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ON THE INVARIANCE OF CLASSES $\Phi BV, \Lambda BV$ UNDER COMPOSITION

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ABSTRACT. The necessary and sufficient condition for $g \circ f$ to be in the class $\Phi BV, \Lambda BV$ for every f of that class whose range is in the domain of g is that g be in Lip 1.

Several different classes of functions arise naturally in the study of the convergence of Fourier series [W1]–[W4], [S], [Y]. We concern ourselves here with two such classes: ΛBV and ΦBV. Waterman [W1] defined the class ΛBV as follows: Suppose $\Lambda = \{\lambda_n\}$ is an increasing sequence such that $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$. We say that $f \in \Lambda BV$ on $[a,b]$ if $\sum_{n=1}^{\infty} |f(I_n)|/\lambda_n < \infty$ for every set $\{I_n\}$ of nonoverlapping intervals in $[a,b]$. If $\{\lambda_n\} = \{n\}$, the resulting class is called HBV, the functions of *harmonic bounded variation.*

To define the class ΦBV, we let φ be a convex function with domain $[0,\infty)$ having the following three properties:

1. $\varphi(0) = 0$, $\varphi(x) > 0$ for $x > 0$;
2. $\frac{\varphi(x)}{x} \to 0$ as $x \to 0$;
3. $\frac{\varphi(x)}{x} \to \infty$ as $x \to \infty$.

Another property which is sometimes assumed is:

4. there exists $a > 0$ and $\delta > 0$ such that
$$\frac{\varphi(2x)}{\varphi(x)} \leq \delta \quad \text{for} \quad x \in (0,a].$$

The last condition above is usually called the condition Δ_2 (often called Δ_2 for small values).

Let $P = \{a = x_0, x_1, \ldots, x_m = b\}$ be a partition of $[a,b]$. We use the notation $I_n = [x_{n-1}, x_n]$, and we write $f(I_n) = f(x_n) - f(x_{n-1})$. Since φ may be represented as an integral of a nondecreasing function, clearly φ itself is a strictly increasing continuous function on $[0,\infty)$.

We say that $f \in \Phi BV$ [MO] on an interval $[a,b]$ if there exists a constant M such that whenever $P = \{I_n\}$ is an arbitrary partition of $[a,b]$, we have
$$\sum_P \phi\left(|f(I_n)|\right) < M.$$

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Alternatively we may require the infinite sum $\sum_{n=1}^{\infty} \phi(|f(I_n)|)$ to be finite whenever $\{I_n\}_{n=1}^{\infty}$ is a collection of *nonoverlapping* intervals in $[a, b]$. These two definitions have been shown to be equivalent. We will henceforth omit all reference to the interval $[a, b]$ and simply write ΦBV to denote this class. We see the importance of the condition Δ_2 in the following theorem.

Theorem (Musielak and Orlicz)**.** *The class ΦBV is linear if and only if Δ_2 is satisfied.*

In what follows we shall assume that φ satisfies Δ_2.

GW represents the class of functions (necessarily having only simple discontinuities) that have a convergent Fourier series for every change of variable, and the class UGW is defined analogously with respect to uniform convergence. Chaika and Waterman [CW] proved the following.

Theorem (Chaika and Waterman)**.** *$g \circ f$ is in one of the classes GW, UGW or HBV for each f of that class whose range is in the domain of g if and only if $g \in \operatorname{Lip} 1$.*

Josephy [J] had proved an analogous theorem for the class BV.

We are interested now in necessary and sufficient conditions for a function f to be preserved as a member of the classes $\Lambda BV, \Phi BV$ when it is composed with a function g on the left. We prove here the following extension of the theorem of Chaika and Waterman:

Theorem. *$g \circ f$ is in the class ΛBV or ΦBV for each f of that class whose range is in the domain of g if and only if $g \in \operatorname{Lip} 1$.*

Proof. We observe that the continuity of g is necessary. Suppose g is discontinuous. We may suppose $g(0) = 0, g(t_n) \geq 1$, where $t_n \searrow 0$ and $\sum_{n=1}^{\infty} t_n < 1$. We will define a function $f \in BV$ such that $g \circ f \notin \Lambda BV \cup \Phi BV$. Let $\{a_n\}$ be a sequence in $(0, 1)$ converging downward to 0. Let f be 0 at 0 and on $[a_1, 1]$. On each interval $[a_{n+1}, a_n]$ we define f to be a tent function such that $f(a_{n+1}) = f(a_n) = 0$ and $f((a_{n+1} + a_n)/2) = t_n$. Note that $f \in BV$ and therefore $f \in \Lambda BV$ and $f \in \Phi BV$. Let $I_n = [(a_n + a_{n+1})/2, a_n]$. Then $(g \circ f)(I_n) = g(t_n) \geq 1$. Hence

$$\sum_{n=1}^{\infty} \frac{|(g \circ f)(I_n)|}{\lambda_n} \geq \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$$

and

$$\sum_{n=1}^{\infty} \varphi(|g \circ f(I_n)|) \geq \sum_{n=1}^{\infty} \varphi(1) = \infty.$$

Let us now suppose $g \in \operatorname{Lip} 1$ and I_n is a collection of nonoverlapping intervals in the domain of f. Then there is a $c > 0$ such that for any interval I_n, $|g \circ f(I_n)| < c|f(I_n)|$. Thus if f is in ΛBV or ΦBV, then so is $g \circ f$.

Now we assume that g is continuous but $g \notin \operatorname{Lip} 1$, and the range of f is in the domain of g. Without loss of generality, we may suppose that the domain of g contains $[0, a]$ for some positive a and that there is a sequence of disjoint intervals $J_n = [p_n, q_n]$ in the domain of g, with $p_n, q_n \searrow 0$, and a sequence of positive numbers $c_n \to \infty$, $c_1 > 1$, such that

$$\sum_{n=1}^{\infty} \frac{1}{c_n} < \infty, \quad |J_n|\, c_n \leq g(J_n),$$

and

$$\sum_{n=1}^{\infty} \varphi(q_n) < \infty.$$

Since g is continuous and $|J_n| \to 0$, we see that $|g(J_n)| \to 0$ implying $c_n|J_n| \to 0$. We may choose $\{J_n\}$ so that $\{c_n|J_n|\}$ is monotone.

We now construct a function $f \in \Lambda BV$ such that $g \circ f \notin \Lambda BV$. To do this, we define a function $L : [0, \infty) \to [0, \infty)$ by setting

$$L(q) = \sum_{k=1}^{q} \frac{1}{\lambda_k} \quad \text{for} \quad q \in Z^+, \quad L(0) = 0,$$

and extending L to be linear on each interval $[q-1, q], q \in Z^+$.

We observe that L is strictly increasing on $[0, \infty)$, $L(x) \to \infty$ as $x \to \infty$, and hence L^{-1} exists. There is no loss of generality if we assume that $\lambda_1 \geq 1$. Let $k_n = \left[L^{-1}(1/(c_n|J_n|) + 1)\right]$, where $[t]$ denotes the greatest integer less than or equal to t. Then

$$k_n \to \infty, \quad L(k_n) > \frac{1}{c_n|J_n|} \quad \text{and} \quad g(J_n)L(k_n) > \frac{g(J_n)}{c_n|J_n|} \geq 1.$$

There is a $C > 1$ such that

$$\frac{1}{c_n} \;\leq\; |J_n| L(k_n) \;<\; C\,\frac{1}{c_n}.$$

For this C we then have

$$\sum_{n=1}^{\infty} |J_n|\, L(k_n) < C \sum_{n=1}^{\infty} \frac{1}{c_n} < \infty. \tag{1}$$

For each $n = 1, 2, \ldots$, let $I_{n,1}, I_{n,2}, \ldots, I_{n,k_n}$ be a collection of disjoint closed intervals in $(2^{-n}, 2^{-(n-1)})$ with $I_{n,m}$ to the left of $I_{n,m+1}$ for each m.

Let f be defined on each $I_{n,m}$ ($m = 1, 2, \ldots, k_n; n = 1, 2, \ldots$) to be the increasing linear map of $I_{n,m}$ onto J_n. Set $f(0) = f(1) = 0$. Define f to be linear on each of the component intervals of the remainder of $[0, 1]$ and continuous on $[0, 1]$. We now claim that $f \in \Lambda BV$ while $g \circ f \notin \Lambda BV$.

To see that $g \circ f \notin \Lambda BV$ we observe that

$$\begin{aligned} V_\Lambda\left(g \circ f, \left[\frac{\pi}{2^n}, \frac{\pi}{2^{n-1}}\right]\right) \quad &\geq \quad \sum_{m=1}^{k_n} \frac{(g \circ f)(I_{n,m})}{\lambda_m} \\ &= \quad g(J_n)L(k_n) \geq 1. \end{aligned}$$

If $g \circ f$ were in ΛBV, then the right continuity of $g \circ f$ at 0 would imply $V_\Lambda(g \circ f, [0, \epsilon]) \to 0$ as $\epsilon \to 0$ (see [W2]). Hence $g \circ f \notin \Lambda BV$. □

To see that $f \in \Lambda BV$ we first recall a definition of Banach:

Definition. If f is a continuous function, then $\mathcal{N}(y) = \mathcal{N}(f; y) = \text{card}\{x | f(x) = y\}$. (This function $\mathcal{N}$ has been called the Banach indicatrix of f.)

This notion is easily extended to regulated functions, i.e., those functions with only simple discontinuities, by adjoining a vertical line connecting $f(x+)$ and $f(x-)$ at each point x of discontinuity (see [W1]). $\mathcal{N}(y)$ is then defined to be the number of intersection points of this "extended" graph of f and the horizontal line of height

y. The following result of Waterman [W2], which generalizes a result of Goffman [G], will prove useful:

Theorem. *If* $\inf f = A$ *and* $\sup f = B$, *f has only simple discontinuities, and $L(x)$ is an increasing function such that*

$$L(n) \sim \sum_{1}^{n} \frac{1}{\lambda_k} \qquad as \qquad n \to \infty,$$

then $\int_A^B L(\mathcal{N}(y))\,dy < \infty$ *implies that f is in the class ΛBV.*

The importance of the class of functions satisfying $\int_A^B L(\mathcal{N}(y))\,dy < \infty$ was first observed by Garsia and Sawyer [GS] for continuous functions and $L(x) = \log(x)$.

Now we need only observe

$$\begin{aligned}
\int L(\mathcal{N}(f;y))\,dy &\leq q_1 L(2) + \sum_{n=1}^{\infty} |J_n|\, L(2k_n) \\
&< q_1 L(2) + 2\sum_{n=1}^{\infty} |J_n|\, L(k_n) \\
&< \infty
\end{aligned}$$

by (1). Hence $f \in \Lambda BV$, and we have the portion of the desired result which is concerned with ΛBV.

To complete the proof for ΦBV, we now define a function P by setting

$$P(n) = \frac{1}{\varphi(c_n |J_n|)} \qquad \text{for } n \in Z^+, \quad P(0) = 0,$$

and extending P to be linear on each interval $[k-1, k], k \in Z^+$. Then P is a one-to-one increasing mapping of $[0, \infty)$ onto $[0, \infty)$. Let $k_n = [P(n) + 1]$. Thus $k_n > P(n) = 1/(\varphi(c_n|J_n|))$.

Let f be defined as in the construction for ΛBV but with this definition of k_n. We now claim that $f \in \Phi BV$ while $g \circ f \notin \Phi BV$.

To see that $g \circ f \notin \Phi BV$ we observe that

$$\begin{aligned}
\sum_{m=1}^{k_n} \varphi(|\,(g \circ f)(I_{n,m})|) &= \sum_{m=1}^{k_n} \varphi(|g(J_n)|) \\
&= k_n \varphi(|g(J_n)|) \\
&\geq k_n \varphi(c_n |J_n|) \\
&> \frac{\varphi(c_n|J_n|)}{\varphi(c_n|J_n|)} \\
&= 1.
\end{aligned}$$

So for this collection $\{I_{n,m}\}$ of intervals, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{m=1}^{k_n} \varphi(|(g \circ f)(I_{n,m})|) &\geq \sum_{n=1}^{\infty} 1 \\
&= \infty,
\end{aligned}$$

implying $g \circ f \notin \Phi BV$.

We now show that $f \in \Phi BV$. Let $\{0 = x_0, x_1, \ldots, x_m = 1\}$ be an arbitrary partition of $[0,1]$. We will find an upper bound for the sum

$$\sum_{i=1}^{m} \varphi\left(|f(x_i) - f(x_{i-1})|\right).$$

Consider the contribution of an arbitrary term, $\varphi(|f(x_i) - f(x_{i-1})|)$. For $i = 1$ we have

$$\varphi\left(|f(x_1) - f(0)|\right) = \varphi\left(|f(x_1)|\right) \leq \varphi(q_1)$$

and similarly for $i = m$ we get $\varphi(|0 - f(x_{m-1})|) \leq \varphi(q_1)$. For $i \neq 1, m$ we have the following cases:

Case 1. There exists n such that $f(x_i), f(x_{i-1}) \in J_n$. If $I_{n,m} = [a_{n,m}, b_{n,m}]$, then $x_i, x_{i-1} \in [a_{n,1}, b_{n,k_n}]$. Clearly,

$$\varphi\left(|f(x_i) - f(x_{i-1})|\right) < \varphi\left(|J_n|\right).$$

We observe that increasing the number of partition points along intervals of monotonicity does not increase the sum, and hence the contribution of all such terms is bounded by $2k_n\varphi(|J_n|)$.

Case 2. There does not exist an n such that $f(x_i), f(x_{i-1}) \in J_n$. Then there is a smallest integer j satisfying $f(x_i) \leq q_j$.

Suppose there are r such intervals having this same smallest integer j satisfying the above. If these intervals are

$$[x_{i_0}, x_{i_0+1}], [x_{i_0+1}, x_{i_0+2}], \ldots, [x_{i_0+r-1}, x_{i_0+r}],$$

we then have, since φ is convex,

$$\begin{aligned} \sum_{l=1}^{r} \varphi(|f(x_{i_0+l}) - f(x_{i_0+l-1})|) &\leq \varphi(|f(x_{i_0+r}) - f(x_{i_0})|) \\ &\leq \varphi(q_j). \end{aligned}$$

We observe that the above situation can arise at most once for each index j, $j = 1, \ldots, m$.

By combining all of the above information we show that an upper bound for $\sum_1^m \varphi(|f(x_i) - f(x_{i-1})|)$ is

$$\begin{aligned} M &= 2\varphi(q_1) + \sum_{n=1}^{\infty} 2k_n\varphi(|J_n|) + \sum_{n=1}^{\infty} \varphi(q_n) \\ &= \quad \mathrm{I} \quad + \quad \mathrm{II} \quad + \quad \mathrm{III}. \end{aligned}$$

Clearly I, II, and III are independent of the choice of the partition. Obviously I is finite and q_n was chosen so that III is finite. Since $c_n \geq 1$, it follows that

$\varphi(c_n|J_n|) > c_n\varphi(|J_n|)$. Thus

$$\begin{aligned}\sum_{n=1}^{\infty} k_n\varphi(|J_n|) &= \sum_{n=1}^{\infty}[P(n)+1]\varphi(|J_n|)\\ &\leq \sum_{n=1}^{\infty}\left(\frac{1}{\varphi(c_n|J_n|)}+1\right)\varphi(|J_n|)\\ &\leq \sum_{n=1}^{\infty}\left(\frac{\varphi(|J_n|)}{c_n\varphi(|J_n|)}+\varphi(|J_n|)\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{c_n}+\sum_{n=1}^{\infty}\varphi(|J_n|)\\ &\leq \sum_{n=1}^{\infty}\frac{1}{c_n}+\sum_{n=1}^{\infty}\varphi(q_n) < \infty.\end{aligned}$$

Thus $f \in \Phi BV$, and the proof is complete.

References

[CW] M. Chaika and D. Waterman, *On the invariance of certain classes of functions under composition*, Proc. Amer. Math. Soc. **43** (1974), 345–348. MR **48:**8704

[G] C. Goffman, *Everywhere convergence of Fourier series*, Indiana Univ. Math. J. **20** (1970), 107–112. MR **42:**4941

[GW] C. Goffman and D. Waterman, *Functions whose Fourier series converge for every change of variable*, Proc. Amer. Math. Soc. **19** (1968), 80–86. MR **36:**4245

[GS] A. Garsia and S. Sawyer, *On some classes of continuous functions with convergent Fourier series*, J. of Math. and Mech. **13** (1964), 586–601. MR **33:**7777

[J] M. Josephy, *Composing functions of bounded variation*, Proc. Amer. Math. Soc. **83** (1981), 354–356. MR **83c:**26009

[MO] J. Musielak and W. Orlicz, *On generalized variations I*, Studia Math. **18** (1959), 11–41. MR **21:**3524

[S] R. Salem, *Sur un test général pour le convergence uniforme des séries de Fourier*, Comptes Rend. Acad. Sci. Paris v. **207** (1938), 662–664.

[W1] D. Waterman, *On convergence of Fourier series of functions of generalized bounded variation*, Studia Math. **44** (1972), 107–117. MR **46:**9623

[W2] D. Waterman, *On Λ-bounded variation*, Studia Math. **57** (1976), 33–45. MR **54:**5408

[W3] D. Waterman, *On the summability of Fourier series of functions of Λ-bounded variation*, Studia Math. **55** (1976), 87–95. MR **53:**6212

[W4] D. Waterman, *Fourier series of functions of Λ-bounded variation*, Proc. Amer. Math. Soc. **74** (1979), 119–123. MR **80j:**42010

[Y] L. C. Young, *Sur une généralisation de la notion de variation de puissance p-ième bornée au sense de M. Wiener, et sur la convergence des séries de Fourier*, Comptes Rend. Acad. Sci. Paris **204** (1937), 470–472.

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