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THE BACKWARD SHIFT ON THE SPACE OF CAUCHY TRANSFORMS

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ABSTRACT. This note examines the subspaces of the space of Cauchy transforms of measures on the unit circle that are invariant under the backward shift operator $f \to z^{-1}(f - f(0))$. We examine this question when the space of Cauchy transforms is endowed with both the norm and weak* topologies.

1. INTRODUCTION AND PRELIMINARIES

In this note, we will examine the invariant subspaces of the backward shift operator

$$(Bf)(z) = \frac{f(z) - f(0)}{z}$$

on the space of Cauchy transforms $\mathcal{K}$ consisting of analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that take the form

$$(K\mu)(z) := \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \overline{\zeta} z}. \tag{1.1}$$

Here $\mu \in M$, the space of finite Borel measures on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

By an "invariant subspace" of $\mathcal{K}$ we will mean a closed linear manifold $\mathcal{M} \subseteq \mathcal{K}$ for which $B\mathcal{M} \subseteq \mathcal{M}$. In using the word "closed", there are two topologies on $\mathcal{K}$ to consider here. The first is the norm topology. For $f \in \mathcal{K}$, let

$$M_f := \{\nu \in M : f = K\nu\}$$

be the set of "representing measures" for f. Define the norm of an element $f \in \mathcal{K}$ by

$$\|f\| := \inf\{\|\nu\| : \nu \in M_f\},$$

where $\|\nu\|$ denotes the total variation norm of the measure ν. The notation $(\mathcal{K}, \|\cdot\|)$ will denote the space $\mathcal{K}$ endowed with the above norm topology. It is well known that $(\mathcal{K}, \|\cdot\|)$ is isometrically isomorphic to the quotient space $M/\overline{H^1_0}$ and is a non-separable Banach space. Here H^1 is the usual Hardy space of the disk [9] and H^1_0 are the functions in H^1 that vanish at the origin. $\overline{H^1_0}$ is regarded as a subspace of M in the natural way as $\{\overline{f} dm : f \in H^1_0\}$ where $dm = |d\zeta|/2\pi$ is normalized Lebesgue measure on the circle. The second topology on $\mathcal{K}$ is the weak* topology

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that arises by identifying the dual space of the disk algebra A (analytic functions on $\mathbb{D}$ that have continuous extensions to $\mathbb{D}^-$) with $\mathcal{K}$ via the pairing

$$\langle f, K\mu\rangle = \int_{\mathbb{T}} \overline{f}\, d\mu, \quad f \in A, \mu \in M.$$

By the F. and M. Riesz theorem [9, p. 41], if $\mu_1, \mu_2 \in M_{K\mu}$, then $d\mu_1 - d\mu_2 = \overline{h} dm$, where $h \in H^1_0$. Thus the above pairing is independent of the representing measure μ. We will use the notation $(\mathcal{K}, *)$ to denote the space $\mathcal{K}$ endowed with the weak* topology. One can show that $(\mathcal{K}, *)$ is separable. Furthermore, every weak* closed subspace of $\mathcal{K}$ is norm closed. See [4], [5], and [6] for a review of these basic facts about $\mathcal{K}$. In this paper, we examine the B-invariant subspaces of $(\mathcal{K}, *)$ and $(\mathcal{K}, \|\cdot\|)$.

To put our results in perspective, we mention some known results about the B-invariant subspaces for other spaces of analytic functions. For example, by Beurling's theorem [9, p. 114], the B-invariant subspaces of the classical Hardy space H^2 all take the form $(\vartheta H^2)^{\perp}$, where ϑ is an inner function. Moreover [8] (see also [6]), f belongs to $(\vartheta H^2)^{\perp}$ if and only if there is a function $G_f \in N^+(\mathbb{D}_e)$ that vanishes at infinity such that

$$\lim_{r \to 1^-} \frac{f}{\vartheta}(r\zeta) = \lim_{r \to 1^-} G_f(\zeta/r) \tag{1.2}$$

for m-almost every $\zeta \in \mathbb{T}$. Here $\mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-$ and $G_f \in N^+(\mathbb{D}_e)$ means $G_f(1/z) \in N^+$ (the Smirnov class of $\mathbb{D}$ [9, p. 25]). The function G_f is called a "pseudocontinuation"[1] of f. If

$$\sigma(\vartheta) := \{z \in \mathbb{D}^- : \varliminf_{\lambda \to z} |\vartheta(\lambda)| = 0\},$$

then, by basic properties of inner functions [11, pp. 68 and 69], ϑ has an analytic continuation to $\widehat{\mathbb{C}} \setminus \sigma(\vartheta)^*$, where $\sigma(\vartheta)^* := \{z \in \widehat{\mathbb{C}} : 1/\overline{z} \in \sigma(\vartheta)\}$. In fact, every $f \in (\vartheta H^2)^{\perp}$ has an analytic continuation to $\widehat{\mathbb{C}} \setminus \sigma(\vartheta)^*$ [8].

For the Bergman space L^2_a (analytic functions f on $\mathbb{D}$ such that $f \in L^2(dx\, dy)$) a theorem of Richter and Sundberg [14] says that every B-invariant subspace takes the form $\mathcal{M}_g := \{f \in L^2_a : f \perp z^n g \ \forall n \in \mathbb{N} \cup \{0\}\}$ for some g in the Dirichlet space (i.e., $g' \in L^2_a$). Here we equate the dual of L^2_a with the Dirichlet space via the "Cauchy" dual pairing

$$\lim_{r \to 1^-} \int f(r\zeta)\overline{g(r\zeta)}\, dm(\zeta).$$

Furthermore, (i) $g\mathcal{M}_g \subseteq H^p$ for all $0 < p < 1$, (ii) for every $f \in \mathcal{M}_g$, the meromorphic function f/ϑ_g (where ϑ_g is the inner factor of g) has a pseudocontinuation as in (1.2), (iii) every $f \in \mathcal{M}_g$ has an analytic continuation to $\widehat{\mathbb{C}} \setminus \sigma(g)^*$. Moreover [2], if g is "sufficiently smooth", then $g\mathcal{M}_g \subseteq H^1$ and $f \in L^2_a$ belongs to $\mathcal{M}_g$ if and only if (a) $fg \in H^1$ and (b) f/ϑ_g has pseudocontinuation as in (1.2). For certain L^p Bergman spaces, the function g can always be chosen to be "sufficiently smooth"; so in this case we have a complete characterization of the B-invariant subspaces. Our purpose here is to get similar-looking results for the space $(\mathcal{K}, *)$ (which can be gleaned from results of Korenblum [13]) and to examine the more difficult problem of characterizing the B-invariant subspaces of $(\mathcal{K}, \|\cdot\|)$.

[1] If h is meromorphic on $\mathbb{D}$ and H is meromorphic on $\mathbb{D}_e$ and the nontangential boundary values of h and H exist and are equal m-almost everywhere, then h and H are "pseudocontinuations" of each other. See [15] for more details.

2. The main results

For a B-invariant subspace $\mathcal{M}$ of $(\mathcal{K}, *)$ let

$$\mathcal{M}_\perp = \{f \in A : \langle f, K\mu\rangle = 0 \text{ for all } K\mu \in \mathcal{M}\}$$

be the pre-annihilator of $\mathcal{M}$. $\mathcal{M}_\perp$ is a norm closed subspace of the disk algebra A. A straightforward calculation shows that

$$\langle f, K\mu\rangle = \int_{\mathbb{T}} \overline{f} d\mu = \lim_{r\to 1^-} \int_{\mathbb{T}} \overline{f(\zeta)}(K\mu)(r\zeta)\, dm(\zeta) = \lim_{r\to 1^-} \sum_{n=0}^{\infty} \overline{\widehat{f}(n)}\widehat{\mu}(n) r^n \tag{2.1}$$

and $\langle f, BK\mu\rangle = \left\langle f, K(\overline{\zeta} d\mu)\right\rangle = \langle zf, K\mu\rangle$. Thus $z\mathcal{M}_\perp \subseteq \mathcal{M}_\perp$ since $B\mathcal{M} \subseteq \mathcal{M}$. Since A is a Banach algebra and polynomials are dense in A [11, p. 17], $\mathcal{M}_\perp$ is an ideal of A. A theorem of Rudin [16] (see also [11, p. 85]) says the following.

Theorem 2.2 (Rudin)**.** *Let I be a norm closed ideal of the disk algebra A. Then there is a closed set $E \subseteq \mathbb{T}$ of Lebesgue measure zero and an inner function ϑ with $\sigma(\vartheta) \cap \mathbb{T} \subseteq E$ such that*

$$I = I(\vartheta, E) := \{f \in A : f/\vartheta \in H^\infty, f|E = 0\}.$$

Furthermore, given a set $E \subseteq \mathbb{T}$ of Lebesgue measure zero and an inner ϑ with $\sigma(\vartheta) \cap \mathbb{T} \subseteq E$, there is an outer function $F \in A$ whose zero set is equal to E and such that $g := \vartheta F$ generates $I(\vartheta, E)$ in the sense that the smallest norm closed ideal of A containing g is equal to $I(\vartheta, E)$.

To describe $\mathcal{M}$, we need (via the Hahn-Banach theorem) to describe the set

$$(\mathcal{M}_\perp)^\perp = I(\vartheta, E)^\perp := \{f \in \mathcal{K} : \langle h, f\rangle = 0 \text{ for all } h \in I(\vartheta, E)\},$$

or equivalently, the set $\{f \in \mathcal{K} : \langle z^n g, f\rangle = 0\, \forall\, n \in \mathbb{N} \cup \{0\}\}$. Korenblum [13] proved the following.

Theorem 2.3 (Korenblum)**.** *If $K\mu \perp I(\vartheta, E)$, then $K\mu$ has an analytic continuation to the set $\widehat{\mathbb{C}} \setminus (\sigma(\vartheta)^* \cup E)$.*

In the process of proving our main theorem (Theorem 2.5), we will give an alternate proof of Korenblum's theorem. Any measure $\mu \in M$ can be decomposed uniquely as

$$d\mu = \phi dm + d\mu_s, \tag{2.4}$$

where $\phi \in L^1(m)$ and $\mu_s \perp m$. Our main theorem describes $I(\vartheta, E)^\perp$.

Theorem 2.5. *For $\mu \in M$, $K\mu \perp I(\vartheta, E)$ if and only if*

1. *the support of μ_s is contained in E;*
2. *$K\mu/\vartheta$ has an analytic continuation across $\mathbb{T} \setminus E$ to a function $F \in N^+(\mathbb{D}_e)$ with $F(\infty) = 0$.*

By the F. and M. Riesz theorem, every measure $\nu \in M_f$ $(f \in \mathcal{K})$ has the same singular part. Thus in condition (1), there is only one singular part to consider.

In H^2, the B-invariant subspace $(\vartheta H^2)^\perp$ is singly generated by the vector $f = B\vartheta$. This next corollary is the analogue of this for $(\mathcal{K}, *)$.

Corollary 2.6. *$I(\vartheta, E)^\perp = \bigvee\{B^n f : n \in \mathbb{N} \cup \{0\}\}$, where $f = B(K\mu)$ for $d\mu = \vartheta dm + d\mu_s$ and $\mu_s \perp m$ with support equal to E.*

Here $\bigvee$ is the closed linear span in $(\mathcal{K}, *)$. This next corollary mimics what happens in the Bergman space setting. By a classical result of Smirnov [9, p. 39], $\mathcal{K} \subseteq H^p$ for all $0 < p < 1$, and so if our B-invariant subspace $\mathcal{M} \subseteq \mathcal{K}$ has the property that $\mathcal{M}_\perp$ is generated by f, i.e., $\mathcal{M}_\perp$ is the closed linear span (in A) of $z^n f$ $(n \in \mathbb{N} \cup \{0\})$, then certainly $f\mathcal{M} \subseteq H^p$ for all $0 < p < 1$. If f is sufficiently smooth, we get the stronger condition $f\mathcal{M} \subseteq H^1$ and even a bit more.

Theorem 2.7. *Suppose $f \in A$ with $f' \in H^\infty$. Let $E = f^{-1}(\{0\}) \cap \mathbb{T}$, and let ϑ_f be the inner factor of f. Then $K\mu \perp z^n f$ for all $n \in \mathbb{N} \cup \{0\}$ if and only if*

(1) *$fK\mu \in H^1$;*
(2) *$K\mu/\vartheta_f$ has an analytic continuation across $\mathbb{T}\backslash E$ to a function $F \in N^+(\mathbb{D}_e)$ with $F(\infty) = 0$.*

If $f \in A$ with $f' \in H^\infty$, then the boundary zero set E of f satisfies the so-called Carleson condition: If (I_n) is the sequence of arcs contiguous to E on the circle, then $\sum_n |I_n| \log |I_n| > -\infty$. Thus, by Theorem 2.2, not every B-invariant subspace of $(\mathcal{K}, *)$ is singly generated by such an f.

Comments about the B-invariant subspaces of $(\mathcal{K}, \|\cdot\|)$ appear at the end of this note.

3. The proofs

Proposition 3.1. *Suppose ϑF is a generator for $I(\vartheta, E)$ and $d\mu = \phi dm + d\mu_s$ as in (2.4). Then $K\mu \perp z^n \vartheta F$ for all $n \in \mathbb{N} \cup \{0\}$ if and only if $\phi \in \vartheta \overline{H^1_0}$ and μ_s is supported in E.*

Proof. Suppose $K\mu \perp z^n \vartheta F$ for all $n \in \mathbb{N} \cup \{0\}$. Then, by (2.1),

$$\int_{\mathbb{T}} \overline{\zeta^n \vartheta F}(\phi dm + d\mu_s) = 0 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

From the F. and M. Riesz theorem, $\overline{\vartheta F} d\mu_s$ is the zero measure (and so μ_s is supported in E) and $\overline{\vartheta F}\phi = \overline{h} \in \overline{H^1_0}$. However, $\phi\overline{\vartheta} = \overline{h/F} \in \overline{N^+}$ and has $L^1(m)$ boundary values, and so $\phi\overline{\vartheta} \in \overline{H^1_0}$ [9, p. 28]. The converse is obvious.

Proof of Theorem 2.5. We start by proving a somewhat weaker result: $K\mu \perp I(E, \vartheta)$ if and only if μ_s is supported in E and $K\mu/\vartheta$ has a pseudocontinuation across $\mathbb{T}$ belonging to $N^+(\mathbb{D}_e)$ and vanishing at infinity. Indeed, suppose $K\mu \perp I(\vartheta, E)$. By Proposition 3.1 we can assume μ takes the form

$$d\mu = \phi dm + d\mu_s, \quad \phi\overline{\vartheta} = \overline{k} \in \overline{H^1_0}, \ \operatorname{supp}(\mu_s) \subseteq E.$$

Since $\overline{k} \in \overline{H^1_0}$, then $\overline{k}(1/\overline{z})$ belongs to $H^1(\mathbb{D}_e)$ and vanishes at infinity. The inner function ϑ is defined on $\mathbb{D}_e$ by $\vartheta(z) = 1/\overline{\vartheta(1/\overline{z})}$. The function

$$\widehat{\mu}(z) := \int \frac{d\mu(\zeta)}{1 - \overline{\zeta} z}, \ \ z \in \mathbb{D}_e$$

belongs to $H^p(\mathbb{D}_e)$ for all $0 < p < 1$ [9, p. 39] and so the function

$$T_{\mu,\vartheta}(z) := \overline{k}(1/\overline{z}) + \frac{\widehat{\mu}(z)}{\vartheta(z)}, \ \ z \in \mathbb{D}_e \tag{3.3}$$

belongs to $N^+(\mathbb{D}_e)$ and vanishes at infinity. By Fatou's jump theorem[2], the boundary function for $T_{\mu,\vartheta}$ is

$$\frac{\phi}{\vartheta}(\zeta) + \frac{(K\mu)(\zeta) - \phi(\zeta)}{\vartheta(\zeta)} = \frac{K\mu}{\vartheta}(\zeta)$$

for m-almost every $\zeta \in \mathbb{T}$. Thus $T_{\mu,\vartheta}$ is the pseudocontinuation of $K\mu/\vartheta$ of the desired type.

Conversely, suppose $d\mu = \phi dm + d\mu_s$, where $\phi \in L^1(m)$ and μ_s is supported in E, and $K\mu/\vartheta$ has a pseudocontinuation $G \in N^+(\mathbb{D}_e)$ with $G(\infty) = 0$. Then, by Fatou's jump theorem,

$$G(\zeta) = \lim_{r\to 1^-} \frac{K\mu}{\vartheta}(r\zeta) = \frac{\phi(\zeta) + \widehat{\mu}(\zeta)}{\vartheta(\zeta)}.$$

Assuming for the moment that $\vartheta(0) \neq 0$, we conclude that $G - \widehat{\mu}/\vartheta \in N^+(\mathbb{D}_e)$ and vanishes at infinity. Then ϕ/ϑ is the boundary function of a function from $N^+(\mathbb{D}_e)$ that vanishes at infinity. But since $\phi/\vartheta \in L^1(m)$, then $\phi/\vartheta \in \overline{H^1_0}$. If $\vartheta(0) = 0$, then use the same argument with ϑ replaced by ϑ/z^n and G replaced by G/z^n for some positive integer n. Now apply Proposition 3.1.

Now we need to show that $K\mu$ has an analytic continuation to $\widehat{\mathbb{C}} \setminus (\sigma(\vartheta)^* \cup E)$. As mentioned earlier, this was originally shown by Korenblum in [13]. Indeed, if $W \subseteq \widehat{\mathbb{C}} \setminus (\sigma(\vartheta)^* \cup E)$ is an open set containing an arc of the circle, then $T_{\mu,\vartheta}$ (as defined in (3.3)) is analytic on $W \cap \mathbb{D}_e$ and by standard estimates,

$$|T_{\mu,\vartheta}(\lambda)| \leqslant C\|\mu\| \frac{1}{|\lambda| - 1}, \quad \lambda \in W \cap \mathbb{D}_e.$$

Since $K\mu \perp I(\vartheta, E)$, we can apply Proposition 3.1 to conclude that μ takes the form

$$d\mu = \phi dm + d\mu_s,$$

where $\phi = \vartheta\overline{h}$ ($h \in H^1_0$) and μ_s is supported in E.

Next, let (h_n) be a sequence of polynomials in H^1_0 that approximates h in norm and set

$$d\mu_n := \vartheta\overline{h_n} dm + d\mu_s.$$

Notice that $\|\mu_n\|$ is uniformly bounded in n. By Proposition 3.1, $K\mu_n \perp I(\vartheta, E)$ and the corresponding pseudocontinuation of $K\mu_n/\vartheta$ is

$$T_{\mu_n,\vartheta}(z) = \overline{h_n}(1/\overline{z}) + \frac{1}{\vartheta(z)} \int \frac{\vartheta(\zeta)\overline{h_n}(\zeta)}{1 - \overline{\zeta}z} dm(\zeta) + \frac{1}{\vartheta(z)} \int \frac{d\mu_s(\zeta)}{1 - \overline{\zeta}z}.$$

Since the functions $\vartheta\overline{h_n}$ are bounded on $\mathbb{T}$, then $K\mu_n/\vartheta$ and $T_{\mu_n,\vartheta}$ are H^1 functions on $W \cap \mathbb{D}$ and $W \cap \mathbb{D}_e$ (respectively) [9, p. 41]. (Note that ϑ has an analytic continuation across $W \cap \mathbb{T}$ as does $\widehat{\mu_s}$ since this $W \cap \mathbb{T}$ avoids the support of μ_s.) Moreover, by what was said earlier, they have equal boundary values almost everywhere on $W \cap \mathbb{T}$. By a standard Morera's theorem argument [10, p. 95], these two functions are analytic continuations of each other across $W \cap \mathbb{T}$.

[2]Fatou's jump theorem: $\lim_{r\to 1^-}(\widehat{\mu}(r\zeta) - \widehat{\mu}(\zeta/r)) = \lim_{r\to 1^-} \int P_{r\zeta} d\mu = d\mu/dm(\zeta)$ m-almost everywhere [9, p. 4].

Finally,

$$|\frac{K\mu_n}{\vartheta}(\lambda)| \leqslant C\|\mu_n\|\frac{1}{1-|\lambda|} \leqslant \frac{C}{1-|\lambda|}, \quad \lambda \in W \cap \mathbb{D},$$
$$|T_{\mu_n,\vartheta}(\lambda)| \leqslant C\|\mu_n\|\frac{1}{|\lambda|-1} \leqslant \frac{C}{|\lambda|-1}, \quad \lambda \in W \cap \mathbb{D}_e.$$

By a theorem of Beurling [7] (see also [15, p. 95]), these functions form a normal family on W and so a subsequence (for which we will use the same index) converges to an analytic function on W. But since $K\mu_n/\vartheta$ converges pointwise to $K\mu/\vartheta$, then $K\mu/\vartheta$, and hence $K\mu$, will have an analytic continuation to W.

Proof of Corollary 2.6. If ϑF is a generator of the ideal $I(\vartheta, E)$, then by our "Cauchy pairing" in (2.1), it is routine to show that

$$\langle z^m \vartheta F, B^n f\rangle = \int_{\mathbb{T}} \overline{\vartheta F \zeta^{n+m+1}}(\vartheta dm + d\mu_s) = 0 \ \ \forall m, n \in \mathbb{N} \cup \{0\}.$$

Thus

$$\bigvee\{B^n f : n = 0, 1, 2, \ldots\} \subseteq I(\vartheta, E)^{\perp}.$$

If $g \in A$ satisfies $\langle g, B^n f\rangle = 0$ for all n, one can use the F. and M. Riesz theorem to show that $g/\vartheta \in H^1$ and g is zero on the support of μ_s (which equals E). Thus $g \in I(\vartheta, E)$. An application of the Hahn-Banach theorem completes the proof.

The proof of Theorem 2.7 requires a few preliminaries. Notice that $K\mu \in L^1(dA)$, where dA is the area measure on $\mathbb{D}$. This follows from Fubini's theorem and the fact that the integral

$$\int_{\mathbb{D}} \frac{1}{|e^{i\theta} - z|}\, dA(z)$$

is uniformly bounded in θ.

For a Cauchy transform $K\mu$, consider the function

$$\int_{\mathbb{D}} \frac{(K\mu)(z)}{z-\lambda}\, dA(z), \quad \lambda \in \mathbb{D}.$$

Since $K\mu \in L^1(dA)$ and is analytic on $\mathbb{D}$, it is not difficult to show, using the fact that $(z-\lambda)^{-1} \in L^1(dA)$ for each fixed $\lambda \in \mathbb{D}$, that the above integral exists for every $\lambda \in \mathbb{D}$. Moreover, the dominated convergence theorem says that the above function is continuous on $\mathbb{D}$.

Proposition 3.4. *For $\mu \in M$,*

$$\sup_{0<r<1} \int_0^{2\pi} \int_{\mathbb{D}} \left| \frac{(K\mu)(z)}{z - re^{i\theta}} \right| dA(z)\, d\theta < \infty.$$

Proof. For fixed $0 < r < 1$,

$$\begin{aligned} &\int_0^{2\pi} \int_{\mathbb{D}} \left| \frac{(K\mu)(z)}{z - re^{i\theta}} \right| dA(z)\, d\theta \\ \leqslant \quad &\int_0^{2\pi} \int_0^1 \int_0^{2\pi} \int_{\mathbb{T}} \frac{1}{|se^{it} - re^{i\theta}|} \frac{1}{|1 - se^{it}e^{-ix}|}\, d|\mu|(e^{ix})\, dt\, ds\, d\theta. \end{aligned}$$

Use the standard inequality

$$\int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - a|} \leqslant C \log(\frac{1}{1-|a|}), \quad |a| < 1$$

to get

$$\int_0^{2\pi} \frac{d\theta}{|se^{it} - re^{i\theta}|} \leqslant C \begin{cases} \frac{1}{r} \log(\frac{1}{1 - s/r}) & \text{for } s < r, \\ \frac{1}{r} \log(\frac{1}{1 - r/s}) & \text{for } s > r, \end{cases}$$

and

$$\int_0^{2\pi} \frac{dt}{|1 - se^{it}e^{-ix}|} \leqslant C \log(\frac{1}{1 - s}).$$

Combine the above two estimates along with Fubini's theorem to show the desired integral is bounded above by

$$\frac{C}{r} \left[\int_0^r \log(\frac{1}{1-s}) \log(\frac{1}{1 - s/r})\, ds + \int_r^1 \log(\frac{1}{1-s}) \log(\frac{1}{1 - r/s})\, ds \right].$$

Standard estimates now show that this quantity is bounded uniformly for r close to 1.

Proof of Theorem 2.7. Suppose $f \in A$ with $f' \in H^\infty$ and $K\mu \perp z^n f$ for all $n \in \mathbb{N} \cup \{0\}$. Theorem 2.5 yields condition (2). Using a power series argument, one can show that

$$\langle f, K\mu \rangle = \lim_{r \to 1^-} \sum_{n=0}^{\infty} \overline{\widehat{f}(n)} \widehat{\mu}(n) r^n = \lim_{r \to 1^-} \int_{\mathbb{D}} (K\mu)(rz) \overline{(zf)'(rz)}\, dm_2(z),$$

where $dm_2 = dA/\pi$. Since $(zf)'$ is a bounded function, we can use the fact that $K\mu \in L^1(dA)$, to rewrite[3] this as

$$\int_{\mathbb{D}} (K\mu)(z) \overline{(zf)'(z)}\, dm_2(z).$$

For fixed $\lambda \in \mathbb{D}$, the function

$$\frac{K\mu - (K\mu)(\lambda)}{z - \lambda}$$

can be written as $K\mu_\lambda$, where $d\mu_\lambda = \overline{\zeta}(1 - \overline{\zeta}\lambda)^{-1} d\mu$. By Proposition 3.1, $K\mu_\lambda$ also annihilates the ideal generated by $f = \vartheta F$. Thus, by what was said above,

$$\int_{\mathbb{D}} \frac{(K\mu)(z) - (K\mu)(\lambda)}{z - \lambda} \overline{(zf)'(z)}\, dm_2(z) = 0, \quad \lambda \in \mathbb{D}. \tag{3.5}$$

Another power series computation yields

$$\int_{\mathbb{D}} \frac{\overline{(zf)'(z)}}{z - \lambda}\, dm_2(z) = -\overline{\lambda f(\lambda)}$$

and so from (3.5),

$$-\overline{\lambda f(\lambda)} (K\mu)(\lambda) = \int_{\mathbb{D}} \frac{(K\mu)(z)}{z - \lambda} \overline{(zf)'(z)}\, dm_2(z).$$

Now use Proposition 3.4 along with the assumption that $(zf)'$ is bounded to show that the integrals

$$\int_0^{2\pi} |f(re^{i\theta})(K\mu)(re^{i\theta})|\, d\theta$$

are uniformly bounded in $0 < r < 1$, that is to say, $fK\mu \in H^1$.

[3]See, for example, the argument used to prove Lemma 2.5 in [3].

Conversely, suppose conditions (1) and (2) are satisfied. Since $\overline{\vartheta_f} K\mu$ and $\overline{F_f}$ (where F_f is the outer factor of f) are the boundary values of functions from $N^+(\mathbb{D}_e)$, then $\overline{f} K\mu$ is also the boundary function of a $N^+(\mathbb{D}_e)$ function that vanishes at infinity. But since $\overline{f} K\mu \in L^1(m)$, then $\overline{f} K\mu \in \overline{H^1_0}$. Thus

$$\int_{\mathbb{T}} (K\mu)(\zeta)\overline{\zeta^n f(\zeta)}\, dm(\zeta) = 0 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Finally, using the notation $g_r(z) := g(rz)$,

$$(K\mu)_r \overline{f_r} - K\mu\, \overline{f} = \Big[\, (K\mu)_r\, f_r - K\mu\, f \,\Big]\, \frac{\overline{f_r}}{f_r} + K\mu\, f\, \Big[\, \frac{\overline{f_r}}{f_r} - \frac{\overline{f}}{f} \,\Big],$$

which goes to zero in the $L^1(m)$ norm as $r \to 1^-$. Thus for any $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \langle z^n f, K\mu \rangle &= \lim_{r \to 1^-} \int_{\mathbb{T}} (K\mu)(r\zeta)\overline{(r\zeta)^n f(r\zeta)}\, dm(\zeta) \\ &= \int_{\mathbb{T}} (K\mu)(\zeta)\overline{\zeta^n f(\zeta)}\, dm(\zeta) \\ &= 0. \end{aligned}$$

4. The norm topology

Recall that $(\mathcal{K}, \|\cdot\|)$ is a nonseparable space, and so a characterization of the B-invariant subspaces is out of reach. In this final section, we will make a few remarks about the subspace $[K\mu]$, which we define to be the smallest B-invariant subspace of $(\mathcal{K}, \|\cdot\|)$ containing $K\mu$.

By the Lebesgue decomposition theorem, the space of measures can be decomposed as $M = M_a \oplus M_s$, where $M_a = \{\phi dm : \phi \in L^1(m)\}$ (the absolutely continuous measures with respect to Lebesgue measure m) and $M_s = \{\mu \in M : \mu \perp m\}$ (the singular ones). Moreover, if $\mu = \mu_a + \mu_s$ ($\mu_a \in M_a$, $\mu_s \in M_s$), then

$$\|\mu\| = \|\mu_a\| + \|\mu_s\|. \tag{4.1}$$

As a consequence of this, the space $(\mathcal{K}, \|\cdot\|)$ can be decomposed as $\mathcal{K} = \mathcal{K}_a \oplus \mathcal{K}_s$, where $\mathcal{K}_a = \{K(\phi dm) : \phi \in L^1(m)\}$ and $\mathcal{K}_s = \{K\mu : \mu \perp m\}$. One can show that $\mathcal{K} \simeq M/\overline{H^1_0}$ (where $\overline{H^1_0}$ is equated with a subspace of M_a in the obvious way) and $\mathcal{K}_a \simeq L^1/\overline{H^1_0}$. This makes the space $(\mathcal{K}_a, \|\cdot\|)$ separable. See [4], [5], and [6] for details.

Although the B-invariant subspaces of $(\mathcal{K}, \|\cdot\|)$ are very much unknown (due to the nonseparability of $\mathcal{K}_s$), the B-invariant subspaces of $(\mathcal{K}_a, \|\cdot\|)$ are known [1] (see also [6, p. 99]).

Theorem 4.2 (Aleksandrov)**.** *If $\mathcal{M}$ is a B-invariant subspace of $(\mathcal{K}_a, \|\cdot\|)$, then there is an inner function ϑ such that $f \in \mathcal{M}$ if and only if f/ϑ has a pseudocontinuation across $\mathbb{T}$ to a function belonging to $N^+(\mathbb{D}_e)$ and vanishing at infinity.*

We now examine $[K\mu]$ (the smallest B-invariant subspace of $(\mathcal{K}, \|\cdot\|)$ containing $K\mu$), where $\mu \in \mathcal{M}$ and whose support is not all of $\mathbb{T}$. First notice the following.

Proposition 4.3. *If $\mu \in M \setminus \{0\}$ with $\mu \ll m$ and* $\operatorname{supp}(\mu) \neq \mathbb{T}$*, then $[K\mu] = \mathcal{K}_a$.*

Proof. Indeed, if the support of μ omits the arc $J \subseteq \mathbb{T}$, then $K\mu$ has an analytic continuation across J given by

$$\widehat{\mu}(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}, \quad z \in \mathbb{D}_e.$$

Moreover, if $[K\mu] \neq \mathcal{K}_a$, then by Aleksandrov's theorem, $K\mu/\vartheta$ will have a pseudocontinuation for some inner function ϑ. But since any inner function ϑ has a pseudocontinuation given by

$$\widetilde{\vartheta}(z) = \frac{1}{\overline{\vartheta(1/\overline{z})}}, \quad z \in \mathbb{D}_e,$$

then $K\mu$ will have a pseudocontinuation F. That is to say, F is meromorphic on $\mathbb{D}_e$ and has nontangential boundary values equal to those of $K\mu$ m-almost everywhere on $\mathbb{T}$. So there are two meromorphic functions on $\mathbb{D}_e$, namely F and $\widehat{\mu}$, that have nontangential boundary values equal to $K\mu$ m-almost everywhere on the arc J. By Privalov's uniqueness theorem [12, pp. 62 - 63][4], $F = \widehat{\mu}$. Thus $\widehat{\mu}$ is a pseudocontinuation of $K\mu$ across $\mathbb{T}$. So

$$\lim_{r\to 1^-} [(K\mu)(r\zeta) - \widehat{\mu}(\zeta/r)] = 0$$

for m-almost every ζ. By Fatou's jump theorem and the absolute continuity of μ, μ must be the zero measure, a contradiction.

If p is an analytic polynomial, then $p(B)K\mu = K(p(\overline{\zeta})d\mu)$. Assuming $\operatorname{supp}(\mu) \neq \mathbb{T}$, we can apply Mergelyan's theorem [17, p. 423] along with the density of the continuous functions in $L^1(\mu)$ as well as the inequality $\|K\mu\| \leqslant \|\mu\|$, to conclude that

$$[K\mu] = \operatorname{clos}_{\mathcal{K}}\{K(fd\mu) : f \in L^1(\mu)\}. \tag{4.4}$$

Recall from the definition of the norm and (4.1) that for $\mu \in M_s$, $\|K\mu\| = \|\mu\|$. It follows now from (4.4) that for $\mu \perp m$ and $\operatorname{supp}(\mu) \neq \mathbb{T}$,

$$[K\mu] = \{K(fd\mu) : f \in L^1(\mu)\}. \tag{4.5}$$

If $\mu_1 \ll \mu_2$ with $\operatorname{supp}(\mu_2) \neq \mathbb{T}$, then $d\mu_1 = g d\mu_2$, where $g \in L^1(\mu_2)$. Thus if $f \in L^1(\mu_1)$, then $K(fd\mu_1) = K(fgd\mu_2)$ and so by (4.4), we have shown the following.

Proposition 4.6. *If* $\mu_1 \ll \mu_2$ *and* $\operatorname{supp}(\mu_2) \neq \mathbb{T}$*, then* $[K\mu_1] \subseteq [K\mu_2]$.

If $\mu \in M$ and is positive with $\operatorname{supp}(\mu) \neq \mathbb{T}$, and $\mu = \mu_a + \mu_s$ ($\mu_a \in M_a$ and $\mu_s \in M_s$), we note that $\mu_a \ll \mu$ and $\mu_s \ll \mu$. We can now apply Proposition 4.6 along with (4.5) and Proposition 4.3 to obtain the following result.

Theorem 4.7. *If* $\mu \in M \setminus \{0\}$ *is positive with* $\operatorname{supp}(\mu) \neq \mathbb{T}$ *and* $\mu = \mu_a + \mu_s$*, then*

$$[K\mu] = \begin{cases} \mathcal{K}_a \oplus \{K(fd\mu_s) : f \in L^1(\mu_s)\} & \text{if } \mu_a \not\equiv 0, \\ \{K(fd\mu_s) : f \in L^1(\mu_s)\} & \text{if } \mu_a \equiv 0. \end{cases}$$

[4]Privalov's uniqueness theorem: If f is meromorphic on $\mathbb{D}$ and has nontangential limits that exist and are equal to zero on a set of positive measure in $\mathbb{T}$, then f is the zero function.

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