THE CAUCHY PROBLEM FOR A CLASS OF KOVALEVSKIAN PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT. We prove the $H^1$ well-posedness of the forward Cauchy problem for a pseudo-differential operator $P$ of order $m \geq 2$ with the Log-Lipschitz continuous symbol in the time variable. The characteristic roots $\lambda_k$ of $P$ are distinct and satisfy the necessary Lax-Mizohata condition $\text{Im} \lambda_k \geq 0$. The Log-Lipschitz regularity has been tested as the optimal one for $H^1$ well-posedness in the case of second-order hyperbolic operators. Our main aim is to present a simple proof which needs only a little of the basic calculus of standard pseudo-differential operators.

INTRODUCTION

Let us consider the Cauchy problem

\begin{align}
Pu(t, x) &= f(t, x), \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^n, \\
\partial_j^t u(0, x) &= g_j(x), \quad 0 \leq j \leq m - 1
\end{align}

for a pseudo-differential operator of Kovalevskian type

\begin{align}
P = D_t^m + \sum_{j=0}^{m-1} A_j(t, x, D_x)D_j, \quad A_j(t) \in OPS^{m-j},
\end{align}

of order $m \geq 2$ in $[0, T] \times \mathbb{R}^n$.

One says that problem (1.1) is well posed in the Sobolev space $H^\infty = H^\infty(\mathbb{R}^n) = \bigcap H^\mu(\mathbb{R}^n)$ if for every $f \in C([0, T]; H^\infty)$ and $g_j \in H^\infty$, $0 \leq j \leq m$, there is a unique solution $u \in C^m([0, T]; H^\infty)$.

In this paper we are concerned with the question of what kind of regularity in the time variable $t$ one has to assume for the $A_j$'s in (1.2) in order to obtain such a well-posedness. From [1] and [2] we know that for second-order strictly hyperbolic differential operators the sharp regularity is the Log-Lipschitz continuity: a function $a(t)$ is said to be Log-Lipschitz continuous, in short $a \in LL([0, T])$, if it satisfies

\[ |a(t) - a(s)| \leq C|t - s| \log |t - s|, \quad 0 < |t - s| < \frac{1}{2}. \]
Here we prove that $H^\infty$ well-posedness holds true for any operator of the type (1.2) with $A_j(t) \in LL([0,T];OPS^{m-j})$, $j = 0, ..., m-1$, provided that the characteristic roots $\lambda_k(t,x,\xi)$ of $P$ are distinct and satisfy the necessary Lax-Mizohata condition

$$\Im \lambda_k(t,x,\xi) \geq 0 \text{ for large } |\xi|, \; k = 1, ..., m.$$  

Our main aim is to give a simple proof which needs only basic calculus of classical pseudo-differential operators.

**Notation.** Throughout this paper, $x, \xi \in \mathbb{R}^n, \langle \xi \rangle$ denotes $\sqrt{1 + |\xi|^2}$ and $D_x = -i\nabla_x$. Since it concerns the notation for pseudo-differential operators, we follow [3], which we refer to also for all the results we need.

Therefore, $S^N$ will denote the class of the symbols $a(x,\xi)$ such that

$$\sup_{x,\xi} |\partial_\xi^\alpha D_\xi^\beta a(x,\xi)| \cdot |\langle \xi \rangle|^{\alpha-N} < \infty$$

for every $\alpha, \beta \in \mathbb{N}^n$. Moreover, $S^N_{\log}$ will denote the class of symbols with the following property:

$$\sup_{x,\xi} |\partial_\xi^\alpha D_\xi^\beta a(x,\xi)| \cdot |\langle \xi \rangle|^{\alpha-N} / \log(1 + |\langle \xi \rangle|) < \infty$$

for every $\alpha, \beta \in \mathbb{N}^n$.

For symbols depending also on a time variable, we introduce the following notation: $a(t,x,\xi) \in LL([0,T];S^N)$, that is, $a(t,x,\xi)$ is Log-Lipschitz with respect to $t$, whenever for every $\alpha, \beta \in \mathbb{N}^n$ there exists $C_{\alpha,\beta} > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |\partial_\xi^\alpha D_\xi^\beta (a(t,x,\xi) - a(s,x,\xi))| \leq C_{\alpha,\beta} |t-s| |\langle \xi \rangle| N-|\alpha|,$$

$$0 < |t-s| < 1/2.$$  

**The Cauchy problem.** Let us consider an operator $P = P(t,x,D_t,D_x)$ in $[0,T] \times \mathbb{R}^n$ given by

$$P = D_t^m + \sum_{j=0}^{m-1} A_j(t,x,D_x)D_t^j$$

where $A_j \in C([0,T];S^{m-j})$. Let $P^0 = D_t^m + \sum_{j=0}^{m-1} A_j^0(t,x,D_x)D_t^j$ be such that

$$A_j - A_j^0 \in C([0,T];S^{m-1-j}) \quad j = 0, ..., m-1,$$

and

$$P^0(t,x,\tau,\xi) = \prod_{k=1}^{m} (\tau - \lambda_k(t,x,\xi))$$

with

$$\lambda_k \in LL([0,T];S^1), \quad k = 1, ..., m.$$  

We assume that the roots are distinct:

$$|\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)| \geq c|\xi| \text{ for large } |\xi|, \; j \neq k, \; c > 0,$$
and that they fulfill the Lax-Mizohata condition:

\[ \text{Im} \lambda_k(t, x, \xi) \geq 0 \quad \text{for large } |\xi|. \]

**Remark 3.1.** If (3.2) holds for \( A_j^0 \) such that each function \( A_j^0(t, x, \xi) \) is homogeneous of degree \( m - j \) in \( \xi \) for large \( |\xi| \), i.e. \( A_j^0(t, x, \xi) = \theta^{m-j} A_j^0(t, x, \xi) \), \( \theta \geq 1 \), \( |\xi| \geq M \), and we assume

\[ A_j^0 \in LL([0, T]; S^{m-j}) , \quad j = 0, ..., m - 1 , \]

then, after a modification in a neighborhood of \( \xi = 0 \), we have (3.4), in view of (3.5).

Our main result is that the Cauchy problem

\[ \begin{cases} Pu(t, x) = f(t, x), & 0 \leq t \leq T, \\ D_t^j u(0, x) = g_j(x), & 0 \leq j \leq m - 1, \end{cases} \]

is well posed in \( H^\infty \) (with a loss of derivatives). In fact, we have the following.

**Theorem 3.1.** Let \( P \) satisfy (3.2), (3.4), (3.5) and (3.6). Then there is \( \delta > 0 \) such that for every \( \mu \in \mathbb{R} \), every \( f \in C([0, T]; H^\mu) \) and \( g_j \in H^{\mu+m-j-1} \), \( 0 \leq j \leq m - 1 \), the Cauchy problem (3.7) has a unique solution \( u \in \bigcap_{j=0}^m C^j([0, T]; H^{\mu-\delta T+m-j-1}). \)

The solution satisfies the inequality

\[ \sum_{j=0}^{m-1} \|D_t^j u(t)\|_{\mu-\delta t+m-j-1}^2 \leq C \left\{ \sum_{j=0}^{m-1} \|g_j\|_{\mu+m-j-1}^2 + \int_0^t \|f(s)\|_{\mu-\delta s}^2 ds \right\}, \quad 0 \leq t \leq T, \]

for some \( C = C_\mu > 0 \).

**Proof.** The first step is to carry the factorization (3.3) to the operator level. Let us introduce the following regularization of \( \lambda_k \) with respect to the variable \( t \):

\[ \tilde{\lambda}_k(t, x, \xi) = \int \lambda_k(s, x, \xi) \rho((t-s) \langle \xi \rangle) \langle \xi \rangle ds, \]

where \( \rho \in C^\infty_0(\mathbb{R}) \), \( 0 \leq \rho \leq 1 \), \( \int \rho = 1 \) and we have set \( \lambda_k(s, x, \xi) = \lambda_k(T, x, \xi) \) for \( s > T \) and \( \lambda_k(s, x, \xi) = \lambda_k(0, x, \xi) \) for \( s < 0 \). It is easy to see that

\[ \tilde{\lambda}_k \in C([0, T]; S^1), \quad \tilde{\lambda}_k - \lambda_k \in C([0, T]; S^h), \quad \partial_t^h \tilde{\lambda}_k \in C([0, T]; S^h) \quad \text{for any } h \geq 1. \]

Thus the operator \( P \) can be written in the form

\[ P = (D_t - \tilde{\lambda}_m(t, x, D_x))...(D_t - \tilde{\lambda}_1(t, x, D_x)) + \sum_{j=0}^{m-1} R_j(t, x, D_x)D_t^j \]
with $R_t \in C([0, T] ; S_t^{m-j-1})$. Next we want to reduce the scalar equation $Pu = f$ to an $m \times m$ system $LU = F$. Let us define $U = (u_0, \ldots, u_{m-1})$ by

$$
\begin{align*}
    u_0 &= \langle D_x \rangle^{m-1}u, \\
    u_1 &= \langle D_x \rangle^{m-2}(D_t - \lambda_1(t, x, D_x))u, \\
    \vdots \\
    u_{m-1} &= \langle D_t - \lambda_{m-1}(t, x, D_x) \rangle \cdots \langle D_t - \lambda_1(t, x, D_x) \rangle u.
\end{align*}
$$

(3.12)

Denoting $\langle \langle D_x \rangle^{m-1}u, \langle D_x \rangle^{m-2}D_t u, \ldots, D_t^{m-1}u \rangle$ by $V$, we immediately have

$$
U = Q(t, x, D_x)V, \quad V = Q_0(t, x, D_x)U,
$$

(3.13)

where the symbols of the entries of the $m \times m$ matrices $Q$ and $Q_0$ belong to $C([0, T]; S^0)$. Thus the equation $Pu = f$ is equivalent to an $m \times m$ system

$$
LU = F,
$$

(3.14)

where $F = (0, \ldots, 0, f)$ and

$$
\mathcal{L} = \partial_t - i\lambda(t, x, D_x) + B(t, x, D_x),
$$

(3.15)

with

$$
\Lambda = \begin{bmatrix}
    \lambda_1 \langle D_x \rangle & \cdots & 0 \\
    0 & \cdots & 0 \\
    0 & \cdots & \tilde{\lambda}_m
\end{bmatrix}
$$

and the entries of $B(t, x, \xi)$ in $C([0, T]; S_t^{0\log})$. The operator $\Lambda$ can be diagonalized by means of

$$
M = \begin{bmatrix}
    1 & d_{ij} \\
    \ddots & \ddots \\
    0 & 1
\end{bmatrix}, \quad d_{ij}(t, x, \xi) = \langle \xi \rangle^{j-1} / \prod_{k=i}^{j-1} (\lambda_j - \tilde{\lambda}_k), \ i < j, \text{ for large } |\xi|.
$$

(3.16)

From (3.10) we have $M, M^{-1} \in C([0, T]; S^0)$, $\partial_t M, \partial_t M^{-1} \in C([0, T]; S_t^{0\log})$. Thus, we have

$$
\mathcal{L}_1 := M^{-1} \mathcal{L} M = \partial_t - i\Delta(t, x, D_x) + B_1(t, x, D_x),
$$

(3.17)

where $\Delta$ is the diagonal matrix of the $\lambda_k$’s and $B_1 \in C([0, T]; S_t^{0\log})$. Now, for $\mu \in \mathbb{R}$ and $\delta > 0$ let us define the operator $\mathcal{L}_2 = \langle D_x \rangle^{\mu - \delta t} \mathcal{L}_1 \langle D_x \rangle^{\mu + \delta t}$. We have

$$
\mathcal{L}_2 = \partial_t - i\Delta(t, x, D_x) + B_1(t, x, D_x) + \delta \log(D_x)I + B_2(t, x, D_x),
$$

(3.18)

where $B_2 \in C([0, T]; S^0)$.

It remains to prove the following.

**Proposition 3.2.** It is possible to fix $\delta > 0$ such that the Cauchy problem

$$
\begin{cases}
    \mathcal{L}_2 U(t, x) = F(t, x), & 0 \leq t \leq T, \\
    U(0, x) = G(x)
\end{cases}
$$

(3.19)
has a unique solution \( \mathcal{U} \in \mathcal{C}([0,T];H^1) \cap C^1([0,T];H^0) \) for any given \( F \in \mathcal{C}([0,T];H^1) \) and \( G \in H^1 \). The solution satisfies the energy inequality

\[
\|\mathcal{U}(t)\|_0^2 \leq C(\|\mathcal{U}(0)\|_0^2 + \int_0^t \|\mathcal{F}(s)\|_0^2 ds), \quad 0 \leq t \leq T,
\]

for some \( C = C_\mu > 0 \).

**Proof.** We have only to prove that it is possible to fix \( \delta > 0 \) such that

\[
\|\mathcal{U}(t)\|_0^2 \leq C(\|\mathcal{U}(0)\|_0^2 + \int_0^t \|\mathcal{L}_2\mathcal{U}(s)\|_0^2 ds), \quad 0 \leq t \leq T,
\]

for every \( \mathcal{U} \in \mathcal{C}([0,T];H^1) \cap C^1([0,T];H^0) \), with a constant \( C > 0 \) depending only on the seminorms of the symbols of \( \Delta, B_1, B_2 \) (hence depending on \( \mu \) because of \( B_2 \)). Then one can apply the usual energy method to solve (3.19), e.g. [3], pp. 236-240. To prove (3.21), we begin by choosing \( \delta \) large enough in order to have positive operators \( B_1(t,x,D_x) + \delta \log(D_x)I \) in (3.18) for \( t \in [0,T] \). This is possible from \( B_1 \in \mathcal{C}([0,T];S^0_{\log}) \). So, we obtain

\[
\frac{d}{dt}\|\mathcal{U}(t)\|_0^2 \leq 2\Re(e(i\Delta \mathcal{U}(t) - B_2 \mathcal{U}(t) + \mathcal{L}_2\mathcal{U},\mathcal{U}(t))
\]

\[
\leq 2\Re(e(i\Delta \mathcal{U}(t),\mathcal{U}(t)) + C_\mu(\|\mathcal{U}(t)\|_0^2 + \|\mathcal{L}_2\mathcal{U}(t)\|_0^2).
\]

Then from (3.6) we can apply the sharp Gårding inequality to the first-order operator \( -i\Delta \):

\[
2\Re(e(-i\Delta \mathcal{U}(t),\mathcal{U}(t)) \geq -C\|\mathcal{U}(t)\|_0^2, \quad C > 0,
\]

which gives

\[
\frac{d}{dt}\|\mathcal{U}(t)\|_0^2 \leq C_\mu^*(\|\mathcal{U}(t)\|_0^2 + \|\mathcal{L}_2\mathcal{U}(t)\|_0^2).
\]

Gronwall’s inequality yields (3.21), completing the proof.

**References**


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