REMARK ABOUT THE SPECTRUM
OF THE $p$-FORM LAPLACIAN UNDER A COLLAPSE
WITH CURVATURE BOUNDED BELOW

JOHN LOTT

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Abstract. We give a lower bound on the number of small positive eigenvalues
of the $p$-form Laplacian in a certain type of collapse with curvature bounded
below.

1. Introduction

A general problem in spectral geometry is to estimate the eigenvalues of the
$p$-form Laplacian on a closed Riemannian manifold $M$ in terms of the geometry of
$M$. From Hodge theory, the number of zero eigenvalues is $b_p(M)$, the $p$-th Betti
number of $M$. Hence the issue is to understand the positive eigenvalues. The
papers [2], [7] and [8] study the case when one assumes an upper bound on the
diameter of the manifold and double-sided bounds on the sectional curvatures. An
important phenomenon is the possible appearance of positive eigenvalues of the $p$-
form Laplacian that approach zero as a manifold collapses with bounded curvature.

The analysis of [7] and [8] uses the results of Cheeger, Fukaya and Gromov on the
geometric structure of manifolds that collapse with double-sided curvature bounds.
If one only assumes a lower sectional curvature bound, then there are some structure
results about collapsing in [4] and [12], but the theory is less developed than in the
bounded curvature case.

In this paper we look at the small positive eigenvalues of the $p$-form Laplacian
in an example of collapse with curvature bounded below. Namely, suppose that a
compact Lie group $G$ acts isometrically on $M$ on the left. Give $G$ a left-invariant
Riemannian metric. For $\epsilon > 0$, let $\epsilon G$ denote $G$ with its Riemannian metric
multiplied by $\epsilon^2$. Let $M_\epsilon$ denote $M = G \backslash (\epsilon G \times M)$ equipped with the quotient
Riemannian metric $g_\epsilon$, where $G$ acts diagonally on $\epsilon G \times M$ on the left. If $G$
is connected, then $\lim_{\epsilon \to 0} M_\epsilon = G \backslash M$ in the Gromov-Hausdorff topology, and as $\epsilon$
goes to zero, the sectional curvatures of $M_\epsilon$ stay uniformly bounded below [12].

For notation, if $M$ is a smooth connected closed manifold with Riemannian
metric $g$, let $\{\lambda_{p,j}(M, g)\}_{j=1}^\infty$ denote the eigenvalues (counted with multiplicity) of
the Laplacian on $\text{Im}(d) \subset \Omega^p_{L^2}(M)$. The projection $M \to G \backslash M$ induces a map
$H^p(G \backslash M; \mathbb{R}) \to H^p(M; \mathbb{R})$. 
Theorem 1.1. If \( j = \dim (\text{Ker}(H^p(G\setminus M; \mathbb{R}) \to H^p(M; \mathbb{R}))) \), then
\[
\lim_{\epsilon \to 0} \lambda_{p,j}(M_\epsilon, g_\epsilon) = 0.
\]

In Section 2 we prove Theorem 1.1. The main points of the proof are the use of a certain variational expression for \( \lambda_{p,j}(M, g) \), due to Cheeger and Dodziuk [3], and the avoidance of dealing with the detailed orbit structure of the group action. We then look at the example of an \( S^1 \)-action on \( S^{2n} \), which is the suspension of the Hopf action of \( S^1 \) on \( S^{2n-1} \), and show that our results slightly improve those of Takahashi [10]. In Section 3 we make some further remarks.

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2. PROOF OF THEOREM 1.1

Let \( \mathfrak{g} \) be the Lie algebra of \( G \). It acquires an inner product from the left-invariant Riemannian metric on \( G \). Given \( \xi \in \mathfrak{g} \), let \( \mathfrak{X} \) be the corresponding vector field on \( M \). Let \( i_X \) denote interior multiplication by \( \mathfrak{X} \).

Let \( \Omega^*(M) \) denote the smooth differential forms on \( M \). Let \( \Omega^*_{L^2}(M) \) be the \( L^2 \)-completion of \( \Omega^*(M) \). Put
\[
\Omega^*_\text{max}(M) = \{ \omega \in \Omega^*_L(M) : d\omega \in \Omega^{*+1}_L(M) \},
\]
where \( d\omega \) is originally defined distributionally.

Put
\[
\Omega^*_G(M) = \{ \omega \in \Omega^*(M) : g \cdot \omega = \omega \text{ for all } g \in G \}
\]
and
\[
\Omega^*_\text{basic}(M) = \{ \omega \in \Omega^*_G(M) : i_x \omega = 0 \text{ for all } x \in \mathfrak{g} \}.
\]

Let \( \Omega^*_G, L^2(M) \) and \( \Omega^*_\text{basic}, L^2(M) \) be the \( L^2 \)-completions of \( \Omega^*_G(M) \) and \( \Omega^*_\text{basic}(M) \), respectively. Put
\[
\Omega^*_\text{basic}, \text{max}(M) = \{ \omega \in \Omega^*_\text{basic}, L^2(M) : d\omega \in \Omega^{*+1}_\text{basic}, L^2(M) \},
\]
where \( d\omega \) is originally defined distributionally. Then \( \Omega^*_\text{basic}, \text{max}(M) \) is a complex.

From [6] and [11], the cohomology of the complex \( \Omega^*_\text{basic}(M) \) is isomorphic to \( H^*(G\setminus M; \mathbb{R}) \).

Lemma 2.1. The cohomology of the complex \( \Omega^*_\text{basic}, \text{max}(M) \) is also isomorphic to \( H^*(G\setminus M; \mathbb{R}) \).

Proof. The proof is essentially the same as that of [11]. For \( U \) an open subset of \( G\setminus M \), let \( \bar{U} \) be its preimage in \( M \). For \( p \geq 0 \), put \( S^p(U) = \Omega^p_{\text{basic}, \text{max}}(\bar{U}) \). If \( V \) is an open subset of \( U \), then there is an obvious restriction homomorphism \( S^p(U) \to S^p(V) \). We obtain a fine sheaf \( S^p \) over \( G\setminus M \). If \( R \) denotes the constant sheaf on \( G\setminus M \) with fiber \( \mathbb{R} \), then we have a complex of sheaves
\[
R \to S^0 \to S^1 \to S^2 \to \ldots.
\]

From sheaf cohomology theory, it suffices to prove that \( (2.5) \) is a resolution of \( R \). As in [11], one can use the slice theorem to reduce this to proving the middle exactness of the complex
\[
\Omega^{p-1}_{\text{basic}, \text{max}}(N) \to \Omega^p_{\text{basic}, \text{max}}(N) \to \Omega^{p+1}_{\text{basic}, \text{max}}(N).
\]
Here \( N \) is a Euclidean space and “basic” refers to a Lie group \( H \) that acts linearly and isometrically on \( N \). As in [11], one can use a homotopy operator \( A \) to prove
the exactness. The only point to note is that the homotopy operator $A$ used in \( \Omega_{\text{basic,max}} \) sends $\Omega_{\text{basic,max}}$ to itself.

The quotient map $p : \epsilon G \times M \to M$, defines a principal $G$-bundle. Pullback gives an isomorphism $p^* : \Omega^*(M) \cong \Omega_{\text{basic}}^*(\epsilon G \times M)$. The parallelism of $G$ gives an isomorphism

\[
(2.9) \quad \Omega^*(\epsilon G \times M) \cong C^\infty(G) \otimes \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M).
\]

Taking $G$-invariants gives isomorphisms

\[
(2.10) \quad \Omega_G^*(\epsilon G \times M) \to (C^\infty(G) \otimes \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M))^G \cong \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M),
\]

where $\beta$ comes from the map that sends $\sum_k f_k \otimes \eta_k \otimes \omega_k \in C^\infty(G) \otimes \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M)$ to $\sum_i f_k(e) \eta_k \otimes \omega_k \in \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M)$.

Let $\{x_j\}_{j=1}^{\text{dim}(G)}$ be a basis of $\mathfrak{g}$. For $x \in \mathfrak{g}$, let $e(x)$ denote exterior multiplication by $x$ on $\Lambda^* \mathfrak{g}^*$.

**Lemma 2.2.** There is an isomorphism of complexes $\mathcal{I} : \Omega^*(M) \to \Omega_{\text{basic}}^*(\epsilon G \times M) \subset \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M)$ given by

\[
\mathcal{I}(\sigma) = \left( \prod_{j=1}^{\text{dim}(G)} (1 - e(x_j^*) \otimes i_{x_j}) \right) (1 \otimes \sigma)
\]

\[
(2.11) \quad = \sum_{k=0}^{\text{dim}(G)} (-1)^k \sum_{1 \leq j_1 < \ldots < j_k \leq \text{dim}(G)} (r_{x_1} \wedge \ldots \wedge r_{x_k}) \otimes i_{x_1} \ldots i_{x_k} \sigma.
\]

**Proof.** If $\sum_k f_k \otimes \eta_k \otimes \omega_k \in (C^\infty(G) \otimes \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M))^G$ is $G$-basic, then for $x \in \mathfrak{g}$, we also have

\[
(2.12) \quad \sum_k \left( f_k \otimes i_x \eta_k \otimes \omega_k + (-1)^{|\eta_k|} f_k \otimes \eta_k \otimes i_x \omega_k \right) = 0.
\]

Then

\[
(2.13) \quad \sum_k \left( f_k(e) i_x \eta_k \otimes \omega_k + (-1)^{|\eta_k|} f_k(e) \eta_k \otimes i_x \omega_k \right) = 0,
\]

i.e., if $\sum_k \eta_k \otimes \omega_k$ lies in the image of $\beta$ restricted to $\Omega_{\text{basic}}^*(\epsilon G \times M)$, then

\[
(2.12) \quad \sum_k \left( i_x \eta_k \otimes \omega_k + (-1)^{|\eta_k|} \eta_k \otimes i_x \omega_k \right) = 0.
\]

It follows that $\sum_k \eta_k \otimes \omega_k$ can be written as $\mathcal{I}(\sigma)$ for some $\sigma \in \Omega^*(M)$. Thus $\mathcal{I}$ is surjective. It is clearly injective.

It remains to show that $\mathcal{I}$ is a morphism of complexes. Let $d^{\text{inv}}$ denote the (finite-dimensional) differential on $\Lambda^* \mathfrak{g}^*$. If an element of $\Omega^*_G(\epsilon G \times M)$ is represented as $\sum_k f_k \otimes \eta_k \otimes \omega_k \in C^\infty(G) \otimes \Lambda^* \mathfrak{g}^* \otimes \Omega^*(M)$, then the $G$-invariance implies that for $x \in \mathfrak{g}$,

\[
(2.13) \quad \sum_k \left( f_k \otimes \eta_k \otimes \omega_k + f_k \otimes \eta_k \otimes \mathcal{L}_x \omega_k \right) = 0.
\]
The differential of $\sum_k f_k \otimes \eta_k \otimes \omega_k$ is represented by

$$
\sum_k \left( \sum_{j=1}^{\dim(G)} \xi_j f_k \otimes e(\xi_j) \eta_k \otimes \omega_k + f_k \otimes d^{inv} \eta_k \otimes \omega_k \right)
+ (-1)^{|\eta_k|} f_k \otimes \eta_k \otimes d\omega_k.
$$

From (2.13), this equals

$$
\sum_k \left( - \sum_{j=1}^{\dim(G)} f_k \otimes e(\xi_j) \eta_k \otimes \mathcal{L}_{X_j} \omega_k + f_k \otimes d^{inv} \eta_k \otimes \omega_k \right)
+ (-1)^{|\eta_k|} f_k \otimes \eta_k \otimes d\omega_k.
$$

Using $\beta$, it follows that the induced differential on $\Lambda^*(g^*) \otimes \Omega^*(M)$ sends $\sum_k \eta_k \otimes \omega_k$ to

$$
\sum_k \left( - \sum_{j=1}^{\dim(G)} e(\xi_j) \eta_k \otimes \mathcal{L}_{X_j} \omega_k + d^{inv} \eta_k \otimes \omega_k + (-1)^{|\eta_k|} \eta_k \otimes d\omega_k \right).
$$

One can check that when this acts on $\mathcal{I}(\sigma)$, the result is $\mathcal{I}(d\sigma)$. Thus $\mathcal{I}$ is an isomorphism of complexes. \qed

In fact, under our identifications, $\mathcal{I}$ is the same as $p^*$.

Let $M^{reg}$ be the union of the principal orbits for the $G$-action on $M$. It is a dense open subset of $M$ with full measure. If $m \in M^{reg}$, let $H \subseteq G$ be its isotropy subgroup, with Lie algebra $\mathfrak{h}$. Define $\alpha : \mathfrak{g} \to T_m M$ by $\alpha(x) = X_m$. It passes to an injection $\pi : \mathfrak{g}/\mathfrak{h} \to T_m M$. For $\epsilon \geq 0$, put $\rho_\epsilon(m) = \det^{1/2}(\epsilon^2 \Id_{\mathfrak{g}/\mathfrak{h}} + \pi^* \pi)$. If $m \notin M^{reg}$, put $\rho_\epsilon(m) = 0$. Note that for $\epsilon > 0$, $\rho_\epsilon^{-1}(m) < \rho_0^{-1}(m)$.

**Lemma 2.3.** $\rho_0^{-1} \in L^1(M, \text{dvol})$.

**Proof.** If $m \in M^{reg}$, then up to an overall constant, $\rho_0(m) \equiv \text{the volume of the orbit } G \cdot m$. Then $\int_{M^{reg}} \rho_0^{-1}(m) \text{dvol}(m)$ is proportionate to the volume of $G \setminus M^{reg} \subset G \setminus M$, which is seen to be finite. \qed

Let $\{\xi_j\}_{j=1}^{\dim(G)}$ be an orthonormal basis of $\mathfrak{g}$.

**Lemma 2.4.** For $\epsilon > 0$, there is a positive constant $C(\epsilon)$ such that $\Omega^*(M_\epsilon)$ is isometrically isomorphic to $\Omega^*(M)$ with the new norm

$$
\| \omega \|^2 = C(\epsilon) \int_M \rho_\epsilon^{-1}(m) \left( \|\omega(m)\|_M^2 + \sum_{k=1}^{\dim(G)} \epsilon^{-2k} \sum_{1 \leq j_1 < \ldots < j_k \leq \dim(G)} \|i_{\xi_{j_1}} \ldots i_{\xi_{j_k}} \omega(m)\|_M^2 \right) \text{dvol}(m).
$$

**Proof.** We can compute the norm squared of $\omega \in \Omega^*(M_\epsilon)$ by taking the local norm squared of $p^* \omega$ on $\epsilon G \times M^{reg}$, dividing by the function that assigns to
(g, m) ∈ εG × Mreg the volume of the orbit G·(g, m), and integrating over εG × Mreg. If m ∈ Mreg, then the relative volume of G·(g, m) is
\[ \det^{1/2}(e^{2} \text{Id})_{g} + α^*α = e^{dim(H)} \rho_{ε}(m). \]

The map β of (2.28) is an isometry, up to a constant. Since \( \{e^{-1}V_{j}\}_{j=1}^{dim(G)} \) is an orthonormal basis for \( T_{εG} \), the lemma follows from Lemma (2.2).

**Proof of Theorem** (2.2). Put \( \lambda_{p, j}(ε) = \lambda_{p, j}(M, g_{ε}). \) From (2.3),
\[ \lambda_{p, j}(ε) = \inf_{V} \sup_{η \in V \to 0} \sup_{θ \in d^{-1}(η)} \frac{\|η\|^{2}}{\|θ\|^{2}}, \]
where \( V \) ranges over the \( j \)-dimensional subspaces of \( \text{Im}(d: Ω^{p-1}(M) → Ω^{p}(M)) \), and \( θ \in d^{-1}(η) \subset Ω^{p-1}(M) \).

Take \( j = \text{dim Ker}(H^{p}(G\setminus M; R) → H^{p}(M; R)) \). From Lemma (2.1), the inclusion of complexes \( Ω_{\text{basic}}^{1}(M) → Ω_{\text{basic,max}}^{1}(M) \) induces an isomorphism on cohomology. Then there is a \( j \)-dimensional subspace \( V \) of
\[ \text{Ker}(d: Ω^{p}_{\text{basic}}(M) → Ω^{p+1}_{\text{basic}}(M)) \cap \text{Im}(d: Ω^{p-1}(M) → Ω^{p}(M)) \]
such that if \( η \in V \to 0 \), then \( η \notin \text{Im}(d: Ω^{p-1}_{\text{basic,max}}(M) → Ω^{p}_{\text{basic},L^{2}}(M)) \). We claim that
\[ \lim_{c → 0} \sup_{η \in V \to 0} \sup_{θ \in d^{-1}(η)} \frac{\|η\|^{2}}{\|θ\|^{2}} = 0. \]
This will suffice to prove the theorem.

Suppose that (2.21) is not true. Then there is a constant \( c > 0 \), a sequence \( \{ε_{r}\}_{r=1}^{∞} \) in \( R^{+} \) approaching zero, a sequence \( \{η_{r}\}_{r=1}^{∞} \) in \( V \to 0 \) and a sequence \( \{θ_{r}\}_{r=1}^{∞} \) in \( Ω^{p-1}(M) \) such that for all \( r \), \( dθ_{r} = η_{r} \) and
\[ \frac{\|η_{r}\|^{2}_{ε_{r}}}{\|θ_{r}\|^{2}_{ε_{r}}} \geq c. \]
Doing a Fourier decomposition of \( θ_{r} \) with respect to \( G \), the ratio in (2.22) does not decrease if we replace \( θ_{r} \) by its \( G \)-invariant component. Thus we may assume that \( θ_{r} \) is \( G \)-invariant.

Without loss of generality, we can replace the norm \( \|·\|_{ε} \) of (2.17) by the same norm divided by \( C(ε) \), which we again denote by \( \|·\|_{ε} \). Since \( η_{r} \) is smooth on \( M \), it follows from Lemma (2.3) that the function \( ρ_{0}^{-1}(m)|η_{r}(m)|^{3}_{M} \) is integrable on \( M \). Without loss of generality, we may assume that
\[ \int_{M} ρ_{0}^{-1}(m)|η_{r}(m)|^{3}_{M} \text{dvol}(m) = 1. \]
Since \( \{η_{r}\}_{r=1}^{∞} \) lies in the sphere of a finite-dimensional space, there will be a subsequence, which we relabel as \( \{η_{r}\}_{r=1}^{∞} \), that converges smoothly to some \( η_{∞} \in V \to 0 \).

From (2.22),
\[ \|θ_{r}\|^{2}_{ε_{r}} \leq c^{-1} \|η_{r}\|^{2}_{ε_{r}} = c^{-1} \int_{M} ρ_{0}^{-1}(m)|η_{r}(m)|^{3}_{M} \text{dvol}(m) \]
\[ \leq c^{-1} \int_{M} ρ_{0}^{-1}(m)|η_{r}(m)|^{3}_{M} \text{dvol}(m) = c^{-1}. \]
For large $r$,  

\begin{equation}
\int_M |\theta_r(m)|_M^2 \, d\text{vol}(m) \leq (\inf_M \rho_r^{-1})^{-1} \int_M \rho_r^{-1}(m) |\theta_r(m)|_M^2 \, d\text{vol}(m) \\
\leq (\inf_M \rho_1^{-1})^{-1} c^{-1}.
\end{equation}

We now work with respect to the metric $g$ on $M$. By weak-compactness of the unit ball in $L^2$, there is a subsequence of $\{\theta_r\}_{r=1}^\infty$, which we relabel as $\{\theta_r\}_{r=1}^\infty$, that converges weakly in $L^2$ to some $\theta_\infty \in \Omega_{G,L^2}^{p-1}(M)$. Then for $\sigma \in \Omega^p(M)$,  

\begin{equation}
\langle \sigma, \eta_\infty \rangle_M - \langle d^* \sigma, \theta_\infty \rangle_M = \lim_{r \to \infty} \langle \sigma, \eta_r \rangle_M - \langle d^* \sigma, \theta_r \rangle_M \\
= \lim_{r \to \infty} \langle \sigma, \eta_r - d\theta_r \rangle_M = 0.
\end{equation}

Thus $\theta_\infty \in \Omega_{\text{basic, max}}^{p-1}(M)$ and $d\theta_\infty = \eta_\infty$.

From (2.22), we also obtain that for each $1 \leq j \leq \dim(G)$,  

\begin{equation}
\int_M |i_x \theta_r(m)|_M^2 \, d\text{vol}(m) \leq (\inf_M \rho_r^{-1})^{-1} \int_M \rho_r^{-1}(m) |i_x \eta_r(m)|_M^2 \, d\text{vol}(m) \\
\leq (\inf_M \rho_1^{-1})^{-1} c^{-1} \varepsilon_r^2.
\end{equation}

Then for all $\sigma \in \Omega^{p-2}(M)$,  

\begin{equation}
\langle \sigma, i_x \theta_\infty \rangle_M = \langle (i_x)^* \sigma, \theta_\infty \rangle_M = \lim_{r \to \infty} \langle (i_x)^* \sigma, \theta_r \rangle_M = \lim_{r \to \infty} \langle \sigma, i_x \theta_r \rangle_M = 0.
\end{equation}

Thus $i_x \theta_\infty = 0$ and $\theta_\infty \in \Omega_{\text{basic, max}}^{p-1}(M)$. Hence

\begin{equation}
\eta_\infty \in \text{Im} \left( d : \Omega_{\text{basic, max}}^{p-1}(M) \to \Omega_{\text{basic, }L^2}^p(M) \right),
\end{equation}

which is a contradiction. \hfill \Box

**Example.** Let $G = U(1)$ act on $M = S^{2n}$ by the suspension of the Hopf action of $U(1)$ on $S^{2n-1}$. Then $G \backslash M = U(1) \backslash S^{2n}$ is the suspension of $\mathbb{C}P^{n-1}$. One finds that $\text{Ker}(H^p(G \backslash M; \mathbb{R}) \to H^p(M; \mathbb{R}))$ is nonzero if and only if $p \in \{3, 5, \ldots, 2n-1\}$.

From Theorem 1.1 as $\epsilon \to 0$, there are small eigenvalues of the $p$-form Laplacian on $\text{Im}(d) \subset \Omega_{L^2}^p(M)$ for $p \in \{3, 5, \ldots, 2n-1\}$. From the Hodge decomposition, there will also be small eigenvalues of the $p$-form Laplacian on $\text{Im}(d^* \sigma) \subset \Omega_{L^2}^p(M)$ for $p \in \{2, 4, \ldots, 2n-2\}$. Then using Hodge duality, one concludes that there are small eigenvalues on

1. $\text{Im}(d^* \sigma) \subset \Omega_{L^2}^1(M)$,
2. $\text{Im}(d) \subset \Omega_{L^2}^p(M)$ and $\text{Im}(d^*) \subset \Omega_{L^2}^p(M)$ for $p \in \{2, 3, 4, \ldots, 2n-3, 2n-2\}$, and
3. $\text{Im}(d) \subset \Omega_{L^2}^{2n-1}(M)$.

This slightly sharpens [11] Theorem 1.2. Note that from eigenvalue estimates for the scalar Laplacian [1], there are no small eigenvalues on $\text{Im}(d^* \sigma) \subset \Omega_{L^2}^0(M)$, $\text{Im}(d) \subset \Omega_{L^2}^1(M)$, $\text{Im}(d^*) \subset \Omega_{L^2}^{2n-1}(M)$ or $\text{Im}(d) \subset \Omega_{L^2}^{2n}(M)$.

3. **Remarks**

1. In the case of a locally-free torus action, there is some intersection between Theorem 1.1 and the results of [2], [7] and [8]. In [8] one deals with the cohomology of a certain $\mathbb{Z}$-graded sheaf $H^p(A_{[r]})$ on the limit space $X$. In the case of a collapsing coming from a locally-free torus action, Theorem 1.1 is a statement about the case...
Then there are a number of properties that $\lim_{i \to \infty} M_i$ possesses. Suppose first that for some $p > 0$, $\lambda_{p,j}(M_i, g_i)$ is positive for all $i$. Then there is a sequence $\{ \lambda_{p,j}(M_i, g_i) \}_{i=1}^{\infty}$ in $M$. Then there is a sequence $\{ (M_i, g_i) \}_{i=1}^{\infty}$ in $M$ with the property that $\lim_{i \to \infty} \lambda_{p,j}(M_i, g_i) = \infty$. A subsequence of $\{ (M_i, g_i) \}_{i=1}^{\infty}$, which we relabel as $\{ (M_i, g_i) \}_{i=1}^{\infty}$, will have finite distance from some $(M_\infty, g_\infty) \in M$. Then there are a number $\epsilon > 0$ and a sequence of bi-Lipschitz homeomorphisms $h_i : M_i \to M$ so that for all $i$,\begin{equation}
abla_{g_\infty} \leq h_i^* g_i \leq e^\epsilon g_\infty.\end{equation}
Here $h_i^* g_i$ is a Lipschitz metric on $M_\infty$. From Hodge theory, we have\begin{equation}
abla_{g_\infty} \leq h_i^* g_i \leq e^\epsilon g_\infty.\end{equation}
where $V$ ranges over the $j$-dimensional subspaces of $\Im (d : \Omega_{\max}^{p-1}(M_i) \to \Omega_{L^2}^p(M_i))$, and $\theta \in d^{-1}(\eta) \subset \Omega_{\max}^{p-1}(M_i)$. By naturality,\begin{equation}
abla_{g_\infty} \leq h_i^* g_i \leq e^\epsilon g_\infty.\end{equation}
where $V$ ranges over the $j$-dimensional subspaces of $\Im (d : \Omega_{\max}^{p-1}(M_\infty) \to \Omega_{L^2}^p(M_\infty))$, and $\theta \in d^{-1}(\eta) \subset \Omega_{\max}^{p-1}(M_\infty)$.

As in $[3]$, it follows from (3.1) and (3.3) that there is a positive integer $J$ which only depends on $n$ so that\begin{equation}
e^{-J \epsilon} \lambda_{p,j}(M_\infty, g_\infty) \leq \lambda_{p,j}(M_i, g_i) \leq e^{J \epsilon} \lambda_{p,j}(M_\infty, g_\infty).\end{equation}
This contradicts the assumption that $\lim_{i \to \infty} \lambda_{p,j}(M_i, g_i) = \infty$.

Now suppose that it is not true that there is a uniform lower bound $a_{p,j}$ on $\{ \lambda_{p,j}(M_i, g_i) \}_{(M_i, g_i) \in M}$ with the property that $\lim_{j \to \infty} a_{p,j} = \infty$. Then there are a number $C > 0$, a sequence $\{ (M_i, g_i) \}_{i=1}^{\infty}$ in $M$ and a sequence of integers $\{ j_i \}_{i=1}^{\infty}$ such that $\lim_{i \to \infty} j_i = \infty$ and for each $i$, $\lambda_{p,j_i}(M_i, g_i) \leq C$. Take a subsequence $\{ (M_i, g_i) \}_{i=1}^{\infty}$ and an $(M_\infty, g_\infty)$ as before. Then for each $j$,$\begin{equation}
\lambda_{p,j}(M_\infty, g_\infty) \leq \sup_{i \to \infty} \lambda_{p,j_i}(M_\infty, g_\infty) \leq \sup_{i \to \infty} e^{J \epsilon} \lambda_{p,j_i}(M_i, g_i) \leq e^{J \epsilon} C.
\end{equation}
This contradicts the fact that the spectrum of the $p$-form Laplacian on $(M_\infty, g_\infty)$ is discrete.

Proposition 4 shows that in a certain sense, one has uniform eigenvalue bounds in the noncollapsing case. It seems possible that for a given $n \in \mathbb{Z}^+$, $K \in \mathbb{R}$ and $v, D > 0$, the collection $\mathcal{M}$ of connected $n$-dimensional Riemannian manifolds $(M, g)$ with sectional curvatures greater than $K$, volume greater than $v$ and diameter less than $D$ satisfies the hypotheses of Proposition 4. It is known that there is a finite number of homeomorphism types in $\mathcal{M}$ [5]. On the other hand, the analogous space of metrics defined with Ricci curvature, instead of sectional curvature, will generally not satisfy the hypotheses of Proposition 4 [9].

REFERENCES


Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109-1109

E-mail address: lott@umich.edu