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INTEGRATION BY PARTS ON THE BROWNIAN MEANDER

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ABSTRACT. We prove infinite-dimensional integration by parts formulae for the laws of the Brownian Meander, of the Bessel Bridge of dimension 3 between $z, z' \geq 0$ and of the Brownian Motion on the set of all paths taking values greater than or equal to a nonpositive constant. We give applications to SPDEs with reflection.

1. Introduction and main result

Let $(B(t))_{t\in[0,1]}$, $(M(t))_{t\in[0,1]}$ and $(b_z^{z'}(t))_{t\in[0,1]}$ be, respectively, a Brownian motion, a Brownian Meander and a Bessel Bridge of dimension 3 between z and $z' \geq 0$ over the time interval $[0,1]$; see [6]. All the processes here are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The aim of this paper is to prove the infinite-dimensional integration by parts formulae (1.2)–(1.4) below for the laws of $(M(t))_{t\in[0,1]}$, $(b_z^{z'}(t))_{t\in[0,1]}$ and $(B(t))_{t\in[0,1]}$ on the convex set of paths $K_\alpha := \{x : [0,1] \mapsto [-\alpha, \infty) \text{ continuous}\}$, $\alpha \geq 0$. In the last section, we also give applications of (1.2)–(1.4) to SPDEs with reflection. In the spirit of [7], we identify the reflecting terms in the SPDEs as families of additive functionals whose Revuz measures are equal to the boundary measures of (1.2)–(1.4).

We introduce the space $E := \{u \in C([0,1]) : u(0) = 0\}$ and define the operator $V : [0,1] \times E \times E \mapsto E$:

$$(1.1)\quad V(r,f,g)(t) := -\sqrt{r}\, f(1) + \sqrt{r}\, 1_{[0,r]}(t)\, f\left(1 - \frac{t}{r}\right) + \sqrt{1-r}\, 1_{[r,1]}(t)\, g\left(\frac{t-r}{1-r}\right), \quad t \in [0,1].$$

Then, the main result of the paper is the following:

Theorem 1.1. *Let $H := L^2(0,1)$, endowed with the canonical scalar product $\langle \cdot, \cdot \rangle$. For all maps $\varphi : H \mapsto \mathbb{R}$, bounded with bounded continuous Fréchet differential, and for all $h \in C^2([0,1])$, we denote by $h'' \in H$ the second derivative of h and by $\partial_h \varphi$, the directional derivative of φ along h.*

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Let (b_0^0, M) *be an independent couple of processes. Then for all* $h \in C_c^2((0,1])$*:*

$$\mathbb{E}\left[\partial_h \varphi(M)\right] = -\,\mathbb{E}\left[\varphi(M)\langle M, h''\rangle\right] - \int_0^1 dr\, h(r)\, \frac{1}{\sqrt{2\pi r^3(1-r)}}\mathbb{E}\left[\varphi\left(V\left(r, b_0^0, M\right)\right)\right]. \tag{1.2}$$

Now let (b_0^z, M) *be an independent couple for all* $z \geq 0$*. Then for all* $h \in C_c^2((0,1])$*:*

$$\mathbb{E}\left[\partial_h \varphi(B) 1_{K_\alpha}(B)\right] = -\,\mathbb{E}\left[\varphi(B)\langle B, h''\rangle 1_{K_\alpha}(B)\right] - \int_0^1 dr\, h(r)\, \frac{\alpha \exp\left(-\frac{\alpha^2}{2r}\right)}{\pi\sqrt{r^3(1-r)}}\mathbb{E}\left[\varphi\left(V\left(r, b_0^{\alpha/\sqrt{r}}, M\right)\right)\right]. \tag{1.3}$$

Finally, let $(b_0^z, \hat{b}_0^{z'})$ *be an independent couple for all* $z, z' \geq 0$*, and assume by convention that* $0/(1-\exp(0)) = 1$*. Then for all* $k \in C_c^2(0,1)$*,*

$$\begin{aligned}\mathbb{E}\left[\partial_k \varphi(b_z^{z'})\right] &= -\,\mathbb{E}\left[\varphi(b_z^{z'})\langle b_z^{z'}, k''\rangle\right] \\ &\quad - \int_0^1 k(r)\, \frac{\exp\left(-\frac{z^2}{2}\frac{r}{1-r} - \frac{(z')^2}{2}\frac{1-r}{r} - 2zz'\right)}{\sqrt{2\pi r^3(1-r)^3}}\, \frac{2zz'}{1-\exp(-2zz')}\cdot \\ &\quad \cdot\, \mathbb{E}\left[\varphi\left(z + V\left(r, b_0^{z/\sqrt{r}}, \hat{b}_0^{z'/\sqrt{1-r}}\right)\right)\right] dr.\end{aligned} \tag{1.4}$$

Integration by parts formulae are important tools in infinite-dimensional analysis; see, e.g., [4]. However, the measures that are usually considered in the literature are supported by suitable topological vector spaces and give strictly positive measure to every open set. The supports of the measures that are considered in (1.2)–(1.4) are instead infinite-dimensional closed convex non-affine sets. From this point of view, formulae (1.2)–(1.4) generalize the integration by parts formulae (1)–(2) of [7], which are, in fact, particular cases of (1.4) with $z = z'$.

Consider, for instance, (1.2). The law of the Brownian Meander M is naturally supported by the set of continuous nonnegative paths x on $[0,1]$ that satisfy $x(0) = 0$ and $x(\tau) > 0$ for all $\tau \in (0,1]$. In the second term of the right-hand side of (1.2), the law of $V\left(r, b_0^0, M\right)$, $r \in (0,1]$, is supported by the set of continuous paths x on $[0,1]$ that satisfy $x(0) = x(r) = 0$ and $x(\tau) > 0$ for all $\tau \in (0,r) \cup (r,1]$. Therefore, the last term in the right-hand side of (1.2) is singular with respect to the law of M and can be interpreted as a boundary term. Notice also that the function $r \mapsto [r^3(1-r)]^{-1/2}$ is not integrable in any neighbourhood of 0; the constraints $M(0) = 0$ and $M \geq 0$ produce a singularity for r close to 0 of the boundary measure, which turns out to have infinite total mass.

Analogous considerations hold for (1.3) and (1.4). The supports of the laws of $V\left(r, b_0^{\alpha/\sqrt{r}}, M\right)$ and respectively $z + V\left(r, b_0^{z/\sqrt{r}}, \hat{b}_0^{z'/\sqrt{1-r}}\right)$ are equal to the supports of the reference measures, i.e., the law of B on K_α and respectively the law of $b_z^{z'}$, with the further condition that $\{x(r) = 0\}$. If $z = 0$ or $z' = 0$ in (1.4), then the function of r that appears in the boundary term is not integrable near 0 or respectively 1, and the boundary measure has again infinite total mass. On the other hand, for $z, z' > 0$ in (1.4) and for all $\alpha > 0$ in (1.3), the singularities disappear and the boundary measures have finite total mass.

2. Proof of Theorem 1.1

We recall the following facts:

Theorem 2.1 (Denisov [1]). *Let $(U, M, \hat{M})$ be an independent triple such that $U : \Omega \to [0,1]$ has the arcsine law and M and $\hat{M}$ are two standard Brownian Meanders. Then $V(U, M, \hat{M}) \overset{d}{=} B$, where V is defined by* (1.1).

Theorem 2.2 (Imhof [3]). *The Brownian Meander M is equal in law to b_0^ρ, where ρ is independent of $(b_0^z)_{z\geq 0}$ and has the Rayleigh density*

$$\mathbb{P}(\rho \in dx) \,=\, x\, e^{-\frac{1}{2}x^2}\, dx, \quad x > 0.$$

Theorem 2.3 (Durrett, Iglehart and Miller [2]). *The law of B conditioned to K_α, $\alpha > 0$, converges weakly in E to the law of M as $\alpha \downarrow 0$.*

Without loss of generality, we can assume that

$$h \geq 0, \quad \varphi \geq 0.$$

In particular, $K_\alpha \subseteq K_\alpha - th$ for all $t \geq 0$. Recall that $\partial_h \varphi(x) = \lim_{t\downarrow 0}(\varphi(x) - \varphi(x - th))/t$. By the Cameron-Martin theorem,

$$\begin{aligned}(2.1)\quad \frac{1}{t}\mathbb{E}\left[1_{K_\alpha}(B)(\varphi(B) - \varphi(B-th))\right] \,=\, &-\frac{1}{t}\mathbb{E}\left[1_{(K_\alpha - th)\setminus K_\alpha}(B)\varphi(B)\right] \\ &+ \frac{1}{t}\mathbb{E}\left[1_{K_\alpha - th}(B)\varphi(B)\,\left(1 - \exp\left(-\frac{1}{2}\|th'\|^2 + t\langle B, h''\rangle\right)\right)\right].\end{aligned}$$

By Theorem 2.1, we have

$$\begin{aligned}\mathbb{E}\left[1_{(K_\alpha - th)\setminus K_\alpha}(B)\varphi(B)\right] \,&=\, \mathbb{E}\left[1_{(K_\alpha - th)\setminus K_\alpha}\left(V(U,M,\hat{M})\right)\,\varphi\left(V(U,M,\hat{M})\right)\right] \\ &= \int_0^1 dr\, \frac{1}{\pi\sqrt{r(1-r)}}\,\mathbb{E}\left[1_{(K_\alpha - th)\setminus K_\alpha}\left(V(r,M,\hat{M})\right)\,\varphi\left(V(r,M,\hat{M})\right)\right].\end{aligned}$$

Let $n \in \mathbb{N}$, $c_n \geq c_{n-1} \geq \cdots \geq c_1 \geq c_0 := 0$, $\{I_1, \ldots, I_n\}$ a Borel partition of $[0,1]$ and $I_0 := \emptyset$, and set

$$h_i := \sum_{j=1}^{n} (c_j \wedge c_i)\, 1_{I_j}, \qquad i = 0, \ldots, n.$$

The key point is the following: for $i = 1, \ldots, n$, since $h_i \geq h_{i-1}$, and $h_i = h_{i-1}$ on $\bigcup_{j=0}^{i-1} I_j$, then for all $r \in (0,1)$,

$$V(r, M, \hat{M}) \,\in\, (K_\alpha - th_i)\setminus(K_\alpha - th_{i-1}) \quad \Longleftrightarrow$$

$$V(r, M, \hat{M}) \,\in\, K_\alpha - th_i, \; r \in \bigcup_{j=i}^{n} I_j \;\text{ and } \sqrt{r}M(1) \in\, [\alpha + tc_{i-1}, \alpha + tc_i).$$

Indeed, $V(r,M,\hat M)$ attains its minimum $-\sqrt{r}M(1)$ only at time r. Then we obtain for all $t\geq 0$ and $i=1,\ldots,n$,

$$\begin{aligned}
&\mathbb{E}\left[1_{(K_\alpha-th_i)\setminus K_\alpha}(B)\varphi(B)\right]\\
&\quad=\int_0^1\frac{dr}{\pi\sqrt{r(1-r)}}\,\mathbb{E}\left[1_{(K_\alpha-th_i)\setminus K_\alpha}\left(V(r,M,\hat M)\right)\,\varphi\left(V(r,M,\hat M)\right)\right]\\
&\quad=\int_0^1\frac{dr}{\pi\sqrt{r(1-r)}}\mathbb{E}\left[\varphi\cdot\left[1_{(K_\alpha-th_{i-1})\setminus K_\alpha\cup(K_\alpha-th_i)\setminus(K_\alpha-th_{i-1})}\right]\left(V(r,M,\hat M)\right)\right]\\
&\quad=\int_0^1\frac{dr}{\pi\sqrt{r(1-r)}}\,\mathbb{E}\left[\varphi\cdot 1_{(K_\alpha-th_{i-1})\setminus K_\alpha}\left(V(r,M,\hat M)\right)\right]\,dr\\
&\qquad+\int_{\cup_{j=i}^n I_j}\frac{dr}{\pi\sqrt{r(1-r)}}\,\mathbb{E}\left[\varphi\cdot 1_{(K_\alpha-th_i)}\left(V(r,M,\hat M)\right)1_{[c_{i-1},c_i)}\left(\tfrac{\sqrt{r}M(1)-\alpha}{t}\right)\right].
\end{aligned}$$

Proceeding by induction on n we obtain by Theorem 2.2,

$$\begin{aligned}
&\mathbb{E}\left[1_{(K_\alpha-th_n)\setminus K_\alpha}(B)\varphi(B)\right]\\
&\quad=\sum_{i=1}^n\int_{\cup_{j=i}^n I_j}\frac{dr}{\pi\sqrt{r(1-r)}}\mathbb{E}\left[\varphi\cdot 1_{(K_\alpha-th_i)}\left(V(r,M,\hat M)\right)1_{[c_{i-1},c_i)}\left(\tfrac{\sqrt{r}M(1)-\alpha}{t}\right)\right]\\
&\quad=\sum_{i=1}^n\sum_{j=i}^n\int_{I_j}\frac{dr}{\pi\sqrt{r(1-r)}}\int_{\frac{\alpha+tc_{i-1}}{\sqrt{r}}}^{\frac{\alpha+tc_i}{\sqrt{r}}}y\,e^{-\frac{y^2}{2}}\,\mathbb{E}\left[\varphi\cdot 1_{(K_\alpha-th_i)}\left(V(r,b_0^y,\hat M)\right)\right]dy.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}\,\mathbb{E}\left[1_{(K_\alpha-th_n)\setminus K_\alpha}(B)\varphi(B)\right]\\
&\qquad=\sum_{i=1}^n\sum_{j=i}^n(c_i-c_{i-1})\int_{I_j}dr\frac{\alpha e^{-\frac{\alpha^2}{2r}}}{\pi\sqrt{r^3(1-r)}}\mathbb{E}\left[\varphi\cdot 1_{(K_\alpha-th_i)}\left(V(r,b_0^y,\hat M)\right)\right]\\
&\qquad=\sum_{j=1}^n\sum_{i=1}^j(c_i-c_{i-1})\int_{I_j}dr\frac{\alpha e^{-\frac{\alpha^2}{2r}}}{\pi\sqrt{r^3(1-r)}}\mathbb{E}\left[\varphi\cdot 1_{(K_\alpha)}\left(V(r,b_0^{\alpha/\sqrt{r}},\hat M)\right)\right]\\
&\qquad=\int_0^1 dr\left(\sum_{j=1}^n c_j\,1_{I_j}(r)\right)\frac{\alpha e^{-\frac{\alpha^2}{2r}}}{\pi\sqrt{r^3(1-r)}}\mathbb{E}\left[\varphi\left(V(r,b_0^{\alpha/\sqrt{r}},\hat M)\right)\right]\\
&\qquad=\int_0^1\frac{dr}{\pi\sqrt{r^3(1-r)}}\,\alpha\,e^{-\frac{\alpha^2}{2r}}\,h_n(r)\,\mathbb{E}\left[\varphi\cdot 1_{(K_\alpha-th_i)}\left(V(r,b_0^{y/\sqrt{r}},\hat M)\right)\right].
\end{aligned}$$

Now set $I_i:=h^{-1}([(i-1)/n,i/n))$, $i\in\mathbb{N}$,

$$f_n:=\sum_{i=1}^\infty\frac{i-1}{n}\,1_{I_i},\quad g_n:=\sum_{i=1}^\infty\frac{i}{n}\,1_{I_i},$$

where both sums are finite, since h is bounded. Then $f_n\leq h\leq g_n$, f_n and g_n converge uniformly on $[0,1]$ to h as $n\to\infty$, and $K_\alpha-tf_n\subseteq K_\alpha-th\subseteq K_\alpha-tg_n$,

$t \geq 0$. Therefore, since $\varphi \geq 0$, we have

$$\begin{aligned}
&\int_0^1 dr\, f_n(r)\, \frac{\alpha \exp\left(-\frac{\alpha^2}{2r}\right)}{\pi\sqrt{r^3(1-r)}} \mathbb{E}\left[\varphi\left(V(r, b_0^{\alpha/\sqrt{r}}, M)\right)\right] \\
&\quad \leq \liminf_{t\downarrow 0} \frac{1}{t} \mathbb{E}\left[1_{(K_\alpha - th)\setminus K_\alpha}(B)\varphi(B)\right] \\
&\quad \leq \limsup_{t\downarrow 0} \frac{1}{t} \mathbb{E}\left[1_{(K_\alpha - th)\setminus K_\alpha}(B)\varphi(B)\right] \\
&\quad \leq \int_0^1 dr\, g_n(r)\, \frac{\alpha \exp\left(-\frac{\alpha^2}{2r}\right)}{\pi\sqrt{r^3(1-r)}} \mathbb{E}\left[\varphi\left(V(r, b_0^{\alpha/\sqrt{r}}, M)\right)\right]
\end{aligned}$$

and by (2.1),

$$\begin{aligned}
&\mathbb{E}\left[\partial_h \varphi(B) 1_{K_\alpha}(B)\right] \\
&\quad = \lim_{t\downarrow 0} \frac{1}{t} \mathbb{E}\left[1_{K_\alpha}(B)(\varphi(B) - \varphi(B - th))\right] \\
&\quad = -\mathbb{E}\left[\varphi(B)\langle B, h''\rangle 1_{K_\alpha}(B)\right] - \int_0^1 dr \frac{h(r)\alpha e^{-\frac{\alpha^2}{2r}}}{\pi\sqrt{r^3(1-r)}} \mathbb{E}\left[\varphi\left(V(r, b_0^{\alpha/\sqrt{r}}, M)\right)\right]
\end{aligned}$$

so that (1.3) is proved. In order to prove (1.2), we recall that

$$\begin{aligned}
\mathbb{P}(B \in K_\alpha) &= \mathbb{P}\left(\inf_{[0,1]} B \geq -\alpha\right) = \mathbb{P}(|B_1| \leq \alpha) \\
&= \sqrt{\frac{2}{\pi}} \int_0^\alpha e^{-\frac{y^2}{2}} dy \sim \sqrt{\frac{2}{\pi}}\, \alpha
\end{aligned}$$

as $\alpha \to 0$. We divide (1.3) by $\mathbb{P}(B \in K_\alpha)$. Since h has compact support in $(0,1)$ and the laws of $e^r_{0,\alpha}$ are weakly continuous in $\alpha \geq 0$, we can let $\alpha \downarrow 0$ in the last term of (1.3). Then we apply Theorem 2.3 to the first and second term in (1.3) and the proof of (1.2) is complete.

Now we prove (1.4). We choose $h = k \in C_c^2(0,1)$. Since k has compact support in $(0,1)$, we have for all $z \geq 0$,

$$\langle B, k''\rangle = \langle B + z, k''\rangle - z(k'(1) - k'(0)) = \langle B + z, k''\rangle.$$

Therefore, we can write (1.3) in the following way:

$$\begin{aligned}
\text{(2.2)}\quad \mathbb{E}\left[\partial_k \varphi(B+z) 1_{K_0}(B+z)\right] &= -\mathbb{E}\left[\varphi(B+z)\langle B+z, k''\rangle 1_{K_0}(B+z)\right] \\
&\quad - \int_0^1 dr\, k(r)\, \frac{z \exp\left(-\frac{z^2}{2r}\right)}{\pi\sqrt{r^3(1-r)}} \mathbb{E}\left[\varphi\left(z + V(r, b_0^{z/\sqrt{r}}, M)\right)\right].
\end{aligned}$$

For all $\epsilon > 0$, let $\gamma^\epsilon \in C^1(\mathbb{R})$ such that

$$0 \leq \gamma^\epsilon \leq 1,\ \gamma^\epsilon \equiv 1 \text{ over } [-\epsilon, \epsilon],\ \gamma^\epsilon \equiv 0 \text{ over } \mathbb{R}\setminus[-\epsilon - \epsilon^2, \epsilon + \epsilon^2].$$

We set for $z, z' \geq 0$ and $\epsilon > 0$,

$$\psi^\epsilon : E \mapsto \mathbb{R}, \quad \psi^\epsilon(x) := \frac{\gamma^\epsilon(x(1) - z')}{\mathbb{E}\left[\gamma^\epsilon(B(1) + z - z') \cdot 1_{K_0}(B+z)\right]}.$$

Then we have

$$\partial_k \left[\varphi \cdot \psi^\epsilon\right](x) \;=\; \left[\partial_k \varphi\right](x) \cdot \psi^\epsilon(x) \;+\; \frac{\left[\gamma^\epsilon\right]'(x(1)-z') \cdot k(1)}{\mathbb{E}\left[\gamma^\epsilon(B(1)+z-z') \cdot 1_{K_0}(B+z)\right]}.$$

Since k has compact support in $(0,1)$, we have $k(1)=0$, and therefore,

$$\begin{aligned}
&\mathbb{E}\left[\left[(\partial_k\varphi)\cdot\psi^\epsilon\right](B+z)1_{K_0}(B+z)\right] \\
&\quad = -\,\mathbb{E}\left[\varphi(B+z)\langle B+z, k''\rangle 1_{K_0}(B+z)\right] \\
&\quad\quad - \int_0^1 dr\, k(r)\, \frac{z\exp\left(-\frac{z^2}{2r}\right)}{\pi\sqrt{r^3(1-r)}} \mathbb{E}\left[\left[\varphi\cdot\psi^\epsilon\right]\left(z+V(r, b_0^{z/\sqrt{r}}, M)\right)\right].
\end{aligned} \tag{2.3}$$

Notice now that

$$\begin{aligned}
&\psi^\epsilon\left(z+V(r,b_0^{z/\sqrt{r}},M)\right) \\
&\qquad = \frac{\gamma^\epsilon(\sqrt{1-r}M(1)-z')}{\mathbb{E}\left[\gamma^\epsilon(\sqrt{1-r}M(1)-z')\right]} \cdot \frac{\mathbb{E}\left[\gamma^\epsilon(\sqrt{1-r}M(1)-z')\right]}{\mathbb{E}\left[\gamma^\epsilon(B(1)+z-z')\cdot 1_{K_0}(B+z)\right]}.
\end{aligned}$$

By the Reflection Principle (see, e.g., III.3.14-(4) in [6]), for all $f:\mathbb{R}\mapsto\mathbb{R}$ bounded and Borel we have

$$\mathbb{E}\left[f(B(1)+z)\cdot 1_{K_0}(B+z)\right] \;=\; \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} f(y)\left(e^{-(z-y)^2/2} - e^{-(z+y)^2/2}\right)dy.$$

Moreover, by Theorem 2.2, $M(1)$ has the Rayleigh density $xe^{-\frac{1}{2}x^2}dx$ on $\{x>0\}$. Then

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{\mathbb{E}\left[\gamma^\epsilon(\sqrt{1-r}M(1)-z')\right]}{\mathbb{E}\left[\gamma^\epsilon(B(1)+z-z')\cdot 1_{K_0}(B+z)\right]} \\
&= \lim_{\epsilon\downarrow 0} \frac{\sqrt{2\pi}\int_{(z'-\epsilon)/\sqrt{1-r}}^{(z'+\epsilon)/\sqrt{1-r}} ye^{-y^2/2}dy}{\int_{z'-\epsilon}^{z'+\epsilon}\left(e^{-(z-y)^2/2}-e^{-(z+y)^2/2}\right)dy} \;=\; \frac{\sqrt{2\pi}z'\exp\left(-\frac{(z')^2}{2(1-r)}+\frac{(z-z')^2}{2}\right)}{(1-r)(1-\exp(-2zz'))}.
\end{aligned}$$

Therefore, by Theorem 2.2, letting $\epsilon\downarrow 0$ in (2.3) for all $z,z'>0$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\partial_k\varphi(b_z^{z'})\right] \;=\; &-\;\mathbb{E}\left[\varphi(b_z^{z'})\langle b_z^{z'}, k''\rangle\right] \\
&- \int_0^1 dr\, k(r)\, \frac{\exp\left(-\frac{z^2}{2}\frac{r}{1-r} - \frac{(z')^2}{2}\frac{1-r}{r} - 2zz'\right)}{\sqrt{2\pi r^3(1-r)^3}}\, \frac{2zz'}{1-\exp(-2zz')} \cdot \\
&\cdot\, \mathbb{E}\left[\varphi\left(z+V\left(r, b_0^{z/\sqrt{r}}, \hat{b}_0^{z'/\sqrt{1-r}}\right)\right)\right]
\end{aligned}$$

and (1.4) is proved. □

3. SPDEs with reflection

Arguing as in [7], the formulae (1.2)–(1.4) find applications to SPDEs with reflection (see [5]). Let $\{W(t,\theta): t\geq 0,\ \theta\in[0,1]\}$ be a Brownian Sheet.

Theorem 3.1. *For all* $\alpha \geq 0$ *let* $E_\alpha := \{x : [0,1] \mapsto [-\alpha, \infty)$ *continuous,* $x(0) = 0\}$, *and for all* $x \in E_\alpha$ *let* (v_α, ζ_α) *be the solution of the following SPDE with reflection:*

$$\begin{cases} \dfrac{\partial v_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v_\alpha}{\partial \theta^2} + \dfrac{\partial^2 W}{\partial t \partial \theta} + \zeta_\alpha, \\ \\ v_\alpha(0,\theta) = x(\theta), \ v_\alpha(t,0) = \dfrac{\partial v_\alpha}{\partial \theta}(t,1) = 0, \\ \\ v_\alpha + \alpha \geq 0, \ d\zeta_\alpha \geq 0, \ \int (v_\alpha + \alpha)\, d\zeta_\alpha = 0, \end{cases} \tag{3.1}$$

where $v_\alpha := [0,\infty) \times [0,1] \mapsto \mathbb{R}$ *is continuous and* ζ_α *is a locally finite positive measure on* $[0,\infty) \times (0,1)$. *We write* $v_\alpha = v_\alpha(t,\theta,x)$, $t \geq 0$, $\theta \in [0,1]$, $x \in E_\alpha$. *Also, let* m_α *be the law of* B *conditioned to* E_α *for* $\alpha > 0$ *and the law of* M *for* $\alpha = 0$. *Then for all* $\alpha \geq 0$:

1. *The process* $(v_\alpha(t,\cdot,x))_{t \geq 0, x \in E_\alpha}$ *is symmetric with respect to its unique invariant probability measure* m_α.
2. *The process* $(v_\alpha(t,\cdot,x))_{t \geq 0, x \in E_\alpha}$ *is the Markov process properly associated with the symmetric Dirichlet Form* $(\Gamma^\alpha, D(\Gamma^\alpha))$ *in* $L^2(m_\alpha)$, *and the closure of the bilinear form*
$$C_b^1(L^2(0,1)) \ni \varphi, \psi \mapsto \frac{1}{2}\int_{E_\alpha} \langle \nabla\varphi, \nabla\psi \rangle \, dm_\alpha.$$
3. *For all Borel sets* $I \subset\subset (0,1)$, *the process* $t \mapsto \zeta_\alpha([0,t] \times I)$ *is an Additive Functional of* v_α, *with Revuz-measure*
$$\alpha > 0: \qquad \int_{E_\alpha} \mathbb{E}\left[\int_0^1 \varphi(v_\alpha(t,\cdot,x))\zeta_\alpha(dt,I)\right] m_\alpha(dx)$$
$$= \frac{1}{2}\,\frac{\alpha}{\int_0^\alpha e^{-\frac{y^2}{2}}\,dy}\int_I \frac{\exp\left(-\frac{\alpha^2}{2r}\right)}{\sqrt{2\pi r^3(1-r)}}\,\mathbb{E}\left[\varphi\left(V\left(r, b_0^{\alpha/\sqrt{r}}, M\right)\right)\right] dr,$$
$$\alpha = 0: \qquad \int_{K_0} \mathbb{E}\left[\int_0^1 \varphi(v_0(t,\cdot,x))\zeta_0(dt,I)\right] m_0(dx)$$
$$= \frac{1}{2}\int_I \frac{1}{\sqrt{2\pi r^3(1-r)}}\,\mathbb{E}\left[\varphi\left(V\left(r, b_0^0, M\right)\right)\right] dr,$$
for $\varphi : L^2(0,1) \mapsto \mathbb{R}$ *Borel and bounded.*
4. *For all* $x \in E_\alpha$, *there exist a random Borel set* $S_\alpha \subset \mathbb{R}_+$ *and a map* $r_\alpha : S_\alpha \mapsto (0,1)$, *such that a.s.*
$$\zeta_\alpha\left([\mathbb{R}_+ \times (0,1)] \setminus \{(s, r_\alpha(s)) : s \in S_\alpha\}\right) = 0$$
$$\forall\, s \in S_\alpha: \quad v_\alpha(s, r_\alpha(s)) = 0, \quad v_\alpha(s,\theta) > 0 \quad \forall\, \theta \in (0,1)\setminus\{r_\alpha(s)\}.$$
5. *Let* δ_r *denote the Dirac mass at* $r \in (0,1)$. *For all* $x \in E_\alpha$, *we have a.s. on* $[0,\infty) \times (0,1)$,
$$\zeta_\alpha(ds, d\theta) = \delta_{r_\alpha(s)}(d\theta)\,\zeta_\alpha(ds,(0,1)).$$

Theorem 3.2. *For all* $z, z' \geq 0$ *let* $D_z^{z'} := \{x : [0,1] \mapsto [0,\infty)$ *continuous,* $x(0) = z, x(1) = z'\}$, *and for all* $x \in D_z^{z'}$ *let* $(w_z^{z'}, \gamma_z^{z'})$ *be the solution of the following*

SPDE with reflection:

$$\begin{cases} \dfrac{\partial w_z^{z'}}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 w_z^{z'}}{\partial \theta^2} + \dfrac{\partial^2 W}{\partial t \partial \theta} + \gamma_z^{z'}, \\ w_z^{z'}(0,\theta) = x(\theta),\ w_z^{z'}(t,0) = z,\ w_z^{z'}(t,1) = z', \\ w_z^{z'} \geq 0,\ d\gamma_z^{z'} \geq 0,\ \int w_z^{z'}\, d\gamma_z^{z'} = 0, \end{cases} \tag{3.2}$$

where $w_z^{z'} := [0,\infty) \times [0,1] \mapsto \mathbb{R}$ *is continuous and* $\gamma_z^{z'}$ *is a locally finite positive measure on* $[0,\infty) \times (0,1)$*. We write* $w_z^{z'} = w_z^{z'}(t,\theta,x)$, $t \geq 0$, $\theta \in [0,1]$, $x \in D_z^{z'}$*. Also, let* $n_z^{z'}$ *be the law of* $b_z^{z'}$*. Then for all* $z, z' \geq 0$*:*

1. *The process* $(w_z^{z'}(t,\cdot,x))_{t\geq 0, x \in D_z^{z'}}$ *is symmetric with respect to its unique invariant probability measure* $n_z^{z'}$*.*
2. *The process* $(w_z^{z'}(t,\cdot,x))_{t\geq 0, x \in D_z^{z'}}$ *is the Markov process properly associated with the symmetric Dirichlet Form* $(\Gamma^{z,z'}, D(\Gamma^{z,z'}))$ *in* $L^2(n_z^{z'})$*, and the closure of the bilinear form:*
$$C_b^1(L^2(0,1)) \ni \varphi, \psi \mapsto \frac{1}{2}\int_{D_z^{z'}} \langle \nabla\varphi, \nabla\psi \rangle \, dn_z^{z'}.$$
3. *For all Borel sets* $I \subset\subset (0,1)$*, the process* $t \mapsto \gamma_z^{z'}([0,t] \times I)$ *is an Additive Functional of* $w_z^{z'}$*, with Revuz-measure*
$$\int_{D_z^{z'}} \mathbb{E}\left[\int_0^1 \varphi(w_z^{z'}(t,\cdot,x))\gamma_z^{z'}(dt,I)\right] n_z^{z'}(dx)$$
$$= \frac{1}{2}\,\frac{2zz'}{1-\exp(-2zz')} \int_I \frac{\exp\left(-\frac{z^2}{2}\frac{r}{1-r} - \frac{(z')^2}{2}\frac{1-r}{r} - 2zz'\right)}{\sqrt{2\pi r^3(1-r)^3}} \cdot$$
$$\cdot\, \mathbb{E}\left[\varphi\left(z + V\left(r, b_0^{z/\sqrt{r}}, \hat{b}_0^{z'/\sqrt{1-r}}\right)\right)\right] dr,$$
for $\varphi : L^2(0,1) \mapsto \mathbb{R}$ *Borel and bounded.*
4. *For all* $x \in D_z^{z'}$*, there exist a random Borel set* $S_z^{z'} \subset \mathbb{R}_+$ *and a map* $r_z^{z'} : S_z^{z'} \mapsto (0,1)$*, such that a.s.*
$$\gamma_z^{z'}\left([\mathbb{R}_+ \times (0,1)] \setminus \{(s, r_z^{z'}(s)) : s \in S_z^{z'}\}\right) = 0$$
$$\forall\, s \in S_z^{z'} : \quad w_z^{z'}(s, r_z^{z'}(s)) \,=\, 0, \quad w_z^{z'}(s,\theta) \,>\, 0 \quad \forall\, \theta \in (0,1)\setminus\{r_z^{z'}(s)\}.$$
5. *Let* δ_r *denote the Dirac mass at* $r \in (0,1)$*. For all* $x \in D_z^{z'}$*, we have a.s. on* $[0,\infty) \times (0,1)$*:*
$$\gamma_z^{z'}(ds, d\theta) \,=\, \delta_{r_z^{z'}(s)}(d\theta)\, \gamma_z^{z'}(ds, (0,1)).$$

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