

## THE BACKWARD SHIFT ON THE SPACE OF CAUCHY TRANSFORMS

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ABSTRACT. This note examines the subspaces of the space of Cauchy transforms of measures on the unit circle that are invariant under the backward shift operator  $f \rightarrow z^{-1}(f - f(0))$ . We examine this question when the space of Cauchy transforms is endowed with both the norm and weak\* topologies.

### 1. INTRODUCTION AND PRELIMINARIES

In this note, we will examine the invariant subspaces of the backward shift operator

$$(Bf)(z) = \frac{f(z) - f(0)}{z}$$

on the space of Cauchy transforms  $\mathcal{K}$  consisting of analytic functions on the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  that take the form

$$(1.1) \quad (K\mu)(z) := \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \bar{\zeta}z}.$$

Here  $\mu \in M$ , the space of finite Borel measures on the unit circle  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ .

By an “invariant subspace” of  $\mathcal{K}$  we will mean a closed linear manifold  $\mathcal{M} \subseteq \mathcal{K}$  for which  $B\mathcal{M} \subseteq \mathcal{M}$ . In using the word “closed”, there are two topologies on  $\mathcal{K}$  to consider here. The first is the norm topology. For  $f \in \mathcal{K}$ , let

$$M_f := \{\nu \in M : f = K\nu\}$$

be the set of “representing measures” for  $f$ . Define the norm of an element  $f \in \mathcal{K}$  by

$$\|f\| := \inf\{\|\nu\| : \nu \in M_f\},$$

where  $\|\nu\|$  denotes the total variation norm of the measure  $\nu$ . The notation  $(\mathcal{K}, \|\cdot\|)$  will denote the space  $\mathcal{K}$  endowed with the above norm topology. It is well known that  $(\mathcal{K}, \|\cdot\|)$  is isometrically isomorphic to the quotient space  $M/\overline{H_0^1}$  and is a non-separable Banach space. Here  $H^1$  is the usual Hardy space of the disk [9] and  $H_0^1$  are the functions in  $H^1$  that vanish at the origin.  $\overline{H_0^1}$  is regarded as a subspace of  $M$  in the natural way as  $\{\bar{f}dm : f \in H_0^1\}$  where  $dm = |d\zeta|/2\pi$  is normalized Lebesgue measure on the circle. The second topology on  $\mathcal{K}$  is the weak\* topology

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that arises by identifying the dual space of the disk algebra  $A$  (analytic functions on  $\mathbb{D}$  that have continuous extensions to  $\mathbb{D}^-$ ) with  $\mathcal{K}$  via the pairing

$$\langle f, K\mu \rangle = \int_{\mathbb{T}} \overline{f} d\mu, \quad f \in A, \mu \in M.$$

By the F. and M. Riesz theorem [9, p. 41], if  $\mu_1, \mu_2 \in M_{K\mu}$ , then  $d\mu_1 - d\mu_2 = \overline{h}dm$ , where  $h \in H_0^1$ . Thus the above pairing is independent of the representing measure  $\mu$ . We will use the notation  $(\mathcal{K}, *)$  to denote the space  $\mathcal{K}$  endowed with the weak\* topology. One can show that  $(\mathcal{K}, *)$  is separable. Furthermore, every weak\* closed subspace of  $\mathcal{K}$  is norm closed. See [4], [5], and [6] for a review of these basic facts about  $\mathcal{K}$ . In this paper, we examine the  $B$ -invariant subspaces of  $(\mathcal{K}, *)$  and  $(\mathcal{K}, \|\cdot\|)$ .

To put our results in perspective, we mention some known results about the  $B$ -invariant subspaces for other spaces of analytic functions. For example, by Beurling's theorem [9, p. 114], the  $B$ -invariant subspaces of the classical Hardy space  $H^2$  all take the form  $(\vartheta H^2)^\perp$ , where  $\vartheta$  is an inner function. Moreover [8] (see also [6]),  $f$  belongs to  $(\vartheta H^2)^\perp$  if and only if there is a function  $G_f \in N^+(\mathbb{D}_e)$  that vanishes at infinity such that

$$(1.2) \quad \lim_{r \rightarrow 1^-} \frac{f}{\vartheta}(r\zeta) = \lim_{r \rightarrow 1^-} G_f(\zeta/r)$$

for  $m$ -almost every  $\zeta \in \mathbb{T}$ . Here  $\mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-$  and  $G_f \in N^+(\mathbb{D}_e)$  means  $G_f(1/z) \in N^+$  (the Smirnov class of  $\mathbb{D}$  [9, p. 25]). The function  $G_f$  is called a “pseudocontinuation”<sup>1</sup> of  $f$ . If

$$\sigma(\vartheta) := \{z \in \mathbb{D}^- : \lim_{\lambda \rightarrow z} |\vartheta(\lambda)| = 0\},$$

then, by basic properties of inner functions [11, pp. 68 and 69],  $\vartheta$  has an analytic continuation to  $\widehat{\mathbb{C}} \setminus \sigma(\vartheta)^*$ , where  $\sigma(\vartheta)^* := \{z \in \widehat{\mathbb{C}} : 1/\overline{z} \in \sigma(\vartheta)\}$ . In fact, every  $f \in (\vartheta H^2)^\perp$  has an analytic continuation to  $\widehat{\mathbb{C}} \setminus \sigma(\vartheta)^*$  [8].

For the Bergman space  $L_a^2$  (analytic functions  $f$  on  $\mathbb{D}$  such that  $f \in L^2(dx dy)$ ) a theorem of Richter and Sundberg [14] says that every  $B$ -invariant subspace takes the form  $\mathcal{M}_g := \{f \in L_a^2 : f \perp z^n g \ \forall n \in \mathbb{N} \cup \{0\}\}$  for some  $g$  in the Dirichlet space (i.e.,  $g' \in L_a^2$ ). Here we equate the dual of  $L_a^2$  with the Dirichlet space via the “Cauchy” dual pairing

$$\lim_{r \rightarrow 1^-} \int f(r\zeta) \overline{g(r\zeta)} dm(\zeta).$$

Furthermore, (i)  $g\mathcal{M}_g \subseteq H^p$  for all  $0 < p < 1$ , (ii) for every  $f \in \mathcal{M}_g$ , the meromorphic function  $f/\vartheta_g$  (where  $\vartheta_g$  is the inner factor of  $g$ ) has a pseudocontinuation as in (1.2), (iii) every  $f \in \mathcal{M}_g$  has an analytic continuation to  $\widehat{\mathbb{C}} \setminus \sigma(g)^*$ . Moreover [2], if  $g$  is “sufficiently smooth”, then  $g\mathcal{M}_g \subseteq H^1$  and  $f \in L_a^2$  belongs to  $\mathcal{M}_g$  if and only if (a)  $f/g \in H^1$  and (b)  $f/\vartheta_g$  has pseudocontinuation as in (1.2). For certain  $L^p$  Bergman spaces, the function  $g$  can always be chosen to be “sufficiently smooth”; so in this case we have a complete characterization of the  $B$ -invariant subspaces. Our purpose here is to get similar-looking results for the space  $(\mathcal{K}, *)$  (which can be gleaned from results of Korenblum [13]) and to examine the more difficult problem of characterizing the  $B$ -invariant subspaces of  $(\mathcal{K}, \|\cdot\|)$ .

<sup>1</sup>If  $h$  is meromorphic on  $\mathbb{D}$  and  $H$  is meromorphic on  $\mathbb{D}_e$  and the nontangential boundary values of  $h$  and  $H$  exist and are equal  $m$ -almost everywhere, then  $h$  and  $H$  are “pseudocontinuations” of each other. See [15] for more details.

2. THE MAIN RESULTS

For a  $B$ -invariant subspace  $\mathcal{M}$  of  $(\mathcal{K}, *)$  let

$$\mathcal{M}_\perp = \{f \in A : \langle f, K\mu \rangle = 0 \text{ for all } K\mu \in \mathcal{M}\}$$

be the pre-annihilator of  $\mathcal{M}$ .  $\mathcal{M}_\perp$  is a norm closed subspace of the disk algebra  $A$ . A straightforward calculation shows that

$$(2.1) \quad \langle f, K\mu \rangle = \int_{\mathbb{T}} \bar{f} d\mu = \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} \overline{f(\zeta)} (K\mu)(r\zeta) dm(\zeta) = \lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} \overline{\widehat{f}(n)} \widehat{\mu}(n) r^n$$

and  $\langle f, BK\mu \rangle = \langle f, K(\bar{\zeta}d\mu) \rangle = \langle zf, K\mu \rangle$ . Thus  $z\mathcal{M}_\perp \subseteq \mathcal{M}_\perp$  since  $B\mathcal{M} \subseteq \mathcal{M}$ . Since  $A$  is a Banach algebra and polynomials are dense in  $A$  [11, p. 17],  $\mathcal{M}_\perp$  is an ideal of  $A$ . A theorem of Rudin [16] (see also [11, p. 85]) says the following.

**Theorem 2.2** (Rudin). *Let  $I$  be a norm closed ideal of the disk algebra  $A$ . Then there is a closed set  $E \subseteq \mathbb{T}$  of Lebesgue measure zero and an inner function  $\vartheta$  with  $\sigma(\vartheta) \cap \mathbb{T} \subseteq E$  such that*

$$I = I(\vartheta, E) := \{f \in A : f/\vartheta \in H^\infty, f|_E = 0\}.$$

Furthermore, given a set  $E \subseteq \mathbb{T}$  of Lebesgue measure zero and an inner  $\vartheta$  with  $\sigma(\vartheta) \cap \mathbb{T} \subseteq E$ , there is an outer function  $F \in A$  whose zero set is equal to  $E$  and such that  $g := \vartheta F$  generates  $I(\vartheta, E)$  in the sense that the smallest norm closed ideal of  $A$  containing  $g$  is equal to  $I(\vartheta, E)$ .

To describe  $\mathcal{M}$ , we need (via the Hahn-Banach theorem) to describe the set

$$(\mathcal{M}_\perp)^\perp = I(\vartheta, E)^\perp := \{f \in \mathcal{K} : \langle h, f \rangle = 0 \text{ for all } h \in I(\vartheta, E)\},$$

or equivalently, the set  $\{f \in \mathcal{K} : \langle z^n g, f \rangle = 0 \forall n \in \mathbb{N} \cup \{0\}\}$ . Korenblum [13] proved the following.

**Theorem 2.3** (Korenblum). *If  $K\mu \perp I(\vartheta, E)$ , then  $K\mu$  has an analytic continuation to the set  $\widehat{\mathbb{C}} \setminus (\sigma(\vartheta)^* \cup E)$ .*

In the process of proving our main theorem (Theorem 2.5), we will give an alternate proof of Korenblum’s theorem. Any measure  $\mu \in M$  can be decomposed uniquely as

$$(2.4) \quad d\mu = \phi dm + d\mu_s,$$

where  $\phi \in L^1(m)$  and  $\mu_s \perp m$ . Our main theorem describes  $I(\vartheta, E)^\perp$ .

**Theorem 2.5.** *For  $\mu \in M$ ,  $K\mu \perp I(\vartheta, E)$  if and only if*

- (1) *the support of  $\mu_s$  is contained in  $E$ ;*
- (2)  *$K\mu/\vartheta$  has an analytic continuation across  $\mathbb{T} \setminus E$  to a function  $F \in N^+(\mathbb{D}_e)$  with  $F(\infty) = 0$ .*

By the F. and M. Riesz theorem, every measure  $\nu \in M_f$  ( $f \in \mathcal{K}$ ) has the same singular part. Thus in condition (1), there is only one singular part to consider.

In  $H^2$ , the  $B$ -invariant subspace  $(\vartheta H^2)^\perp$  is singly generated by the vector  $f = B\vartheta$ . This next corollary is the analogue of this for  $(\mathcal{K}, *)$ .

**Corollary 2.6.**  *$I(\vartheta, E)^\perp = \vee\{B^n f : n \in \mathbb{N} \cup \{0\}\}$ , where  $f = B(K\mu)$  for  $d\mu = \vartheta dm + d\mu_s$  and  $\mu_s \perp m$  with support equal to  $E$ .*

Here  $\vee$  is the closed linear span in  $(\mathcal{K}, *)$ . This next corollary mimics what happens in the Bergman space setting. By a classical result of Smirnov [9, p. 39],  $\mathcal{K} \subseteq H^p$  for all  $0 < p < 1$ , and so if our  $B$ -invariant subspace  $\mathcal{M} \subseteq \mathcal{K}$  has the property that  $\mathcal{M}_\perp$  is generated by  $f$ , i.e.,  $\mathcal{M}_\perp$  is the closed linear span (in  $A$ ) of  $z^n f$  ( $n \in \mathbb{N} \cup \{0\}$ ), then certainly  $f\mathcal{M} \subseteq H^p$  for all  $0 < p < 1$ . If  $f$  is sufficiently smooth, we get the stronger condition  $f\mathcal{M} \subseteq H^1$  and even a bit more.

**Theorem 2.7.** *Suppose  $f \in A$  with  $f' \in H^\infty$ . Let  $E = f^{-1}(\{0\}) \cap \mathbb{T}$ , and let  $\vartheta_f$  be the inner factor of  $f$ . Then  $K\mu \perp z^n f$  for all  $n \in \mathbb{N} \cup \{0\}$  if and only if*

- (1)  $fK\mu \in H^1$ ;
- (2)  $K\mu/\vartheta_f$  has an analytic continuation across  $\mathbb{T} \setminus E$  to a function  $F \in N^+(\mathbb{D}_e)$  with  $F(\infty) = 0$ .

If  $f \in A$  with  $f' \in H^\infty$ , then the boundary zero set  $E$  of  $f$  satisfies the so-called Carleson condition: If  $(I_n)$  is the sequence of arcs contiguous to  $E$  on the circle, then  $\sum_n |I_n| \log |I_n| > -\infty$ . Thus, by Theorem 2.2, not every  $B$ -invariant subspace of  $(\mathcal{K}, *)$  is singly generated by such an  $f$ .

Comments about the  $B$ -invariant subspaces of  $(\mathcal{K}, \|\cdot\|)$  appear at the end of this note.

### 3. THE PROOFS

**Proposition 3.1.** *Suppose  $\vartheta F$  is a generator for  $I(\vartheta, E)$  and  $d\mu = \phi dm + d\mu_s$  as in (2.4). Then  $K\mu \perp z^n \vartheta F$  for all  $n \in \mathbb{N} \cup \{0\}$  if and only if  $\phi \in \overline{\vartheta H_0^1}$  and  $\mu_s$  is supported in  $E$ .*

*Proof.* Suppose  $K\mu \perp z^n \vartheta F$  for all  $n \in \mathbb{N} \cup \{0\}$ . Then, by (2.1),

$$(3.2) \quad \int_{\mathbb{T}} \overline{\zeta^n \vartheta F} (\phi dm + d\mu_s) = 0 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

From the F. and M. Riesz theorem,  $\overline{\vartheta F} d\mu_s$  is the zero measure (and so  $\mu_s$  is supported in  $E$ ) and  $\overline{\vartheta F} \phi = \overline{h} \in \overline{H_0^1}$ . However,  $\phi \overline{\vartheta} = \overline{h/F} \in \overline{N^+}$  and has  $L^1(m)$  boundary values, and so  $\phi \overline{\vartheta} \in \overline{H_0^1}$  [9, p. 28]. The converse is obvious.

*Proof of Theorem 2.5.* We start by proving a somewhat weaker result:  $K\mu \perp I(E, \vartheta)$  if and only if  $\mu_s$  is supported in  $E$  and  $K\mu/\vartheta$  has a pseudocontinuation across  $\mathbb{T}$  belonging to  $N^+(\mathbb{D}_e)$  and vanishing at infinity. Indeed, suppose  $K\mu \perp I(E, \vartheta)$ . By Proposition 3.1 we can assume  $\mu$  takes the form

$$d\mu = \phi dm + d\mu_s, \quad \phi \overline{\vartheta} = \overline{k} \in \overline{H_0^1}, \quad \text{supp}(\mu_s) \subseteq E.$$

Since  $\overline{k} \in \overline{H_0^1}$ , then  $\overline{k}(1/\overline{z})$  belongs to  $H^1(\mathbb{D}_e)$  and vanishes at infinity. The inner function  $\vartheta$  is defined on  $\mathbb{D}_e$  by  $\vartheta(z) = 1/\overline{\vartheta(1/\overline{z})}$ . The function

$$\widehat{\mu}(z) := \int \frac{d\mu(\zeta)}{1 - \overline{\zeta}z}, \quad z \in \mathbb{D}_e$$

belongs to  $H^p(\mathbb{D}_e)$  for all  $0 < p < 1$  [9, p. 39] and so the function

$$(3.3) \quad T_{\mu, \vartheta}(z) := \overline{k}(1/\overline{z}) + \frac{\widehat{\mu}(z)}{\vartheta(z)}, \quad z \in \mathbb{D}_e$$

belongs to  $N^+(\mathbb{D}_e)$  and vanishes at infinity. By Fatou’s jump theorem<sup>2</sup>, the boundary function for  $T_{\mu,\vartheta}$  is

$$\frac{\phi}{\vartheta}(\zeta) + \frac{(K\mu)(\zeta) - \phi(\zeta)}{\vartheta(\zeta)} = \frac{K\mu}{\vartheta}(\zeta)$$

for  $m$ -almost every  $\zeta \in \mathbb{T}$ . Thus  $T_{\mu,\vartheta}$  is the pseudocontinuation of  $K\mu/\vartheta$  of the desired type.

Conversely, suppose  $d\mu = \phi dm + d\mu_s$ , where  $\phi \in L^1(m)$  and  $\mu_s$  is supported in  $E$ , and  $K\mu/\vartheta$  has a pseudocontinuation  $G \in N^+(\mathbb{D}_e)$  with  $G(\infty) = 0$ . Then, by Fatou’s jump theorem,

$$G(\zeta) = \lim_{r \rightarrow 1^-} \frac{K\mu}{\vartheta}(r\zeta) = \frac{\phi(\zeta) + \widehat{\mu}(\zeta)}{\vartheta(\zeta)}.$$

Assuming for the moment that  $\vartheta(0) \neq 0$ , we conclude that  $G - \widehat{\mu}/\vartheta \in N^+(\mathbb{D}_e)$  and vanishes at infinity. Then  $\phi/\vartheta$  is the boundary function of a function from  $N^+(\mathbb{D}_e)$  that vanishes at infinity. But since  $\phi/\vartheta \in L^1(m)$ , then  $\phi/\vartheta \in \overline{H_0^1}$ . If  $\vartheta(0) = 0$ , then use the same argument with  $\vartheta$  replaced by  $\vartheta/z^n$  and  $G$  replaced by  $G/z^n$  for some positive integer  $n$ . Now apply Proposition 3.1.

Now we need to show that  $K\mu$  has an analytic continuation to  $\widehat{\mathbb{C}} \setminus (\sigma(\vartheta)^* \cup E)$ . As mentioned earlier, this was originally shown by Korenblum in [13]. Indeed, if  $W \subseteq \widehat{\mathbb{C}} \setminus (\sigma(\vartheta)^* \cup E)$  is an open set containing an arc of the circle, then  $T_{\mu,\vartheta}$  (as defined in (3.3)) is analytic on  $W \cap \mathbb{D}_e$  and by standard estimates,

$$|T_{\mu,\vartheta}(\lambda)| \leq C \|\mu\| \frac{1}{|\lambda| - 1}, \quad \lambda \in W \cap \mathbb{D}_e.$$

Since  $K\mu \perp I(\vartheta, E)$ , we can apply Proposition 3.1 to conclude that  $\mu$  takes the form

$$d\mu = \phi dm + d\mu_s,$$

where  $\phi = \vartheta \overline{h}$  ( $h \in H_0^1$ ) and  $\mu_s$  is supported in  $E$ .

Next, let  $(h_n)$  be a sequence of polynomials in  $H_0^1$  that approximates  $h$  in norm and set

$$d\mu_n := \vartheta \overline{h_n} dm + d\mu_s.$$

Notice that  $\|\mu_n\|$  is uniformly bounded in  $n$ . By Proposition 3.1,  $K\mu_n \perp I(\vartheta, E)$  and the corresponding pseudocontinuation of  $K\mu_n/\vartheta$  is

$$T_{\mu_n,\vartheta}(z) = \overline{h_n}(1/\overline{z}) + \frac{1}{\vartheta(z)} \int \frac{\vartheta(\zeta) \overline{h_n}(\zeta)}{1 - \overline{\zeta}z} dm(\zeta) + \frac{1}{\vartheta(z)} \int \frac{d\mu_s(\zeta)}{1 - \overline{\zeta}z}.$$

Since the functions  $\vartheta \overline{h_n}$  are bounded on  $\mathbb{T}$ , then  $K\mu_n/\vartheta$  and  $T_{\mu_n,\vartheta}$  are  $H^1$  functions on  $W \cap \mathbb{D}$  and  $W \cap \mathbb{D}_e$  (respectively) [9, p. 41]. (Note that  $\vartheta$  has an analytic continuation across  $W \cap \mathbb{T}$  as does  $\widehat{\mu}_s$  since this  $W \cap \mathbb{T}$  avoids the support of  $\mu_s$ .) Moreover, by what was said earlier, they have equal boundary values almost everywhere on  $W \cap \mathbb{T}$ . By a standard Morera’s theorem argument [10, p. 95], these two functions are analytic continuations of each other across  $W \cap \mathbb{T}$ .

<sup>2</sup>Fatou’s jump theorem:  $\lim_{r \rightarrow 1^-} (\widehat{\mu}(r\zeta) - \widehat{\mu}(\zeta/r)) = \lim_{r \rightarrow 1^-} \int P_{r\zeta} d\mu = d\mu/dm(\zeta)$   $m$ -almost everywhere [9, p. 4].

Finally,

$$\begin{aligned} \left| \frac{K\mu_n}{\vartheta}(\lambda) \right| &\leq C \|\mu_n\| \frac{1}{1-|\lambda|} \leq \frac{C}{1-|\lambda|}, \quad \lambda \in W \cap \mathbb{D}, \\ |T_{\mu_n, \vartheta}(\lambda)| &\leq C \|\mu_n\| \frac{1}{|\lambda|-1} \leq \frac{C}{|\lambda|-1}, \quad \lambda \in W \cap \mathbb{D}_e. \end{aligned}$$

By a theorem of Beurling [7] (see also [15, p. 95]), these functions form a normal family on  $W$  and so a subsequence (for which we will use the same index) converges to an analytic function on  $W$ . But since  $K\mu_n/\vartheta$  converges pointwise to  $K\mu/\vartheta$ , then  $K\mu/\vartheta$ , and hence  $K\mu$ , will have an analytic continuation to  $W$ .

*Proof of Corollary 2.6.* If  $\vartheta F$  is a generator of the ideal  $I(\vartheta, E)$ , then by our ‘‘Cauchy pairing’’ in (2.1), it is routine to show that

$$\langle z^m \vartheta F, B^n f \rangle = \int_{\mathbb{T}} \overline{\vartheta F \zeta^{n+m+1}} (\vartheta dm + d\mu_s) = 0 \quad \forall m, n \in \mathbb{N} \cup \{0\}.$$

Thus

$$\bigvee \{B^n f : n = 0, 1, 2, \dots\} \subseteq I(\vartheta, E)^\perp.$$

If  $g \in A$  satisfies  $\langle g, B^n f \rangle = 0$  for all  $n$ , one can use the F. and M. Riesz theorem to show that  $g/\vartheta \in H^1$  and  $g$  is zero on the support of  $\mu_s$  (which equals  $E$ ). Thus  $g \in I(\vartheta, E)$ . An application of the Hahn-Banach theorem completes the proof.

The proof of Theorem 2.7 requires a few preliminaries. Notice that  $K\mu \in L^1(dA)$ , where  $dA$  is the area measure on  $\mathbb{D}$ . This follows from Fubini’s theorem and the fact that the integral

$$\int_{\mathbb{D}} \frac{1}{|e^{i\theta} - z|} dA(z)$$

is uniformly bounded in  $\theta$ .

For a Cauchy transform  $K\mu$ , consider the function

$$\int_{\mathbb{D}} \frac{(K\mu)(z)}{z - \lambda} dA(z), \quad \lambda \in \mathbb{D}.$$

Since  $K\mu \in L^1(dA)$  and is analytic on  $\mathbb{D}$ , it is not difficult to show, using the fact that  $(z - \lambda)^{-1} \in L^1(dA)$  for each fixed  $\lambda \in \mathbb{D}$ , that the above integral exists for every  $\lambda \in \mathbb{D}$ . Moreover, the dominated convergence theorem says that the above function is continuous on  $\mathbb{D}$ .

**Proposition 3.4.** *For  $\mu \in M$ ,*

$$\sup_{0 < r < 1} \int_0^{2\pi} \int_{\mathbb{D}} \left| \frac{(K\mu)(z)}{z - re^{i\theta}} \right| dA(z) d\theta < \infty.$$

*Proof.* For fixed  $0 < r < 1$ ,

$$\begin{aligned} &\int_0^{2\pi} \int_{\mathbb{D}} \left| \frac{(K\mu)(z)}{z - re^{i\theta}} \right| dA(z) d\theta \\ &\leq \int_0^{2\pi} \int_0^1 \int_0^{2\pi} \int_{\mathbb{T}} \frac{1}{|se^{it} - re^{i\theta}|} \frac{1}{|1 - se^{it}e^{-ix}|} d|\mu|(e^{ix}) dt ds d\theta. \end{aligned}$$

Use the standard inequality

$$\int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - a|} \leq C \log\left(\frac{1}{1 - |a|}\right), \quad |a| < 1$$

to get

$$\int_0^{2\pi} \frac{d\theta}{|se^{it} - re^{i\theta}|} \leq C \begin{cases} \frac{1}{r} \log\left(\frac{1}{1-s/r}\right) & \text{for } s < r, \\ \frac{1}{r} \log\left(\frac{1}{1-r/s}\right) & \text{for } s > r, \end{cases}$$

and

$$\int_0^{2\pi} \frac{dt}{|1 - se^{it}e^{-ix}|} \leq C \log\left(\frac{1}{1-s}\right).$$

Combine the above two estimates along with Fubini’s theorem to show the desired integral is bounded above by

$$\frac{C}{r} \left[ \int_0^r \log\left(\frac{1}{1-s}\right) \log\left(\frac{1}{1-s/r}\right) ds + \int_r^1 \log\left(\frac{1}{1-s}\right) \log\left(\frac{1}{1-r/s}\right) ds \right].$$

Standard estimates now show that this quantity is bounded uniformly for  $r$  close to 1.

*Proof of Theorem 2.7.* Suppose  $f \in A$  with  $f' \in H^\infty$  and  $K\mu \perp z^n f$  for all  $n \in \mathbb{N} \cup \{0\}$ . Theorem 2.5 yields condition (2). Using a power series argument, one can show that

$$\langle f, K\mu \rangle = \lim_{r \rightarrow 1^-} \sum_{n=0}^\infty \widehat{f}(n) \widehat{\mu}(n) r^n = \lim_{r \rightarrow 1^-} \int_{\mathbb{D}} (K\mu)(rz) \overline{(zf)'(rz)} dm_2(z),$$

where  $dm_2 = dA/\pi$ . Since  $(zf)'$  is a bounded function, we can use the fact that  $K\mu \in L^1(dA)$ , to rewrite<sup>3</sup> this as

$$\int_{\mathbb{D}} (K\mu)(z) \overline{(zf)'(z)} dm_2(z).$$

For fixed  $\lambda \in \mathbb{D}$ , the function

$$\frac{K\mu - (K\mu)(\lambda)}{z - \lambda}$$

can be written as  $K\mu_\lambda$ , where  $d\mu_\lambda = \bar{\zeta}(1 - \bar{\zeta}\lambda)^{-1}d\mu$ . By Proposition 3.1,  $K\mu_\lambda$  also annihilates the ideal generated by  $f = \vartheta F$ . Thus, by what was said above,

$$(3.5) \quad \int_{\mathbb{D}} \frac{(K\mu)(z) - (K\mu)(\lambda)}{z - \lambda} \overline{(zf)'(z)} dm_2(z) = 0, \quad \lambda \in \mathbb{D}.$$

Another power series computation yields

$$\int_{\mathbb{D}} \frac{\overline{(zf)'(z)}}{z - \lambda} dm_2(z) = -\overline{\lambda f(\lambda)}$$

and so from (3.5),

$$-\overline{\lambda f(\lambda)}(K\mu)(\lambda) = \int_{\mathbb{D}} \frac{(K\mu)(z)}{z - \lambda} \overline{(zf)'(z)} dm_2(z).$$

Now use Proposition 3.4 along with the assumption that  $(zf)'$  is bounded to show that the integrals

$$\int_0^{2\pi} |f(re^{i\theta})(K\mu)(re^{i\theta})| d\theta$$

are uniformly bounded in  $0 < r < 1$ , that is to say,  $fK\mu \in H^1$ .

<sup>3</sup>See, for example, the argument used to prove Lemma 2.5 in [3].

Conversely, suppose conditions (1) and (2) are satisfied. Since  $\overline{\vartheta_f}K\mu$  and  $\overline{F_f}$  (where  $F_f$  is the outer factor of  $f$ ) are the boundary values of functions from  $N^+(\mathbb{D}_e)$ , then  $\overline{f}K\mu$  is also the boundary function of a  $N^+(\mathbb{D}_e)$  function that vanishes at infinity. But since  $\overline{f}K\mu \in L^1(m)$ , then  $\overline{f}K\mu \in \overline{H_0^1}$ . Thus

$$\int_{\mathbb{T}} (K\mu)(\zeta) \overline{\zeta^n f(\zeta)} dm(\zeta) = 0 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Finally, using the notation  $g_r(z) := g(rz)$ ,

$$(K\mu)_r \overline{f_r} - K\mu \overline{f} = [(K\mu)_r f_r - K\mu f] \frac{\overline{f_r}}{f_r} + K\mu f \left[ \frac{\overline{f_r}}{f_r} - \frac{\overline{f}}{f} \right],$$

which goes to zero in the  $L^1(m)$  norm as  $r \rightarrow 1^-$ . Thus for any  $n \in \mathbb{N} \cup \{0\}$ ,

$$\begin{aligned} \langle z^n f, K\mu \rangle &= \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} (K\mu)(r\zeta) \overline{(r\zeta)^n f(r\zeta)} dm(\zeta) \\ &= \int_{\mathbb{T}} (K\mu)(\zeta) \overline{\zeta^n f(\zeta)} dm(\zeta) \\ &= 0. \end{aligned}$$

#### 4. THE NORM TOPOLOGY

Recall that  $(\mathcal{K}, \|\cdot\|)$  is a nonseparable space, and so a characterization of the  $B$ -invariant subspaces is out of reach. In this final section, we will make a few remarks about the subspace  $[K\mu]$ , which we define to be the smallest  $B$ -invariant subspace of  $(\mathcal{K}, \|\cdot\|)$  containing  $K\mu$ .

By the Lebesgue decomposition theorem, the space of measures can be decomposed as  $M = M_a \oplus M_s$ , where  $M_a = \{\phi dm : \phi \in L^1(m)\}$  (the absolutely continuous measures with respect to Lebesgue measure  $m$ ) and  $M_s = \{\mu \in M : \mu \perp m\}$  (the singular ones). Moreover, if  $\mu = \mu_a + \mu_s$  ( $\mu_a \in M_a, \mu_s \in M_s$ ), then

$$(4.1) \quad \|\mu\| = \|\mu_a\| + \|\mu_s\|.$$

As a consequence of this, the space  $(\mathcal{K}, \|\cdot\|)$  can be decomposed as  $\mathcal{K} = \mathcal{K}_a \oplus \mathcal{K}_s$ , where  $\mathcal{K}_a = \{K(\phi dm) : \phi \in L^1(m)\}$  and  $\mathcal{K}_s = \{K\mu : \mu \perp m\}$ . One can show that  $\mathcal{K} \simeq M/\overline{H_0^1}$  (where  $\overline{H_0^1}$  is equated with a subspace of  $M_a$  in the obvious way) and  $\mathcal{K}_a \simeq L^1/\overline{H_0^1}$ . This makes the space  $(\mathcal{K}_a, \|\cdot\|)$  separable. See [4], [5], and [6] for details.

Although the  $B$ -invariant subspaces of  $(\mathcal{K}, \|\cdot\|)$  are very much unknown (due to the nonseparability of  $\mathcal{K}_s$ ), the  $B$ -invariant subspaces of  $(\mathcal{K}_a, \|\cdot\|)$  are known [1] (see also [6, p. 99]).

**Theorem 4.2** (Aleksandrov). *If  $\mathcal{M}$  is a  $B$ -invariant subspace of  $(\mathcal{K}_a, \|\cdot\|)$ , then there is an inner function  $\vartheta$  such that  $f \in \mathcal{M}$  if and only if  $f/\vartheta$  has a pseudocontinuation across  $\mathbb{T}$  to a function belonging to  $N^+(\mathbb{D}_e)$  and vanishing at infinity.*

We now examine  $[K\mu]$  (the smallest  $B$ -invariant subspace of  $(\mathcal{K}, \|\cdot\|)$  containing  $K\mu$ ), where  $\mu \in M$  and whose support is not all of  $\mathbb{T}$ . First notice the following.

**Proposition 4.3.** *If  $\mu \in M \setminus \{0\}$  with  $\mu \ll m$  and  $\text{supp}(\mu) \neq \mathbb{T}$ , then  $[K\mu] = \mathcal{K}_a$ .*

*Proof.* Indeed, if the support of  $\mu$  omits the arc  $J \subseteq \mathbb{T}$ , then  $K\mu$  has an analytic continuation across  $J$  given by

$$\widehat{\mu}(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta z}, \quad z \in \mathbb{D}_e.$$

Moreover, if  $[K\mu] \neq \mathcal{K}_a$ , then by Aleksandrov’s theorem,  $K\mu/\vartheta$  will have a pseudocontinuation for some inner function  $\vartheta$ . But since any inner function  $\vartheta$  has a pseudocontinuation given by

$$\widetilde{\vartheta}(z) = \frac{1}{\vartheta(1/\bar{z})}, \quad z \in \mathbb{D}_e,$$

then  $K\mu$  will have a pseudocontinuation  $F$ . That is to say,  $F$  is meromorphic on  $\mathbb{D}_e$  and has nontangential boundary values equal to those of  $K\mu$   $m$ -almost everywhere on  $\mathbb{T}$ . So there are two meromorphic functions on  $\mathbb{D}_e$ , namely  $F$  and  $\widehat{\mu}$ , that have nontangential boundary values equal to  $K\mu$   $m$ -almost everywhere on the arc  $J$ . By Privalov’s uniqueness theorem [12, pp. 62 - 63]<sup>4</sup>,  $F = \widehat{\mu}$ . Thus  $\widehat{\mu}$  is a pseudocontinuation of  $K\mu$  across  $\mathbb{T}$ . So

$$\lim_{r \rightarrow 1^-} [(K\mu)(r\zeta) - \widehat{\mu}(\zeta/r)] = 0$$

for  $m$ -almost every  $\zeta$ . By Fatou’s jump theorem and the absolute continuity of  $\mu$ ,  $\mu$  must be the zero measure, a contradiction.

If  $p$  is an analytic polynomial, then  $p(B)K\mu = K(p(\bar{\zeta})d\mu)$ . Assuming  $\text{supp}(\mu) \neq \mathbb{T}$ , we can apply Mergelyan’s theorem [17, p. 423] along with the density of the continuous functions in  $L^1(\mu)$  as well as the inequality  $\|K\mu\| \leq \|\mu\|$ , to conclude that

$$(4.4) \quad [K\mu] = \text{clos}_{\mathcal{K}}\{K(fd\mu) : f \in L^1(\mu)\}.$$

Recall from the definition of the norm and (4.1) that for  $\mu \in M_s$ ,  $\|K\mu\| = \|\mu\|$ . It follows now from (4.4) that for  $\mu \perp m$  and  $\text{supp}(\mu) \neq \mathbb{T}$ ,

$$(4.5) \quad [K\mu] = \{K(fd\mu) : f \in L^1(\mu)\}.$$

If  $\mu_1 \ll \mu_2$  with  $\text{supp}(\mu_2) \neq \mathbb{T}$ , then  $d\mu_1 = gd\mu_2$ , where  $g \in L^1(\mu_2)$ . Thus if  $f \in L^1(\mu_1)$ , then  $K(fd\mu_1) = K(fgd\mu_2)$  and so by (4.4), we have shown the following.

**Proposition 4.6.** *If  $\mu_1 \ll \mu_2$  and  $\text{supp}(\mu_2) \neq \mathbb{T}$ , then  $[K\mu_1] \subseteq [K\mu_2]$ .*

If  $\mu \in M$  and is positive with  $\text{supp}(\mu) \neq \mathbb{T}$ , and  $\mu = \mu_a + \mu_s$  ( $\mu_a \in M_a$  and  $\mu_s \in M_s$ ), we note that  $\mu_a \ll \mu$  and  $\mu_s \ll \mu$ . We can now apply Proposition 4.6 along with (4.5) and Proposition 4.3 to obtain the following result.

**Theorem 4.7.** *If  $\mu \in M \setminus \{0\}$  is positive with  $\text{supp}(\mu) \neq \mathbb{T}$  and  $\mu = \mu_a + \mu_s$ , then*

$$[K\mu] = \begin{cases} \mathcal{K}_a \oplus \{K(fd\mu_s) : f \in L^1(\mu_s)\} & \text{if } \mu_a \neq 0, \\ \{K(fd\mu_s) : f \in L^1(\mu_s)\} & \text{if } \mu_a \equiv 0. \end{cases}$$

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<sup>4</sup>Privalov’s uniqueness theorem: If  $f$  is meromorphic on  $\mathbb{D}$  and has nontangential limits that exist and are equal to zero on a set of positive measure in  $\mathbb{T}$ , then  $f$  is the zero function.

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