PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 3, Pages 669–679
S 0002-9939(03)07093-X
Article electronically published on August 13, 2003

ASYMPTOTICS FOR THE MULTIPLICITIES IN THE COCHARACTERS OF SOME PI-ALGEBRAS

FRANCESCA BENANTI, ANTONIO GIAMBRUNO, AND IRINA SVIRIDOVA

(Communicated by Martin Lorenz)

Abstract. We consider associative PI-algebras over a field of characteristic zero. We study the asymptotic behavior of the sequence of multiplicities of the cocharacters for some significant classes of algebras. We also give a characterization of finitely generated algebras for which this behavior is linear or quadratic.

1. Introduction

Let F be a field of characteristic zero and $F\langle X\rangle$ the free associative algebra over F of countable rank with a set of generators $X = \{x_1, x_2, \ldots\}$. If A is an associative algebra over F, we denote by $Id(A) \subseteq F\langle X\rangle$ the T-ideal of all polynomial identities of A. It is well known that in characteristic zero every T-ideal is completely determined by its multilinear elements. Hence, if V_n is the space of all multilinear polynomials of degree n in $x_1, \ldots, x_n$, we study the sequence of spaces $V_n \cap Id(A)$, $n = 1, 2, \ldots$.

A useful approach to this study is through the representation theory of the symmetric group S_n. In fact, there is a natural action of S_n on V_n leaving $V_n \cap Id(A)$ invariant: if $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in V_n$, then one defines $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. This in turn makes $V_n(A) = V_n/V_n \cap Id(A)$ an S_n-module.

The S_n-character of $V_n(A)$, denoted $\chi_n(A)$, is called the n-th cocharacter of A or of $Id(A)$. By complete reducibility $\chi_n(A)$ decomposes into irreducibles and allows us to write $\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$, where χ_λ is the irreducible S_n-character associated to the partition λ of n and m_λ is the corresponding multiplicity. Through the sequence of cocharacters $\{\chi_n(A)\}_{n\geq 1}$ one can attach to A three numerical sequences. The first, called the sequence of codimensions, is given by

$$c_n(A) = \chi_n(A)(1) = \dim_F V_n/V_n \cap Id(A),$$

$n = 1, 2, \ldots$. The second sequence is

$$m_n(A) = \max_{\lambda\vdash n} m_\lambda,$$

Received by the editors March 22, 2002 and, in revised form, July 31, 2002 and October 30, 2002.

2000 *Mathematics Subject Classification.* Primary 16R10, 16P90.

Key words and phrases. Polynomial identities, multiplicities, codimensions, growth.

The first and the second authors were partially supported by MURST of Italy.

The third author was partially supported by the scientific program "The Universities of Russia".

$n = 1, 2, \ldots$ and we call it the sequence of multiplicities. The third, called the sequence of colengths, is

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda,$$

$n = 1, 2, \ldots$.

We are interested in the asymptotic behaviour of these three sequences and in their interrelations.

For the sequence of codimensions it was first proved by Regev in [24] that if A is a PI-algebra, then $c_n(A)$ is exponentially bounded. The sequence $c_n(A)$ or even its asymptotic behaviour has only been computed for some special algebras [6], [9], [18], [19], [23], [25]; it turns out that it is in general a very hard problem to determine the precise asymptotic behaviour of such a sequence.

It was recently proved in [10], [11] that for a PI-algebra A,

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is an integer; $\exp(A)$ is called the PI-exponent of the algebra A, and it has been computed for some classes of algebras [2], [7], [12], [25].

For the other sequences, the most important result is due to Berele and Regev who proved in [1] that for a PI-algebra A, the sequence $\{l_n(A)\}_{n\geq 1}$ is polynomial bounded. Hence $\{m_n(A)\}_{n\geq 1}$ is also polynomial bounded. Notice that if A is a nilpotent algebra, then for n large, $V_n(A)$ is the zero module. We then define for any nonnilpotent PI-algebra A,

$$\mathrm{mlt}(A) = \limsup_{n\to\infty} \log_n m_n(A)$$

and

$$\mathrm{col}(A) = \limsup_{n\to\infty} \log_n l_n(A).$$

Note that for any numerical sequence $p(n)$, if $b \cdot n^k \leq p(n) \leq a \cdot n^k$ for some constants $a, b > 0$, then $\limsup_{n\to\infty} \log_n(p(n)) = k$. Hence $\mathrm{mlt}(A)$ and $\mathrm{col}(A)$ actually capture the polynomial behaviour of the sequences $\{m_n(A)\}_{n\geq 1}$ and $\{l_n(A)\}_{n\geq 1}$ respectively. In this paper we find the precise value of $\mathrm{mlt}(A)$ for some significant minimal algebras A of small PI-exponent. Also, we characterize finitely generated algebras (or better the corresponding T-ideals of identities) for which $\mathrm{mlt}(A) = 1$, and we find precise relations between $\mathrm{mlt}(A)$ and $\exp(A)$.

2. Preliminaries

Throughout the paper we will denote by F a field of characteristic zero. Recall that an algebra A is a PI-algebra if it satisfies a nontrivial polynomial identity. If f is a polynomial identity on A we usually write $f \equiv 0$ in A. Let $Id(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ in } A\}$ be the ideal of identities of A. Recall that $Id(A)$ is a T-ideal, i.e., an ideal invariant under all endomorphisms of $F\langle X\rangle$. If $\mathcal{V}$ is a variety of associative algebras, $\mathcal{V}$ determines uniquely a T-ideal $I = Id(\mathcal{V})$. Also, if $\mathcal{V}$ is generated by the algebra A we write $\mathcal{V} = \mathrm{var}(A) = \mathrm{var}(I)$ and $I = Id(\mathcal{V}) = Id(A)$. In case $\mathcal{V} = \mathrm{var}(A) = \mathrm{var}(B)$, i.e., $Id(A) = Id(B)$, we also say that A and B are PI-equivalent.

Let G be the infinite-dimensional Grassmann algebra. Recall that G is the algebra generated by a countable set $\{e_1, e_2, \ldots\}$ subject to the conditions $e_ie_j = -e_je_i$ for all $i, j = 1, 2, \ldots$. Let $G = G_0 \oplus G_1$ be the natural $\mathbb{Z}_2$-grading on G where

G_0 and G_1 are the spaces generated by all monomials in the generators e_i of even and odd length, respectively.

Now, if $B = B_0 \oplus B_1$ is any $\mathbb{Z}_2$-graded algebra, then $G(B) = B_0 \otimes G_0 \oplus B_1 \otimes G_1$ is called the Grassmann envelope of B. The importance of such algebras is given by a well-known result of Kemer ([17, Theorem 2.3]) which states that for any proper variety $\mathcal{V}$, there exists a finite-dimensional superalgebra B such that $\mathcal{V} = \mathrm{var}(G(B))$.

Let $V_n = \mathrm{Span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ be the space of multilinear polynomials in $x_1, \ldots, x_n$. If A is a PI-algebra, let $c_n(A) = \dim_F V_n / V_n \cap Id(A)$ be the n-th codimension of A. As mentioned in the introduction, the PI-exponent of A is defined as $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$. In [10], [11] it was proved that $\exp(A)$ always exists and is a nonnegative integer; in that paper the authors also gave a constructive way for computing the exponent as follows: let B be a finite-dimensional $\mathbb{Z}_2$-graded algebra such that $\mathrm{var}(A) = \mathrm{var}(G(B))$ and suppose that F is algebraically closed. Let $B = B_1 \oplus \cdots \oplus B_k + J$ be the Wedderburn-Malcev decomposition of the algebra B where $B_1, \ldots, B_k$ are simple subalgebras and J is the Jacobson radical of B (see [4, Theorem 72.19]). It is also well known (see, for instance, [17]) that in such a decomposition we may take $B_1, \ldots, B_k$ to be stable under the $\mathbb{Z}_2$-grading. Then

$$\exp(A) = \max\nolimits_{i_1 < \cdots < i_t} \dim_F (B_{i_1} \oplus \cdots \oplus B_{i_t})$$

where $B_{i_1}, \ldots, B_{i_t}$ satisfy the condition $B_{i_1} J B_{i_2} J \cdots J B_{i_t} \neq 0$.

We remark that the codimensions of a PI-algebra do not change if we extend the base field (see, for instance, [10, Remark 1]).

As an example, that we shall use in the next section, we now compute the exponent of the algebra $UT_2(G_0, G) = \left(\begin{smallmatrix} G_0 & G \\ 0 & G \end{smallmatrix}\right)$. Notice that $UT_2(G_0, G)$ is the Grassmann envelope of the algebra $\left(\begin{smallmatrix} F & F\oplus tF \\ 0 & F\oplus tF \end{smallmatrix}\right)$, where $t^2 = 1$ with $\mathbb{Z}_2$-grading $\left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & tF \\ 0 & tF \end{smallmatrix}\right)$. As we remarked above we may assume F to be algebraically closed. Hence, since $J = (F \oplus tF)e_{12}$ and $Fe_{11} J (F \oplus tF) e_{22} \neq 0$, we get $\exp(UT_2(G_0, G)) = 3$.

3. Growth of the multiplicities of some PI-algebras

In this section we shall compute $\mathrm{mlt}(A)$ for some significant algebras.

In PI-theory an important role is played by the so-called minimal algebras of given PI-exponent (see [13] and [14]). Recall that an algebra A is minimal of PI-exponent $d \geq 2$ if $\exp(A) = d$ and $\exp(B) < d$ for all algebras B such that $Id(B) \supsetneqq Id(A)$.

If $\mathcal{V} = \mathrm{var}(A)$ is the variety generated by A we shall write $\exp(\mathcal{V}) = \exp(A)$ and, in case A is minimal of PI-exponent d, we shall say that $\mathcal{V}$ is minimal of exponent d.

A complete list of minimal algebras of given PI-exponent was determined (up to PI-equivalence) in [14] in case A is a finitely generated algebra (see also [13]). Moreover, in [12], by giving a characterization of algebras of PI-exponent > 2, the authors determined a list of algebras minimal of PI-exponent ≤ 4. We next describe these algebras and we compute for them the function $\mathrm{mlt}(A)$.

Let $UT_k(F)$ be the algebra of $k \times k$ upper triangular matrices over F. In [16] it was essentially proved that G and $UT_2(F)$ generate the only two minimal varieties of exponent 2. We shall see below that the algebras

$$UT_3(F), \quad UT_2(G_0, G) = \begin{pmatrix} G_0 & G \\ 0 & G \end{pmatrix} \text{ and } UT_2(G, G_0) = \begin{pmatrix} G & G \\ 0 & G_0 \end{pmatrix}$$

generate the only minimal varieties of exponent 3. We shall also see that $M_2(F)$, the algebra of 2×2 matrices over F, and $M_{1,1}(G) = \left(\begin{smallmatrix} G_0 & G_1 \\ G_1 & G_0 \end{smallmatrix}\right)$ are also minimal of PI-exponent 4.

If $f_1, \dots, f_n \in F\langle X\rangle$, let $\langle f_1, \dots, f_n\rangle_T$ denote the T-ideal of $F\langle X\rangle$ generated by the polynomials $f_1, \dots, f_n$.

In the next lemma we shall need some results on the proper polynomial identities of a PI-algebra. Recall that the space of multilinear proper polynomials in $x_1, \dots, x_n$ is the subspace P_n of V_n spanned by all products of Lie commutators of length ≥ 2. Then one has in a natural way the notion of proper polynomial identity for an algebra A (we refer to [5] for the basic properties of the proper identities). Let $Id^{pr}(A)$ be the space of proper identities of the algebra A. The S_n-action on V_n induces a structure of S_n-module on P_n and on $P_n(A) = P_n/P_n \cap Id^{pr}(A)$. Then one considers in a natural way the S_n-character of $P_n(A)$, denoted $\psi_n(A)$, which is called the n-th proper cocharacter of A. A result of Drensky [6, Theorem 2.6] gives the precise relation between the ordinary cocharacters and the proper cocharacters of any PI-algebra A: if $\psi_p(A) = \sum_{\lambda\vdash p} m_\lambda \chi_\lambda$, where χ_λ is the irreducible S_n-character associated to the partition $\lambda \vdash n$, then

$$\chi_n(A) = \sum_{p=0}^{n} \psi_p(A) \otimes \chi_{(n-p)} = \sum_{p=0}^{n}\sum_{\lambda\vdash p} m_\lambda(\chi_\lambda \otimes \chi_{(n-p)})$$

and the tensor product $\chi_\lambda \otimes \chi_{(n-p)}$ is computed according to Young's rule (see [15, Theorem 2.8.2]). We now apply this result in the following

Lemma 3.1. $\mathrm{mlt}(M_{1,1}(G)) = 1$.

Proof. It is well known (see [17, p. 24]) that $M_{1,1}(G)$ is PI-equivalent to $G\otimes G$. Popov in [22] proved that $Id(G\otimes G) = \langle [x_1, x_2, [x_3, x_4], x_5], [[x_1, x_2]^2, x_1]\rangle_T$, and he also described the S_n-module structure of the proper multilinear polynomial identities of $G\otimes G$. In particular, if $\psi_n(G\otimes G)$ denotes the n-th proper cocharacter of this algebra, then he proved the following decomposition [22, Theorem 7.1]:

$$\psi_n(G\otimes G) = \sum_{\lambda=(r,1^p)\vdash n} \chi_\lambda \; + \sum_{\lambda=(r,2^p,1^q)\vdash n} \chi_\lambda .$$

Now, by Drensky's result mentioned above and by applying Young's rule, we obtain that in the decomposition of the n-th cocharacter of A only partitions of the type $\lambda = (r, 1^p)$, $\lambda = (r, 2^p, 1^q)$, $\lambda = (r, s, 1^p)$, $\lambda = (r, s, 2^p, 1^q)$ appear. Moreover, for each such partition λ, $m_\lambda \leq t(n+1)$, for some constant $t \in \mathbb{Z}^+$. In particular, if $\lambda_0 = (2[\frac{n}{6}], [\frac{n}{6}], 2^{[\frac{n}{6}]}, 1^{n-5[\frac{n}{6}]})$, where $[a]$ denotes the integral part of a, then we have that $m_{\lambda_0} = t([\frac{n}{6}]+1)$. Therefore, it turns out that

$$t\frac{n}{6} < m_{\lambda_0} \leq m_n(M_{1,1}(G)) \leq t(n+1) \leq 2tn$$

and, so, $\mathrm{mlt}(M_{1,1}(G)) = \limsup_{n\to\infty} \log_n m_n(M_{1,1}(G)) = 1$. □

In the next lemmas we shall use a result of Berele and Regev which allows us to compute the n-th cocharacter of a product of T-ideals. The result is the following.

Theorem 3.2 ([3], Theorem 1.1)**.** *Let A, A_1, A_2 be PI-algebras such that $Id(A) = Id(A_1)Id(A_2)$. Then*

$$\chi_n(A) = \chi_n(A_1)+\chi_n(A_2)+\chi_{(1)}\otimes\sum_{j=0}^{n-1}\chi_j(A_1)\otimes\chi_{n-j-1}(A_2)-\sum_{j=0}^{n}\chi_j(A_1)\otimes\chi_{n-j}(A_2).$$

Lemma 3.3. $\mathrm{mlt}(UT_2(G_0,G)) = \mathrm{mlt}(UT_2(G,G_0)) = 1$.

Proof. It is easy to check that $UT_2(G_0,G)$ satisfies the identity $[x_1,x_2][x_3,x_4,x_5] \equiv 0$. Hence $\mathrm{var}(UT_2(G_0,G)) \subseteq \mathrm{var}(\langle[x_1,x_2][x_3,x_4,x_5]\rangle_T)$. Now, in [26] it was proved that the identity $[x_1,x_2][x_3,x_4,x_5] \equiv 0$ defines a minimal variety of exponent 3. As seen in the previous section, $\exp(UT_2(G_0,G)) = 3$; hence we obtain $\mathrm{var}(UT_2(G_0,G)) = \mathrm{var}(\langle[x_1,x_2][x_3,x_4,x_5]\rangle_T)$.

By applying Theorem 3.2, we obtain

$$\chi_n(UT_2(G_0,G)) = \sum_{\lambda=(r,s,1^t)\vdash n} m_\lambda\chi_\lambda,$$

where $m_{(n)} = 1$ and $m_\lambda = q_1(r-s+q_2) \leq q_1(n+q_2) \leq 2q_1n$, for some constants $q_1,q_2 \in \mathbb{Z}^+$. In particular, for $\lambda_0 = (2[\frac{n}{4}],[\frac{n}{4}],1^{n-3[\frac{n}{4}]})$, we obtain $m_{\lambda_0} = q_1([\frac{n}{4}]+q_2)$. As in the proof of Lemma 3.1, we obtain $\mathrm{mlt}(UT_2(G_0,G)) = 1$. Similarly, one can prove that $\mathrm{var}(UT_2(G,G_0)) = \mathrm{var}(\langle[x_1,x_2,x_3][x_4,x_5]\rangle_T)$ and, as above, it follows that $\mathrm{mlt}(UT_2(G,G_0)) = 1$. □

In the proof of the previous lemma we saw that $UT_2(G_0,G)$ and $UT_2(G,G_0)$ are minimal algebras of exponent 3. But then, by [12, Theorem 1], it follows that $UT_3(F)$, $UT_2(G_0,G)$ and $UT_2(G,G_0)$ are, up to PI-equivalence, the only minimal algebras of PI-exponent 3.

Next, we deal with the cases $A = M_2(F)$ and $A = UT_k(F)$. Notice first that from the definition of the exponent, $\exp(M_k(F)) = k^2$ and $\exp(UT_k(F)) = k$, for all $k \geq 1$. Moreover, by [7] both $M_k(F)$ and $UT_k(F)$ are minimal algebras.

Next, we compute the functions mlt(A) for $M_2(F)$ and $UT_k(F)$, $k \geq 1$.

Lemma 3.4. $\mathrm{mlt}(M_2(F)) = 3$.

Proof. The multiplicities in the cocharacter sequence of $M_2(F)$ were computed in [6], [9], [23]. An inspection of these multiplicities shows that for all n, $\max_{\mu\vdash n} m_\mu = m_\lambda$ for some $\lambda = (\lambda_1,\ldots,\lambda_4)$ with $\lambda_4 > 0$. Since by [5, Theorem 12.6.5] when $\lambda_4 > 0$, $m_\lambda = (\lambda_1-\lambda_2+1)(\lambda_2-\lambda_3+1)(\lambda_3-\lambda_4+1)$ it turns out that $m_\lambda \leq (n+1)^3 \leq 8n^3$. Now let $\lambda' \vdash n$ be the partition $\lambda' = (4[\frac{n}{10}],3[\frac{n}{10}],2[\frac{n}{10}],n-9[\frac{n}{10}])$. Then $m_{\lambda'} = ([\frac{n}{10}]+1)^3$ and this implies

$$\frac{n^3}{10^3} \leq m_{\lambda'} \leq 8n^3.$$

Hence $\mathrm{mlt}(M_2(F)) = 3$. □

Lemma 3.5. $\mathrm{mlt}(UT_k(F)) = \binom{k}{2}$.

Proof. It is well known (see [20]) that the T-ideal of identities of $UT_k(F)$ is generated by the polynomial $[x_1,x_2]\cdots[x_{2k-1},x_{2k}]$. Then

$$Id(UT_k(F)) = Id(UT_{k-1}(F))Id(F).$$

In particular, for $k=2$, $Id(UT_2(F)) = Id(F)Id(F)$. Then, by applying Theorem 3.2,

$$\chi_n(UT_2(F)) = \chi_n(F)+\chi_n(F)+\chi_{(1)}\otimes\sum_{j=0}^{n-1}\chi_j(F)\otimes\chi_{n-j-1}(F)-\sum_{j=0}^{n}\chi_j(F)\otimes\chi_{n-j}(F).$$

In this decomposition the irreducible character corresponding to the partition $\lambda=(n)$ appears in the terms

$$2\chi_{(n)}+\chi_{(1)}\otimes\sum_{j=0}^{n-1}\chi_{(j)}\otimes\chi_{(n-j-1)}-\sum_{j=0}^{n}\chi_{(j)}\otimes\chi_{(n-j)}$$

and its multiplicity is $m_{(n)} = 2+n-n-1 = 1$.

The irreducible character corresponding to $\lambda=(\lambda_1,\lambda_2)\vdash n$ appears in the terms

$$\chi_{(1)}\otimes\sum_{j=\lambda_2-1}^{\lambda_1}\chi_{(j)}\otimes\chi_{(n-j-1)}+\sum_{j=\lambda_2}^{\lambda_1-1}\chi_{(j)}\otimes\chi_{(n-j-1)}-\sum_{j=\lambda_2}^{\lambda_1}\chi_{(j)}\otimes\chi_{(n-j)}$$

and $m_{(\lambda_1,\lambda_2)} = [\lambda_1-(\lambda_2-1)+1]+[(\lambda_1-1)-\lambda_2+1]-[\lambda_1-\lambda_2+1] = \lambda_1-\lambda_2+1$.

The irreducible character corresponding to $\lambda=(\lambda_1,\lambda_2,\lambda_3)\vdash n$ appears only if $\lambda_3=1$ in the term

$$\chi_{(1)}\otimes\sum_{j=\lambda_2}^{\lambda_1}\chi_{(j)}\otimes\chi_{(n-j)}.$$

Then $m_{(\lambda_1,\lambda_2,1)} = \lambda_1-\lambda_2+1$.

Moreover, $m_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)} = 0$ since $\dim_{\mathrm{F}}(UT_2) = 3$.

Similarly, it is possible to calculate $\chi_n(UT_k(F))$ for $k>2$ and then to prove that the characters corresponding to partitions $\lambda=(\lambda_1,\dots,\lambda_k)\vdash n$ have multiplicity larger than those of any other shape and for any such λ,

$$m_\lambda = q\prod_{\substack{i<j\\ i,j=1,\dots,k}}(\lambda_i-\lambda_j+q_{ij}),$$

for some constants $q\in\mathbb{Q}^+$, $q_{ij}\in\mathbb{N}$. Hence for any $\lambda\vdash n$ and for large enough n,

$$m_\lambda \le q\prod_{\substack{i<j\\ i,j=1,\dots,k}}(n+q_{ij}) \le q2^d n^d,$$

where $d=\frac{k(k-1)}{2}=\binom{k}{2}$. Let $d'=\frac{k(k+1)}{2}$. If we now consider the partition $\lambda'=(k[\frac{n}{d'}],(k-1)[\frac{n}{d'}],\dots,n-(d'-1)[\frac{n}{d'}])$ we have that

$$m_{\lambda'} = q\prod_{\substack{i<j\\ i,j=1,\dots,k}}([\frac{n}{d'}]+q_{ij}) \ge \overline{q}n^d,$$

with $\overline{q}\in\mathbb{Q}^+$. Hence

$$\overline{q}n^d \le m_{\lambda'} \le m_n(A) \le q2^d n^d$$

and $m(UT_k(F)) = d = \binom{k}{2}$. □

We now state and prove a remark that will be used throughout the paper.

Remark 3.6. Let A and B be PI-algebras. If $B\in\mathrm{var}(A)$, then $\exp(B)\le\exp(A)$, $\mathrm{mlt}(B)\le\mathrm{mlt}(A)$ and $\mathrm{col}(B)\le\mathrm{col}(A)$.

Proof. Since $Id(B) \supseteq Id(A)$, it follows that for all $n \geq 1$, $V_n/V_n \cap Id(B)$ is isomorphic to a quotient module of $V_n/V_n \cap Id(A)$. Thus $c_n(B) \leq c_n(A)$ and, thus, $\exp(B) \leq \exp(A)$. Moreover, by complete reducibility, if $\chi_n(B) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ and $\chi_n(A) = \sum_{\lambda \vdash n} m'_\lambda \chi_\lambda$, then we must have $m_\lambda \leq m'_\lambda$ for all $\lambda \vdash n$. Thus $\mathrm{mlt}(B) \leq \mathrm{mlt}(A)$ and $\mathrm{col}(B) \leq \mathrm{col}(A)$ follows. □

In the next corollary we combine Lemmas 3.1, 3.3, 3.4, and 3.5 with a characterization of varieties of exponent $d \geq 2$ given in [12].

Corollary 3.7. *For any PI-algebra A, if* $\exp(A) > 2$*, then* $\mathrm{mlt}(A) \geq 1$.

Proof. Since $\exp(A) > 2$, by [12, Theorem 1] one of the algebras $UT_2(G_0, G)$, $UT_2(G, G_0)$, $UT_3(F)$, $M_2(F)$, $M_{1,1}(G)$ lies in $\mathrm{var}(A)$. Call B such an algebra. Since by Lemmas 3.1, 3.3, 3.4, and 3.5, $\mathrm{mlt}(UT_2(G_0, G)) = \mathrm{mlt}(UT_2(G, G_0)) = \mathrm{mlt}(M_{1,1}(G)) = 1$ and $\mathrm{mlt}(UT_3(F)) = \mathrm{mlt}(M_2(F)) = 3$, by the previous remark we obtain that $\mathrm{mlt}(A) \geq 1$. □

Corollary 3.8. $\mathrm{col}(UT_2(F)) = 2$.

Proof. From the definition of colength and the proof of Lemma 3.5, we have

$$\begin{aligned} l_n(UT_2(F)) &= \sum_{\lambda \vdash n} m_\lambda \\ &= m_{(n)} + \sum_{\lambda_1+\lambda_2=n} m_{(\lambda_1,\lambda_2)} + \sum_{\lambda_1+\lambda_2=n-1} m_{(\lambda_1,\lambda_2,1)} \\ &= 1 + \sum_{\lambda_1+\lambda_2=n} (\lambda_1 - \lambda_2 + 1) + \sum_{\lambda_1+\lambda_2=n-1} (\lambda_1 - \lambda_2 + 1) \\ &= 1 + \sum_{\lambda_1=n/2}^{n} (\lambda_1 - (n - \lambda_1) + 1) + \sum_{\lambda_1=n/2}^{n} (\lambda_1 - (n - 1 - \lambda_1) + 1) \\ &= 1 + \frac{1}{4}(n+2)^2 + \frac{1}{4}(n+4)(n+2) = \frac{1}{2}n^2 + \frac{5}{2}n + 4. \end{aligned}$$

Hence $\mathrm{col}(UT_2(F)) = 2$. □

4. Growth of the multiplicities of finitely generated algebras

It is well known (see, for instance, [10, Remark 1] for its proof) that if A is a PI-algebra over a field F and $K \supseteq F$ is an extension field, then $Id(A) \otimes_F K = Id(A \otimes_F K)$. Hence we shall assume from now on that F is an algebraically closed field of characteristic zero. Also, throughout this section we shall assume that A is a finitely generated PI-algebra over F,

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda \tag{1}$$

is its n-th cocharacter and

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda \tag{2}$$

its colength.

Recently in [8] Drensky and Kassabov obtained a characterization of a finitely generated algebra satisfying a nonmatrix identity, i.e., an identity not satisfied by 2×2 matrices over F. Here we state a special case of their result which will be essential in the proof of the next theorem. Recall that if $\lambda \vdash n$, we write $\lambda = (\lambda_1, \ldots, \lambda_s)$.

Theorem 4.1 ([8]). *Let A be a finitely generated algebra satisfying a nonmatrix polynomial identity. Then the multiplicities m_λ, in the n-th cocharacter of A, are bounded by a linear function of n if and only if there exists a positive integer q such that $m_\lambda = 0$ whenever $\lambda_3 > q$.*

For $\lambda \vdash n$ let $h(\lambda)$ be the number of nonzero parts of λ. Hence $h(\lambda)$ is the height of the Young diagram corresponding to λ. Also, if $n \geq m$ and $\lambda \vdash n, \mu \vdash m$, we write $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all i. Recall that the group algebra FS_n decomposes into the direct sum of its minimal two-sided ideals $FS_n = \bigoplus_{\lambda \vdash n} I_\lambda$ where I_λ is the ideal corresponding to the partition λ.

Another result needed in the next theorem is the following lemma essentially proved in [10].

Lemma 4.2. *Let A be a finitely generated PI-algebra and let $\exp(A) = d$. Then there exists a constant $q \geq 0$ such that*

$$\chi_n(A) = \sum_{\substack{\lambda \vdash n \\ n-(\lambda_1+\cdots+\lambda_d) \leq q}} m_\lambda \chi_\lambda .$$

Proof. By a theorem of Kemer (see [17, Theorem 2.3]) there exists a finite-dimensional algebra B such that $Id(A) = Id(B)$. Hence, we may assume that A is a finite-dimensional algebra over the algebraically closed field F. Let $\dim_F A = r$. It is well known (see, for instance, [10, Lemma 1]) that

$$\chi_n(A) = \sum_{\substack{\lambda \vdash n \\ h(\lambda) \leq r}} m_\lambda \chi_\lambda , \tag{4.1}$$

i.e., if χ_λ participates in $\chi_n(A)$ with nonzero multiplicity, then the diagram of λ lies in a strip of height r. Now, by [11, Corollary 1], since $\exp(A) = d$, there exists an integer $k \geq 0$ such that $\bigoplus_{\lambda \geq (k^d)} I_\lambda \subseteq Id(A)$. This says that in ((4.1)), $m_\lambda = 0$ whenever $\lambda \geq (k^d)$. Also, notice that by the characterization of the exponent, $d \leq r$ (see [10]). By combining the above results, we obtain $m_\lambda = 0$ whenever $\lambda_{d+1} \geq k$. It follows that if we set $(r-d)k = q$, then $m_\lambda = 0$ whenever $n-(\lambda_1+\cdots+\lambda_d) \leq q$. □

Theorem 4.3. *For a finitely generated PI-algebra A the following are equivalent:*

(1) $\mathrm{mlt}(A) \leq 1$,
(2) $\exp(A) \leq 2$,
(3) $UT_3(F),\ M_2(F) \notin \mathrm{var}(A)$.

Proof. Since by Lemmas 3.4 and 3.5, $\mathrm{mlt}(M_2(F)) = \mathrm{mlt}(UT_3(F)) = 3$, it is clear that if $\mathrm{mlt}(A) \leq 1$, then $UT_3(F),\ M_2(F) \notin \mathrm{var}(A)$.

Suppose that $UT_3(F)$ and $M_2(F)$ do not belong to the variety generated by A. As we remarked above, we may assume that A is a finite-dimensional algebra over F and that F is algebraically closed. Let $A = A_1 \oplus \cdots \oplus A_t + J$ be the Wedderburn-Malcev decomposition of A, where $J = J(A)$ is the Jacobson radical of A and $A_i \cong M_{n_i}(F)$ is a simple subalgebra of A for $i = 1, \ldots, t$.

If there exists j such that $A_j \cong M_{n_j}(F)$ with $n_j \geq 2$, then $M_2(F) \subseteq M_{n_j}(F) \subseteq A$ and we obtain a contradiction. Hence $A_i \cong F$ for all $i = 1, \ldots, t$. Suppose that there exist $A_i,\ A_l,\ A_m$ distinct such that $A_i J A_l J A_m \neq 0$. Then we can choose

$j_1, j_2 \in J$ such that $1_1 j_1 1_2 j_2 1_3 \neq 0$, where 1_1, 1_2, 1_3 are the unit elements of A_i, A_l, A_m, respectively. Now set $u_{11} = 1_1$, $u_{22} = 1_2$, $u_{33} = 1_3$, $u_{12} = 1_1 j_1 1_2$, $u_{23} = 1_2 j_2 1_3$ and $u_{13} = 1_1 j_1 1_2 j_2 1_3$. If B is the subalgebra of A generated by these elements, it is clear that $B \cong UT_3(F)$; thus $UT_3(F) \in \mathrm{var}(A)$, contrary to the assumption. Hence $A = F_1 \oplus \cdots \oplus F_t + J$ and $F_i J F_l J F_m = 0$ for all i, l, m distinct. By the characterization of the exponent given in [11], it follows that $\exp(A) \leq 2$. Notice that since J is a nilpotent ideal, say $J^k = 0$, and for all $a, b \in A$, $[a, b] \in J$, it follows that A satisfies the identity $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}] \equiv 0$. Hence A satisfies a nonmatrix identity and $\exp(A) \leq 2$; but then, from Lemma 4.2 and Theorem 4.1 it follows that $\mathrm{mlt}(A) \leq 1$. □

As a consequence of the previous theorem, Lemmas 3.4 and 3.5 we get

Corollary 4.4. *Let A be a finitely generated PI-algebra. Then the following properties are equivalent:*

(1) $\mathrm{mlt}(A) = 1$,
(2) $\exp(A) = 2$,
(3) $UT_3(F)$, $M_2(F) \notin \mathrm{var}(A)$ *and* $UT_2(F) \in \mathrm{var}\ (A)$.

Proof. Observe that $\mathrm{mlt}(A) = 0$ means that for all n, the multiplicities m_λ are bounded by a constant. This is equivalent to $UT_2(F) \notin \mathrm{var}\ (A)$. Then for any finitely generated PI-algebra A, $G \notin \mathrm{var}\ (A)$. Hence by a result of Kemer, $G, UT_2(F) \notin \mathrm{var}\ (A)$ is equivalent to $\exp(A) \leq 1$. We get the desired conclusion. □

Notice that in the previous corollary we actually proved that $\mathrm{mlt}(A) = 0$, $\exp(A) = 1$ and $UT_2(F) \notin \mathrm{var}\ (A)$ are equivalent properties.

Next, we show that there exists no finitely generated PI-algebra such that $\mathrm{mlt}(A) = 2$.

Theorem 4.5. *For any finitely generated PI-algebra A, $\mathrm{mlt}(A) \neq 2$.*

Proof. Let $\mathrm{mlt}\ (A) = 2$. As we remarked above we may assume that A is a finite-dimensional algebra and let $A = A_1 \oplus \cdots \oplus A_t + J$ be its Wedderburn-Malcev decomposition. If for some $i \geq 1$, $A_i \cong M_{n_i}(F)$ and $n_i \geq 2$, then $M_2(F) \subseteq \mathrm{var}(A)$ and $3 = \mathrm{mtl}(M_2(F)) \leq \mathrm{mtl}(A) = 2$ leads to a contradiction. Hence $A = F_1 \oplus \cdots \oplus F_t + J$. If $F_i J F_l J F_m \neq 0$, then, as in the proof of Theorem 4.3, we can construct a subalgebra B of A isomorphic to $UT_3(F)$ and by Lemma 3.5 we get a contradiction. Consequently $M_2(F)$ and $UT_3(F)$ do not belong to the variety generated by A and by Theorem 4.3, $\mathrm{mtl}(A) \leq 1$, a contradiction. □

We next study finitely generated PI-algebras for which $\mathrm{col}(A) \leq 1$.

Theorem 4.6. *Let A be a finitely generated PI-algebra. Then* $\mathrm{col}(A) \leq 1$ *if and only if* $UT_2(F) \notin \mathrm{var}(A)$.

Proof. If $\mathrm{col}(A) \leq 1$, then, by Remark 3.6 and Corollary 3.8, $UT_2(F) \notin \mathrm{var}(A)$. Conversely, if $UT_2(F) \notin \mathrm{var}(A)$, then by [21, Theorem 1] in the decomposition of the n-th cocharacter $\chi_n(A)$ of A, $m_\lambda \leq q$, for some constant q; also there exists a constant M such that

$$\chi_n(A) = \sum_{\substack{\lambda \vdash n \\ n - \lambda_1 \leq M}} m_\lambda \chi_\lambda.$$

Hence we can write

$$l_n(A) = \sum_{\substack{\lambda \vdash n \\ n-\lambda_1 \le M}} m_\lambda \le n\overline{q},$$

for some constant $\overline{q}$ and $\mathrm{col}(A) \le 1$. □

We finish by relating the functions $\mathrm{col}(A)$ and $\exp(A)$, at least when they take small values.

Remark 4.7. For any finitely generated PI-algebra A we have $\mathrm{col}(A) \le 1$ if and only if $\exp(A) = 1$.

Proof. By Theorem 4.6, $\mathrm{col}(A) \le 1$ if and only if $UT_2(F) \notin \mathrm{var}(A)$. By [21, Theorem 1] we have that $UT_2(F) \notin \mathrm{var}(A)$ if and only if $\exp(A) = 1$ and the property is proved. □

References

1. A. Berele and A. Regev, *Applications of hook Young diagrams to P.I. algebras*, J. Algebra **82** (1983), 559–567. MR **84g:**16012
2. A. Berele and A. Regev, *On the codimensions of the verbally prime P.I. algebras*, Israel J. Math. **91** (1995), 239–247. MR **96g:**16028
3. A. Berele and A. Regev, *Codimensions of products and of intersections of verbally prime T-ideals*, Israel J. Math. **103** (1998), 17–28. MR **99b:**16037
4. C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, John Wiley and Sons, New York, 1962. MR **26:**2519
5. V. Drensky, *Free Algebras and PI-Algebras*, Springer-Verlag, Berlin-Heidelberg-Singapore, 1999. MR **2000j:**16002
6. V. S. Drensky, *Codimensions of T-ideals and Hilbert series of relatively free algebras*, J. Algebra **91** (1984), 1–17. MR **86b:**16010
7. V. S. Drensky, *Extremal varieties of algebras. I, II*, Serdica **13** (1987), 320–332; **14** (1988), 20–27 (Russian). MR **90e:**08010, MR **90e:**08011
8. V. S. Drensky and M. Kassabov, *Growth of nonmatrix varieties of algebras*, preprint.
9. E. Formanek, *Invariants and the ring of generic matrices*, J. Algebra **89** (1984), 178–223. MR **85g:**15031
10. A. Giambruno and M. Zaicev, *On codimension growth of finitely generated associative algebras*, Adv. Math. **140** (1998), 145–155. MR **99k:**16049
11. A. Giambruno and M. Zaicev, *Exponential codimension growth of P.I. algebras: an exact estimate*, Adv. Math. **142** (1999), 221–243. MR **2000a:**16048
12. A. Giambruno and M. Zaicev, *A characterization of varieties of associative algebras of exponent two*, Serdica Math. J. **26** (2000), 245–252. MR **2001i:**16041
13. A. Giambruno and M. Zaicev, *Minimal varieties of algebras of exponential growth*, Electron. Res. Announc. Amer. Math. Soc. **6** (2000), 40–44 (electronic). MR **2001e:**16037
14. A. Giambruno and M. Zaicev, *Minimal varieties of algebras of exponential growth*, Adv. Math. **174** (2003), 33–42.
15. G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, Vol. 16, Addison-Wesley, London, 1981. MR **83k:**20003
16. A. R. Kemer, *The Spechtian nature of T-ideals whose codimensions have power growth*, Sibirskii Matematicheskii **19** (1978), 54–69 (Russian). Translation: Siberian Math. J. **19** (1978), 37–48. MR **57:**6070
17. A. R. Kemer, *Ideals of Identities of Associative Algebras*, Translations of Math. Monographs **87**, Amer. Math. Soc., Providence, RI, 1991. MR **92f:**16031
18. D. Krakowski and A. Regev, *The polynomial identities of the Grassmann algebra*, Trans. Amer. Math. Soc. **181** (1973), 429–438. MR **48:**4005
19. V. N. Latyshev, *The complexity of nonmatrix varieties of associative algebras. I, II*, Algebra i Logika **16** (1977), 149–183, 184–199 (Russian). Translation: Algebra and Logic **16** (1977), 48–122, 122–133. MR **58:**27695

20. Yu. N. Mal'cev, *A basis for the identities of the algebra of upper triangular matrices*, Algebra i Logika **10** (1971), 393–400 (Russian). Translation: Algebra and Logic **10** (1971), 242–247. MR **46:**3561
21. S. P. Mishchenko, A. Regev and M. V. Zaicev, *A characterization of PI-algebras with bounded multiplicities of the cocharacters*, J. Algebra **219** (1999), 356–368. MR **2000e:**16026
22. A. P. Popov, *Identities of the tensor square of a Grassmann algebra*, Algebra i Logika **21** (1982), 442–471 (Russian). Translation: Algebra and Logic **21** (1982), 296–316. MR **85j:**15018
23. C. Procesi, *Computing with* 2×2 *matrices*, J. Algebra **87** (1984), 342–359. MR **86g:**16022
24. A. Regev, *Existence of identities in* $A \otimes B$, Israel J. Math. **11** (1972), 131–152. MR **47:**3442
25. A. Regev, *Codimensions and trace codimensions of matrices are asymtotically equal*, Israel J. Math. **47** (1984), 246-250. MR **85j:**16024
26. A. N. Stoyanova-Venkova, *Some lattices of varieties of associative algebras defined by identities of the fifth degree*, C. R. Acad. Bulgare Sci. **35** (1982), 865–868. MR **84b:**08007

Dipartimento di Matematica ed Applicazioni, Università di Palermo, via Archirafi 34, 90123 Palermo, Italy
E-mail address: fbenanti@math.unipa.it

Dipartimento di Matematica ed Applicazioni, Università di Palermo, via Archirafi 34, 90123 Palermo, Italy
E-mail address: a.giambruno@unipa.it

Department of Algebra and Geometric Computations, Faculty of Mathematics and Mechanics, Ulyanovsk State University, Ulyanovsk 4327000, Russia
E-mail address: sviridova_i@rambler.ru