_{n+1}, A can be brought into diagonal form, and then all its entries are d-th roots of unity. □

Lemma 3.3. *Let $\bar{X}$ be a canonical Calabi–Yau threefold with a finite number $m \geq 2$ of isolated $\frac{1}{5}(1,1,3)$ quotient singularities and Picard number one. Let $\pi : X \to \bar{X}$ be the Calabi–Yau resolution. Then $\mathrm{Aut}(X) \cong \mathrm{Aut}(\bar{X})$.*

Proof. The Picard group of the resolution X is

$$\mathrm{Pic}_{\mathbb{Q}}(X) \cong \mathbb{Q}H \oplus \mathbb{Q}E_1 \oplus \mathbb{Q}F_1 \oplus \ldots \oplus \mathbb{Q}E_m \oplus \mathbb{Q}F_m,$$

where $H = \pi^*(\mathcal{O}_{\bar{X}}(1))$ and E_i, F_i are the classes of the exceptional divisors as described in Proposition 1.4. The intersection numbers are as follows:

$H^3 = d > 0$	the degree of $\bar{X}$,
$H \cdot E_i = H \cdot F_i = 0$	as H is a pullback,
$E_i \cdot E_j = E_i \cdot F_j = F_i \cdot F_j = 0$	unless $i = j$,
$E_i^3 = (K_{E_i})^2 = 8$	as $E_i \cong \mathbb{F}_3$,
$F_i^3 = (K_{F_i})^2 = 9$	as $F_i \cong \mathbb{P}^2$,
$E_i^2 F_i = 1$,	
$F_i^2 E_i = -3$.	

Introducing the basis $H_0 = H$, $H_{2i-1} = E_i + \frac{1}{3}F_i$, $H_{2i} = F_i$ of $\mathrm{Pic}_{\mathbb{Q}}(X)$, the cubic form takes the shape

$$\left(\sum_{i=0}^{2m} \alpha_i H_i\right)^3 = d\alpha_0^3 + 8\frac{1}{3}\sum_{i=1}^{m} \alpha_{2i-1}^3 + 9\sum_{i=1}^{m} \alpha_{2i}^3.$$

Finally, the values of the second Chern class are

$$c_2(X) \cdot E_i = -4,\ c_2(X) \cdot F_i = -6,\ c_2(X) \cdot H = c \geq 0,$$

where the last inequality follows from a result of Miyaoka [11, Theorem 1.1].

Let $\sigma \in \mathrm{Aut}(X)$ be an automorphism. It acts via pullback on $\mathrm{Pic}_{\mathbb{Q}}(X)$, fixing the cubic form together with the linear form given by cup product with $c_2(X)$. We claim that the element $H_0 = H$ of $\mathrm{Pic}_{\mathbb{Q}}(X)$ must be fixed under the action. To see this, note that the cubic form has been manufactured to take the shape of the Fermat cubic. Every automorphism of $\mathrm{Pic}_{\mathbb{Q}}(X)$ must fix the associated (projectivized) hypersurface. The possible automorphisms are known from Lemma 3.2. Moreover, in the present case, the multiplications by roots of unity are excluded since σ must fix a *rational* vector space. The possible permutations are constrained by the fact that c_2 has to be fixed as well. As c_2 is negative on the H_i for $i > 0$ and nonnegative on $H = H_0$, the latter is fixed and this proves the claim.

For large and divisible m, the divisor class mH is base-point free and, since the torsion in $\mathrm{Pic}(X)$ is finite, is the unique representative of its numerical equivalence class. As $H \in \mathrm{Pic}_{\mathbb{Q}}(X)$ is fixed by the induced action of σ, for large and divisible m the space of sections of the linear system $\mid mH \mid$ is also acted on by σ. In other words, the automorphism σ descends to the image of the associated morphism which is exactly $\bar{X}$.

For the converse, note that the quotient singularity $\frac{1}{5}(1,1,3)$ has a unique crepant resolution. Hence every automorphism $\bar{\sigma} \in \mathrm{Aut}(\bar{X})$ extends to a biregular automorphism $\sigma \in \mathrm{Aut}(X)$ of the resolution. The lemma follows. □

Lemma 3.4. *Let X be a smooth algebraic variety with finite fundamental group F. Let Y be the universal cover of X, a smooth algebraic variety with an action of F by automorphisms. Then*

$$\mathrm{Aut}(X) \cong N_{\mathrm{Aut}(Y)}(F)/F.$$

Proof. Obvious. □

To finish the proof of Theorem 3.1, let Q_0^0 be the open set of the Fermat quintic Q_0 on which the action of G is free. Let $Y_0^0 = Q_0^0/G$. There is a sequence of maps

$$\mathrm{Aut}(\bar{Y}_0) \hookrightarrow \mathrm{Aut}(Y_0^0) \cong N_{\mathrm{Aut}(Q_0^0)}(G)/G \cong N_{\mathrm{Aut}(Q_0)}(G)/G.$$

The first isomorphism follows from Lemma 3.4. The second isomorphism uses $\mathrm{Aut}(Q_0^0) \cong \mathrm{Aut}(Q_0)$; here $\mathrm{Aut}(Q_0^0) \subset \mathrm{Aut}(Q_0)$ is proved by the argument used already in Proposition 1.7 and the other direction is clear by Lemma 3.2.

On the other hand, by Lemma 3.2, the automorphism group of Q_0 is the semidirect product $G_{4,5}$ of the permutation and diagonal symmetries. Finding the normalizer of G in $G_{4,5}$ is a finite search best done using a computer; a short Mathematica routine computes this normalizer to be

$$N_{\mathrm{Aut}(Q_0)}(G)/G \cong \langle G, g_4, g_5 \rangle / G$$

with g_4, g_5 as in the statement of Theorem 3.1. So we obtain

$$\mathrm{Aut}(\bar{Y}_0) \hookrightarrow \langle G, g_4, g_5 \rangle / G$$

and it is easy to see that this is in fact an isomorphism. Finally, by Lemma 3.3, $\mathrm{Aut}(\bar{Y}_0) \cong \mathrm{Aut}(Y_0)$. This proves the first statement. The second statement follows by inspection: every generator of the normalizer fixes Q_t. □

4. The proof of Theorem 0.1

The proof is based on the following rather standard result, a version of which was already used in [13]:

Theorem 4.1. *Let $\mathcal{X}_i \to B$, $i = 1, 2$, be families of canonical Calabi–Yau varieties over a base scheme B, having simultaneous resolutions $\mathcal{Y}_i \to \mathcal{X}_i$ over B. Let $\mathcal{L}_i$ be relatively ample relative Cartier divisors on $\mathcal{X}_i$. Let $\mathrm{Isom}_B(\mathcal{X}_i, \mathcal{L}_i)$ be the functor*

$$\mathrm{Isom}_B(\mathcal{X}_i, \mathcal{L}_i) : \underline{\mathrm{Schemes}} \to \underline{\mathrm{Sets}}$$

defined by

$$\mathrm{Isom}_B(\mathcal{X}_i, \mathcal{L}_i)(S) = \{\textit{polarized } S\textit{–isomorphisms } (\mathcal{X}_1)_S \to (\mathcal{X}_2)_S\},$$

where the pullback families $(\mathcal{X}_i)_S$ are polarized by the relatively ample line bundles $(\mathcal{L}_i)_S$. This functor is represented by a scheme $\boldsymbol{Isom}_B(\mathcal{X}_i, \mathcal{L}_i)$, proper and unramified over B.

Proof. By Grothendieck's theory of the representability of Hilbert schemes and related functors, the above functor is represented by a scheme $\mathbf{Isom}_B(\mathcal{X}_i, \mathcal{L}_i)$, separated and of finite type over B. The fact that the fibres have no infinitesimal automorphisms implies that $\mathbf{Isom}_B(\mathcal{X}_i, \mathcal{L}_i)$ is unramified over B. Properness follows from the valuative criterion along the lines of [4, Proposition 4.4]; the existence of a simultaneous resolution is needed for this final step. □

Theorem 4.2. *Let $\mathcal{Y} \to B$ be the family constructed in Section 1, ξ a primitive fifth root of unity. Then there is a Zariski dense subset $U \subset B$, such that the fibres Y_t and $Y_{\xi t}$ are not isomorphic as algebraic varieties for $t \in U$.*

Proof. First we work with the singular family $\bar{\mathcal{Y}}$; for ease of notation, let $\bar{\mathcal{Y}}_1 = \bar{\mathcal{Y}}$. Fixing an ample divisor L on $\mathbb{P}_\Delta/K$ gives by restriction a relatively ample divisor $\mathcal{L}$ on $\bar{\mathcal{Y}}_1$. Let $\mathcal{L}_1 = \mathcal{L}^{\otimes 5}$.

Let $\gamma : B \to B$ be the map of the base which is multiplication by ξ^{-1}. Let $\bar{\mathcal{Y}}_2 \to B$ denote the pullback of $\bar{\mathcal{Y}}_1 \to B$ by γ. The family $\bar{\mathcal{Y}}_2 \to B$ is equipped with the relatively ample line bundle $\mathcal{L}_2 = \gamma^*(\mathcal{L}_1)$ and its fibre over $t \in B$ is $\bar{Y}_{\xi t}$.

Lemma 4.3. *Let $t \in B$, and let $\bar{Y}_{i,t}$ be the fibres of the two families polarized by the ample divisors $L_{i,t}$. Then every isomorphism*

$$\varphi : \bar{Y}_{1,t} \xrightarrow{\sim} \bar{Y}_{2,t}$$

satisfies $\varphi^*(L_{2,t}) \sim L_{1,t}$.

Proof. The fibres have Picard number one, and multiplication by five annihilates every torsion element in their Picard groups. So the divisors $L_{i,t}$ are canonical elements of the respective Picard groups. The lemma follows. □

Continuing the proof of Theorem 4.2, consider the relative isomorphism scheme

$$\mathbf{Isom} = \mathbf{Isom}_B(\bar{\mathcal{Y}}_i, \mathcal{L}_i)$$

together with the natural map $\mathbf{Isom} \to B$. By Theorem 4.1, this map is proper, so its image V is a closed subvariety of the quasi-projective variety B.

Assume first that $V = B$. Then $\mathbf{Isom}$ has a component $\mathbf{I}$ with a surjective unramified map onto a Zariski neighbourhood of $0 \in B$. Now switch to the complex topology; let Δ be a disc in $\mathbf{I}$ mapping isomorphically onto a neighbourhood of

$0 \in B$. Consider the pullback families $\bar{\mathcal{Y}}_{i,\Delta} \to \Delta$. By the definition of $\mathbf{I}$, these families are isomorphic under an isomorphism φ over Δ.

Consider the composition

$$\bar{\mathcal{Y}}_{1,\Delta} \xrightarrow{\varphi} \bar{\mathcal{Y}}_{2,\Delta} \xrightarrow{(\gamma^{-1})^*} \bar{\mathcal{Y}}_{1,\Delta}.$$

Its restriction to the central fibre $\bar{Y}_0$ is a polarized automorphism σ.

By Proposition 1.6, $\bar{\mathcal{Y}}_1 \to \Delta$ is the universal deformation space of $\bar{Y}_0$ in the analytic category. The automorphism σ acts on the base of the deformation space by universality. This action equals the composite of the actions of φ and $(\gamma^{-1})^*$ on the base Δ. However, φ is an isomorphism over Δ, so the action of σ on Δ is multiplication by a primitive fifth root of unity, i.e. a rotation of the disc.

On the other hand, by Theorem 3.1, the action of every automorphism of $\bar{Y}_0$ on the base of the universal deformation space is *trivial*. Thus σ cannot exist. So the assumption $V = B$ leads to a contradiction.

Thus V is a proper closed subset of B. Let $U = B \setminus V$, a Zariski open subset of B. Over $t \in U$ the scheme **Isom** has no points. Using Lemma 4.3, this implies that for $t \in U$ there cannot exist any isomorphism between $\bar{Y}_t$ and $\bar{Y}_{\xi t}$.

Finally, if $Y_t \cong Y_{\xi t}$ for some $t \in B$, then an argument analogous to the proof of Lemma 3.3 shows that the singular Calabi–Yau models $\bar{Y}_t$, $\bar{Y}_{\xi t}$ are also isomorphic. This concludes the proof of Theorem 4.2. □

Applying this theorem for ξ^i, $i = 1, \ldots, 4$, and taking the intersection of the resulting open sets concludes the proof of Theorem 0.1 announced in the Introduction.

Remark 4.4. Theorem 0.1 is also argued for in the paper [1]. Aspinwall and Morrison write down a power series in the coordinate t of the base B, following [2], related to higher genus Gromov–Witten invariants of the family mirror family $\mathcal{X}$. This series is a function of t rather than t^5, and this is a strong indication of the validity of Theorem 0.1. As a matter of fact, I believe that this is also an indication of the validity of Conjecture 0.2. However, a solid mathematical definition, let alone computation, of this power series has not been given to date.

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