PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 132, Number 3, Pages 761–768
S 0002-9939(03)07113-2
Article electronically published on July 28, 2003

PHRAGMÉN-LINDELÖF THEOREMS

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(Communicated by Juha M. Heinonen)

ABSTRACT. Results of Phragmén-Lindelöf type are obtained for subharmonic functions in sectorial domains of bounded angular extent.

1. Introduction

The main results here have to do with the least harmonic majorants of subharmonic functions in sectorial domains of bounded angular extent. We will say a domain D is *sectorial* if its boundary consists of two simple curves Γ_1 and Γ_2 joining 0 to ∞, which are nonintersecting unless they are identical, and D has *angular extent* at most 2η, where $0 < \eta \le \pi$, if, for all positive r, the measure of $\{\arg z : z \in D_r\}$ is at most 2η. Here $D_r = D \cap \{z : |z| = r\}$.

For a function $u(z)$, subharmonic in D, we write

$$B_D(r,u) = \sup_{z\in D_r} u(z), \quad A_D(r,u) = \inf_{z\in D_r} u(z),$$

and drop the "D" if D is the whole plane. As usual, we denote the Riesz measure associated with $u(z)$ by $\mu(z,u)$, and write $\mu^*(r,u) = \mu(D \cap \{z : |z| \le r\}, u)$ and

$$N(r,u) = \int_0^r \frac{\mu^*(t,u)}{t} dt.$$

$N(r,u)$ is well defined if $u(z)$ is harmonic near the origin, which may be assumed in what follows without loss of generality. We will also assume throughout that $u(z)$ is not harmonic in the whole of D, so that $\mu(z,u)$ is not identically 0, and $N(r,u) \to \infty$ as $r \to \infty$. As will be apparent, this too entails no loss of generality in our results.

We shall prove:

Theorem 1. *Suppose that $u(z)$ is subharmonic in the plane and such that*

$$\frac{B(r,u)}{r^\rho} \to 0 \ \text{ as } r \to \infty, \tag{1}$$

where $0 < \rho < 1/2$, and that D is a sectorial domain of angular extent at most 2η, where $0 < \eta \le \pi$. Given $\alpha > 0$, let r_α be a point (which need not be unique) at which $N(r,u) - \alpha r^\rho$ attains its maximum value, a_α say, on $[0,\infty)$. Then $a_\alpha \to \infty$

Received by the editors August 28, 2002 and, in revised form, October 22, 2002.
2000 *Mathematics Subject Classification.* Primary 30D15.
The first author completed part of this work while visiting Virginia Tech. He would like to thank the Department of Mathematics for its warm and generous hospitality and financial support.

and $r_\alpha \to \infty$ *as* $\alpha \to 0$ *and there exist numbers* $b_\alpha \to \infty$ *as* $\alpha \to 0$ *such that, if* $H_D(z,u)$ *is the least harmonic majorant of* $u(z)$ *in* D,

$$H_D(z,u) < (1 + (\tan \pi\rho)(\tan \eta\rho))u(z) - b_\alpha, \tag{2}$$

for all $z \in D_{r_\alpha}$.

Condition (1) cannot be relaxed. The function $u(re^{i\theta}) = kr^\rho \cos(\pi - |\theta|)\rho$, where $k = (\cos \eta\rho)\sec(\pi - \eta)\rho$, is subharmonic in $D = \{z : |\arg z| < \eta\}$, and $B(r,u)/r^\rho = \cos \eta\rho$. Moreover, $H_D(z,u) = r^\rho \cos \theta\rho$ and, for all real positive values of z,

$$\frac{H_D(z,u)}{u(z)} = \frac{\cos(\pi - \eta)\rho}{(\cos \pi\rho)(\cos \eta\rho)} = 1 + (\tan \pi\rho)(\tan \eta\rho).$$

The proof of Theorem 1 follows the method of a paper by Hinkkanen and Rossi [7], but with the addition of a technique used in the proof of a theorem that Boas [1, p. 4] and Cartwright [2, p. 34] refer to as Beurling's theorem. This technique is applied repeatedly here, simply mimicking its original use in the proof of Theorem 2 but using it in what appears to be a new way in proving Theorems 1 and 3. Theorem 2 extends Beurling's theorem to sectorial domains.

Theorem 2. *Suppose that* D *is a sectorial domain bounded by* Γ_1 *and* Γ_2, *of angular extent at most* 2η, *and that* $u(z)$ *is subharmonic in* D *and has a continuous extension to* Γ_1 *and* Γ_2. *Suppose also that*

(i) $\liminf_{r\to\infty} B_D(r,u)/r^{\pi/(2\eta)} = 0$, *and*

(ii) $u(z) \le \phi(|z|)$ *on* Γ_1 *and* Γ_2, *where* $\phi(r)$ *is continuous and unbounded above and such that* $\phi(r)/r^\rho \to 0$ *as* $r \to \infty$, *for some* ρ *satisfying* $0 < \rho < 1/2$.

Given $\alpha > 0$, *let* r_α *be a point (which need not be unique) at which* $\phi(r) - \alpha r^\rho$ *attains its maximum value,* a_α *say, on* $[0,\infty)$. *Then* $a_\alpha \to \infty$ *and* $r_\alpha \to \infty$ *as* $\alpha \to 0$ *and there exist numbers* $b_\alpha \to \infty$ *as* $\alpha \to 0$ *such that, if* $H_D(z,u)$ *is the least harmonic majorant of* $u(z)$ *in* D,

$$H_D(z,u) < (\sec \eta\rho)\phi(|z|) - b_\alpha,$$

for all $z \in D_{r_\alpha}$.

The condition (ii) cannot be relaxed. If $D = \{z : |\arg z| < \eta\}$, ρ is such that $0 < \rho < \min\{1/2, \pi/(2\eta)\}$ and $u(re^{i\theta}) = r^\rho = \phi(r)$, then (i) holds and (ii) just fails. Moreover, $H_D(re^{i\theta},u) = (\sec \eta\rho)(\cos \theta\rho)r^\rho$, and thus, for all real, positive values of z, $H_D(z,u) = (\sec \eta\rho)\phi(|z|)$.

To prove Theorem 1 we need a result that is not without interest in itself.

Theorem 3. *Suppose that* $u(z)$ *is subharmonic in the plane and that* $B(r,u)/r^\rho \to 0$ *as* $r \to \infty$, *where* $0 < \rho < 1$. *Given* $\alpha > 0$, *let* r_α *be a point (which need not be unique) at which* $N(r,u) - \alpha r^\rho$ *attains its maximum value,* a_α *say, on* $[0,\infty)$. *Then* $a_\alpha \to \infty$ *and* $r_\alpha \to \infty$ *as* $\alpha \to 0$ *and there exist numbers* $b_\alpha \to \infty$ *as* $\alpha \to 0$ *such that the following inequalities hold:*

$$A(r_\alpha,u) > \pi\rho(\cot \pi\rho)N(r_\alpha,u) + b_\alpha, \tag{3}$$

$$B(r_\alpha,u) < \pi\rho(\csc \pi\rho)N(r_\alpha,u) - b_\alpha, \tag{4}$$

$$A(r_\alpha,u) + B(r_\alpha,u) > \pi\rho(\cot(\pi\rho/2))N(r_\alpha,u) + b_\alpha, \tag{5}$$

and, if $0 < \rho < 1/2$ *and* $\beta > \rho$,

$$\int_0^\infty \log\left|\frac{r_\alpha^\beta + t^\beta}{r_\alpha^\beta - t^\beta}\right| d\mu^*(t,u) < \pi\rho(\tan(\pi\rho/(2\beta)))N(r_\alpha,u) - b_\alpha. \tag{6}$$

A version of the $\cos \pi\rho$ theorem follows from (3), (4) and (5). Eremenko, Shea and Sodin [4] proved that the weaker inequalities

$$A(r,u) > (1+o(1))\pi\rho(\cot \pi\rho)N(r,u), \quad B(r,u) < (1+o(1))\pi\rho(\csc \pi\rho)N(r,u),$$

hold simultaneously on a sequence of strong Polya peaks $r \to \infty$ under the weaker assumption that ρ is the lower order of u.

The paper concludes with a discussion of the main result of [7], and an example that shows it to be sharp.

2. Proof of Theorem 3

Since [6, (3.9.5), (3.9.6) and Theorem 3.19, p. 127] $N(r,u) \le B(r,u^+) - u(0)$, we have $N(r,u)/r^\rho \to 0$ as $r \to \infty$. It follows that, given $\alpha > 0$, if $a_\alpha = \max_{0 \le r < \infty}(N(r,u) - \alpha r^\rho) = N(r_\alpha, u) - \alpha r_\alpha^\rho$, then $a_\alpha \to \infty$ and $r_\alpha \to \infty$ as $\alpha \to 0$. We have:

Lemma 4. *$\mu^*(r,u)$ is continuous at r_α and $\mu^*(r_\alpha,u) = \alpha\rho r_\alpha^\rho$. Thus $N'(r_\alpha,u)$ exists and equals $\alpha\rho r_\alpha^{\rho-1}$.*

To see this, write $l = \lim_{t \to r_\alpha^-} \mu^*(t,u)$, $L = \lim_{t \to r_\alpha^+} \mu^*(t,u)$, which exist since $\mu^*(t,u)$ is nondecreasing. We have $L \ge l$. Also, for $t > r_\alpha$, $N(t,u) - N(r_\alpha,u) \le \alpha(t^\rho - r_\alpha^\rho)$; that is, $(L + o(1))\log(t/r_\alpha) \le (\alpha\rho r_\alpha^{\rho-1} + o(1))(t - r_\alpha)$, giving $L \le \alpha\rho r_\alpha^\rho$. In a similar way, $l \ge \alpha\rho r_\alpha^\rho$, and the lemma follows.

As usual, in proving (3), (4) and (5), we may assume that $u(z)$ is harmonic off the negative real axis. With this assumption, we have the representation

$$A(r_\alpha,u) = \int_0^\infty \frac{\mu^*(t,u)}{t}\frac{r_\alpha}{r_\alpha - t}dt, \tag{7}$$

the integral (a Cauchy principal value [8, *SF*, 4.23]) being the limit of $I_1 + I_2$ as $\epsilon \to 0^+$, where

$$I_1 = \int_0^{r_\alpha-\epsilon} \frac{\mu^*(t,u)}{t}\frac{r_\alpha}{r_\alpha - t}dt, \quad I_2 = \int_{r_\alpha+\epsilon}^\infty \frac{\mu^*(t,u)}{t}\frac{r_\alpha}{r_\alpha - t}dt.$$

Integrating by parts,

$$\begin{aligned}
I_1 &= \left[N(t,u)\frac{r_\alpha}{r_\alpha - t}\right]_0^{r_\alpha-\epsilon} - \int_0^{r_\alpha-\epsilon} N(t,u)\frac{r_\alpha}{(r_\alpha - t)^2}dt \\
&\ge \frac{r_\alpha}{\epsilon}N(r_\alpha - \epsilon, u) - \int_0^{r_\alpha-\epsilon}(\alpha t^\rho + a_\alpha)\frac{r_\alpha}{(r_\alpha - t)^2}dt \\
&= \frac{r_\alpha}{\epsilon}(N(r_\alpha - \epsilon,u) - \alpha(r_\alpha - \epsilon)^\rho - a_\alpha) + a_\alpha + \alpha\rho\int_0^{r_\alpha-\epsilon} t^{\rho-1}\frac{r_\alpha}{r_\alpha - t}dt, \qquad (8)
\end{aligned}$$

and similarly,

$$I_2 \ge \frac{r_\alpha}{\epsilon}(N(r_\alpha + \epsilon,u) - \alpha(r_\alpha + \epsilon)^\rho - a_\alpha) + \alpha\rho\int_{r_\alpha+\epsilon}^\infty t^{\rho-1}\frac{r_\alpha}{r_\alpha - t}dt.$$

Now, as $\epsilon \to 0$,

$$\begin{aligned}
&\frac{r_\alpha}{\epsilon}(N(r_\alpha - \epsilon,u) - \alpha(r_\alpha - \epsilon)^\rho - a_\alpha) \\
&\quad = \frac{r_\alpha}{\epsilon}(\alpha(r_\alpha^\rho - (r_\alpha - \epsilon)^\rho) - (N(r_\alpha,u) - N(r_\alpha - \epsilon,u))) = o(1), \qquad (9)
\end{aligned}$$

from Lemma 4, and similarly,

$$\frac{r_\alpha}{\epsilon}(N(r_\alpha+\epsilon,u)-\alpha(r_\alpha+\epsilon)^\rho-a_\alpha)=o(1).$$

So, combining the inequalities and allowing ϵ to tend to 0, we get

$$\begin{aligned} A(r_\alpha,u) &\geq a_\alpha+\alpha\rho\int_0^\infty t^{\rho-1}\frac{r_\alpha}{r_\alpha-t}dt \\ &= a_\alpha+\alpha\rho r_\alpha^\rho\int_0^\infty \frac{t^{\rho-1}}{1-t}dt \\ &= a_\alpha+\alpha\pi\rho(\cot\pi\rho)r_\alpha^\rho. \qquad (10)\end{aligned}$$

Since $\alpha r_\alpha^\rho = N(r_\alpha,u)-a_\alpha$, we obtain

$$A(r_\alpha,u)\geq \pi\rho(\cot\pi\rho)N(r_\alpha,u)+a_\alpha(1-\pi\rho\cot\pi\rho),$$

and (3) follows, since $a_\alpha\to\infty$ as $\alpha\to 0$.

(4) is approached in a similar way:

$$\begin{aligned} B(r_\alpha,u) &= \int_0^\infty \frac{\mu^*(t,u)}{t}\frac{r_\alpha}{r_\alpha+t}dt \\ &= \int_0^\infty N(t,u)\frac{r_\alpha}{(r_\alpha+t)^2}dt \\ &\leq \int_0^\infty (\alpha t^\rho+a_\alpha)\frac{r_\alpha}{(r_\alpha+t)^2}dt \\ &= a_\alpha+\alpha\rho r_\alpha^\rho\int_0^\infty \frac{t^{\rho-1}}{1+t}dt \\ &= a_\alpha+\alpha\pi\rho(\csc\pi\rho)r_\alpha^\rho. \qquad (11)\end{aligned}$$

Since $\alpha r_\alpha^\rho = N(r_\alpha,u)-a_\alpha$, we obtain

$$B(r_\alpha,u)\leq \pi\rho(\csc\pi\rho)N(r_\alpha,u)+a_\alpha(1-\pi\rho\csc\pi\rho),$$

and (4) follows, since $a_\alpha\to\infty$ as $\alpha\to 0$.

For (5), we have

$$A(r_\alpha,u)+B(r_\alpha,u)=\int_0^\infty \frac{\mu^*(t,u)}{t}\frac{2{r_\alpha}^2}{r_\alpha^2-t^2}dt=\int_0^\infty \frac{\mu^*(\sqrt{t},u)}{t}\frac{r_\alpha^2}{r_\alpha^2-t}dt,$$

an integral of the same form as (7), and following a similar path we obtain

$$A(r_\alpha,u)+B(r_\alpha,u)\geq 2a_\alpha+\alpha\pi\rho(\cot(\pi\rho/2))r_\alpha^\rho.$$

(5) follows from this by substituting $\alpha r_\alpha^\rho = N(r_\alpha,u)-a_\alpha$.

The left-hand side of (6) is, from the monotone convergence theorem, the limit as $\epsilon\to 0^+$ of $J_1+J_1'+J_2+J_2'$, where

$$J_1=\int_0^{r_\alpha-\epsilon}\log\left|\frac{r_\alpha^\beta+t^\beta}{r_\alpha^\beta-t^\beta}\right|d\mu^*(t,u), \quad J_2=\int_{r_\alpha+\epsilon}^\infty\log\left|\frac{r_\alpha^\beta+t^\beta}{r_\alpha^\beta-t^\beta}\right|d\mu^*(t,u),$$

$$J_1'=(\mu^*(r_\alpha,u)-\mu^*(r_\alpha-\epsilon,u))\log\left|\frac{r_\alpha^\beta+(r_\alpha-\epsilon)^\beta}{r_\alpha^\beta-(r_\alpha-\epsilon)^\beta}\right|,$$

$$J_2'=(\mu^*(r_\alpha+\epsilon,u)-\mu^*(r_\alpha,u))\log\left|\frac{r_\alpha^\beta+(r_\alpha+\epsilon)^\beta}{r_\alpha^\beta-(r_\alpha+\epsilon)^\beta}\right|.$$

Integrating by parts twice, using the fact that $N(t,u) \le \alpha t^\rho + a_\alpha$, and then reversing both integrations by parts, we obtain

$$J_1 \le \left[(\mu^*(t,u) - \alpha\rho t^\rho) \log \left| \frac{r_\alpha^\beta + t^\beta}{r_\alpha^\beta - t^\beta} \right| + (N(t,u) - \alpha t^\rho - a_\alpha) \frac{2\beta r_\alpha^\beta t^\beta}{t^{2\beta} - r_\alpha^{2\beta}} \right]_0^{r_\alpha - \epsilon}$$

$$+ \alpha\rho^2 \int_0^{r_\alpha - \epsilon} \log \left| \frac{r_\alpha^\beta + t^\beta}{r_\alpha^\beta - t^\beta} \right| t^{\rho-1} dt. \tag{12}$$

When combined with J_1', the first part of the evaluation term in (12) is

$$\begin{aligned} &(\mu^*(r_\alpha,u) - \alpha\rho(r_\alpha - \epsilon)^\rho) \log \left| \frac{r_\alpha^\beta + (r_\alpha - \epsilon)^\beta}{r_\alpha^\beta - (r_\alpha - \epsilon)^\beta} \right| \\ &= \alpha\rho(r_\alpha^\rho - (r_\alpha - \epsilon)^\rho) \log \left(\frac{2r_\alpha}{\beta\epsilon}(1 + o(1)) \right) \\ &= O(\epsilon \log(1/\epsilon)) = o(1), \end{aligned} \tag{13}$$

as $\epsilon \to 0$, using Lemma 4. For the second part of the evaluation term,

$$\begin{aligned} &(N(r_\alpha - \epsilon, u) - \alpha(r_\alpha - \epsilon)^\rho - a_\alpha) \frac{2\beta r_\alpha^\beta (r_\alpha - \epsilon)^\beta}{(r_\alpha - \epsilon)^{2\beta} - r_\alpha^{2\beta}} \\ &= (1 + o(1)) \frac{r_\alpha}{\epsilon} ((N(r_\alpha, u) - N(r_\alpha - \epsilon, u) - \alpha(r_\alpha^\rho - (r_\alpha - \epsilon)^\rho)) \\ &= (r_\alpha + o(1))(N'(r_\alpha, u) - \alpha\rho r_\alpha^{\rho-1} + o(1)) = o(1), \end{aligned} \tag{14}$$

again using Lemma 4. Thus

$$J_1 + J_1' \le \alpha\rho^2 \int_0^{r_\alpha - \epsilon} \log \left| \frac{r_\alpha^\beta + t^\beta}{r_\alpha^\beta - t^\beta} \right| t^{\rho-1} dt + o(1).$$

$J_2 + J_2'$ is treated in the same way (the condition $\beta > \rho$ is used at this point to ensure that the contributions at ∞ vanish) and, combining the inequalities for $J_1 + J_1'$ and $J_2 + J_2'$ and allowing ϵ to tend to 0, we obtain [see [7, p. 158]]

$$\begin{aligned} \int_0^\infty \log \left| \frac{r_\alpha^\beta + t^\beta}{r_\alpha^\beta - t^\beta} \right| d\mu^*(t,u) &\le \alpha\rho^2 \int_0^\infty \log \left| \frac{r_\alpha^\beta + t^\beta}{r_\alpha^\beta - t^\beta} \right| t^{\rho-1} dt \\ &= \alpha\pi\rho(\tan(\pi\rho/(2\beta))) r_\alpha^\rho. \end{aligned} \tag{15}$$

(6) follows from this, after substituting $\alpha r_\alpha^\rho = N(r_\alpha, u) - a_\alpha$.

3. Proof of Theorem 1

For $z \in D$,

$$H_D(z,u) = u(z) + \int_D g_D(\zeta, z) d\mu(\zeta, u),$$

where $g_D(\zeta, z)$ is Green's function for D. As in [7], with $\beta = \pi/(2\eta)$, if $z \in D_{r_\alpha}$,

$$\begin{aligned} \int_D g_D(\zeta, z) d\mu(\zeta, u) &\le \int_0^\infty \log \left| \frac{r_\alpha^\beta + t^\beta}{r_\alpha^\beta - t^\beta} \right| d\mu^*(t,u) \\ &< \pi\rho(\tan\eta\rho) N(r_\alpha, u) - b_\alpha \\ &< (\tan\eta\rho)(\tan\pi\rho) A(r_\alpha, u) - b_\alpha, \end{aligned} \tag{16}$$

from Theorem 3, and Theorem 1 follows.

4. Proof of Theorem 2

First a lemma:

Lemma 5. *Suppose that D is the domain of Theorem 2 and, given ρ such that $0 < \rho < 1/2$, let $H_D(z,v)$ be the least harmonic majorant of $v(z) = |z|^\rho$ in D. Then, for all z in D,*

$$H_D(z,v) \le (\sec \eta\rho)|z|^\rho.$$

In this case, when $v(z) = |z|^\rho$, we have $\mu^*(t,u) = \rho t^\rho$, so that $d\mu(\zeta,v) = \frac{1}{2\pi}\rho^2|\zeta|^{\rho-1}d\theta d|\zeta|$. Thus

$$\begin{aligned} H_D(z,v) &= v(z) + \int_D g_D(\zeta,z)d\mu(\zeta,u) \\ &= v(z) + \frac{\rho^2}{2\pi}\int_0^\infty t^{\rho-1}dt\int_{D_t} g_D(te^{i\theta},z)d\theta. \end{aligned} \tag{17}$$

Also, if D^* is the circular symmetrization of D, then [5, (9.2.10)]

$$\begin{aligned} \int_{D_t} g_D(te^{i\theta},z)d\theta &\le \int_{D_t^*} g_{D^*}(te^{i\theta},|z|)d\theta \\ &\le \int_{\Omega_t} g_\Omega(te^{i\theta},|z|)d\theta, \end{aligned} \tag{18}$$

using the monotonicity of Green's functions, where $\Omega = \{z : |\arg z| < \eta\}$. This last integral, when added to $v(z)$, is the least harmonic majorant of $v(z)$ in Ω—that is, $H_\Omega(re^{i\theta},v) = r^\rho(\sec\eta\rho)(\cos\theta\rho)$—evaluated at $|z|$, and the lemma follows.

Turning to the proof of Theorem 2, let $a_\alpha = \max_{0\le r<\infty}(\phi(r)-\alpha r^\rho) = \phi(r_\alpha)-\alpha r_\alpha^\rho$ (note that, since $\phi(r)$ is unbounded above, $a_\alpha \to \infty$ and $r_\alpha \to \infty$ as $\alpha \to 0$), and consider $U(z) = u(z) - \alpha H_D(z,v) - a_\alpha$, where $H_D(z,v)$ is the least harmonic majorant of $v(z) = |z|^\rho$ in D, as in Lemma 5. For z on Γ_1 or Γ_2, $U(z) \le \phi(|z|) - \alpha|z|^\rho - a_\alpha \le 0$, and we deduce from the Carleman-Tsuji-Heins inequality ([9, p. 116] or [5, p. 548, where $\pi/2\eta \le \alpha(\rho)$; see pp. 535-536 for the definition of $\alpha(\rho)$]) that either $\liminf_{r\to\infty} B_D(r,u)/r^{\pi/(2\eta)} > 0$, which is ruled out by the hypotheses, or else $U(z) \le 0$ in D; that is,

$$u(z) \le \alpha H_D(z,v) + a_\alpha \le \alpha(\sec\eta\rho)|z|^\rho + a_\alpha,$$

for all z in D. Since $\alpha H_D(z,v) + a_\alpha$ is harmonic in D, the same inequality holds with u replaced by its least harmonic majorant, $H_D(z,u)$, and in particular, if $z \in D_{r_\alpha}$, so that $\alpha|z|^\rho = \alpha r_\alpha^\rho = \phi(r_\alpha) - a_\alpha$, we obtain

$$H_D(z,u) \le \alpha H_D(z,v) + a_\alpha \le (\sec\eta\rho)\phi(|z|) + a_\alpha(1-\sec\eta\rho),$$

and Theorem 2 follows.

5. Asymptotic functions: an example

The context of Hinkkanen and Rossi's paper is that of analytic functions having asymptotic functions. To be specific, it is supposed that Γ_1 and Γ_2 are simple curves that join 0 to ∞ and intersect at most finitely often in any bounded region (unless they are identical). Γ_1 and Γ_2 enclose a sequence of domains D_1, D_2, D_3,…; the sequence may be finite or infinite. We set $D = \bigcup_j D_j$ and suppose that $f(z)$ is analytic in D, continuous in $\overline{D}$, and such that $f(z) = O(1)$ as $z \to \infty$ along Γ_1 and $f(z) = a(z) + O(1)$ as $z \to \infty$ along Γ_2, where $a(z)$ is a non-constant entire function of order $\rho < 1/2$. The conclusion is that, if the angular extent of D is no more than 2η, where $0 < \eta \le \pi$, and if $\rho < 1/(2+2\eta/\pi)$, then the sequence D_1, D_2, $D_3,\ldots$, is finite (so that the last domain in the sequence is unbounded) and

$$\liminf_{r\to\infty} B_D(r, \log|f|)/r^{\pi/(2\eta)} > 0. \tag{19}$$

(Fenton and Dudley Ward [3] had shown that (19) holds when D is a sector of opening 2η.) Actually, showing that Γ_1 and Γ_2 cannot intersect infinitely often requires no assumption about the growth of $f(z)$. Taking $\eta = \pi$ leads to the curious observation that no entire function can have distinct asymptotic functions of order less than $1/4$ along paths that intersect at a sequence of points tending to ∞.

Not surprisingly, given the origins of Theorem 1 in the method of [7], (19) can be deduced from Theorem 1. Rather than pursue this, however, we conclude with an example that shows that the condition $\rho < 1/(2+2\eta/\pi)$ is best possible. (This was known in the case $\eta = \pi$ [3].)

Given $\eta > 0$, where $0 < \eta \le \pi$, we construct an analytic function $f(z)$ in $D = \{z : 0 < \arg z < 2\eta\}$, continuous in $\overline{D}$, such that $f(r) = a(r) + O(1)$ as $r \to \infty$, where $a(z)$ is a non-constant entire function of order $\rho = 1/(2+2\eta/\pi)$, and $f(re^{2i\eta}) = O(1)$ as $r \to \infty$, while $B_D(r, f) = O(r^\rho)$. Thus, since $\pi/(2\eta) \ge 1/2 > \rho$, (19) fails.

With $\rho = 1/(2+2\eta/\pi)$, where $0 < \eta \le \pi$, let $\lambda = 1/(4\rho)$ (so that $1/2 < \lambda \le 1$) and let $E_\lambda(z)$ be the Mittag-Leffler function [2, p. 50]. $E_\lambda(z)$ is an entire function of order $1/\lambda$ mean type that is bounded in the sector $\lambda\pi/2 \le \arg z \le 2\pi - \lambda\pi/2$. Let $c = e^{i(1-\lambda)\pi/2}$, and let $b_1 = 1$, $b_2 = i$, $b_3 = -1$, $b_4 = -i$ be the fourth roots of unity. We define

$$a(z) = \sum_{j=1}^{4} E_\lambda(cb_j z^{1/4}),$$

where $0 \le \arg z \le 2\pi$, an entire function of order ρ mean type, and also

$$f(z) = E_\lambda(cz^{1/4}),$$

which is analytic in D and continuous in $\overline{D}$. Evidently $B_D(r, \log|f|) = O(r^\rho)$.

When $z = r$, $a(z) = E_\lambda(cz^{1/4}) + O(1) = f(z) + O(1)$ as $r \to \infty$, since $cb_2 = e^{i(2-\lambda)\pi/2}$, $cb_3 = e^{i(3-\lambda)\pi/2}$, $cb_4 = e^{i(4-\lambda)\pi/2}$, and

$$\lambda\pi/2 \le (2-\lambda)\pi/2 < (4-\lambda)\pi/2 = 2\pi - \lambda\pi/2.$$

When $z = re^{2i\eta}$, on the other hand, $cz^{1/4} = r^{1/4}e^{\lambda\pi i/2}$, and so $f(z) = O(1)$ as $r \to \infty$.

References

[1] Boas, R. P., *Entire Functions*, Academic Press, New York (1954). MR **16:**914f

[2] Cartwright, M. L., *Integral Functions*, Cambridge University Press (1956). MR **17:**1067c

[3] Dudley Ward, N. F. and Fenton, P. C., Some results of Phragmén-Lindelöf type, J. Math. Anal. Appl., **192** (1995), 63-70. MR **96f:**30026

[4] Eremenko, A., Shia, D. and Sodin, M., The minimum of the modulus of an entire function on a sequence of Pólya peaks (in Russian), Teor. Funktsii Funktsional Anal. i Prilozhen **45** (1986), 26-40. English transl. in J. Soviet Math. **48** (1990), 386-398. MR **88f:**30045

[5] Hayman, W. K., *Subharmonic Functions*, vol. 2, London Mathematical Society Monographs, No. 20, Academic Press (1989). MR **91f:**31001

[6] Hayman, W. K. and Kennedy, P. B., *Subharmonic Functions*, vol. 1, London Mathematical Society Monographs, No. 9, Academic Press (1976). MR **57:**665

[7] Hinkkanen, A. and Rossi, John, Entire functions with asymptotic functions, Math. Scand. **77** (1995), 153-160. MR **97j:**30010

[8] Rado, T., *On the Problem of Plateau/Subharmonic Functions*, Springer-Verlag, New York (1971). MR **49:**9718

[9] Tsuji, M., *Potential Theory in Modern Function Theory*, Maruzen Co., Tokyo (1959). MR **22:**5712

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