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ON INVARIABILITY OF SELF-INJECTIVE ALGEBRAS OF TILTED TYPE UNDER STABLE EQUIVALENCES

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Abstract. We prove that a large class of self-injective algebras of tilted type is invariant under stable equivalences of module categories.

1. Introduction and the main result

Throughout the paper, by an algebra is meant a basic, connected, Artin algebra (associative, with an identity) over a fixed commutative Artinian ring K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\underline{\operatorname{mod}}A$ the stable module category of A. Recall that the objects of $\underline{\operatorname{mod}}A$ are the objects of $\operatorname{mod} A$ without nonzero projective direct summands, and for any two objects M and N in $\underline{\operatorname{mod}}A$ the K-module $\underline{\operatorname{Hom}}_A(M,N)$ of morphisms from M to N is the quotient $\operatorname{Hom}_A(M,N)/P(M,N)$, where $P(M,N)$ is the submodule of $\operatorname{Hom}_A(M,N)$ consisting of all A-homomorphisms that factorize through projective A-modules. Two algebras A and Λ are said to be stably equivalent if their stable module categories $\underline{\operatorname{mod}}A$ and $\underline{\operatorname{mod}}\Lambda$ are equivalent. Recall also that $D = \operatorname{Hom}_K(-,E)$, where E is a minimal injective cogenerator in $\operatorname{mod} K$, defines a duality between the categories of left and right modules. An algebra A is called self-injective if $A \cong D(A)$ in $\operatorname{mod} A$, that is, the projective A-modules are injective. An important class of self-injective algebras is formed by the algebras of the form $\widehat{B}/G$ where $\widehat{B}$ is the repetitive algebra [5] (locally bounded, without identity)

$$\widehat{B} = \bigoplus_{k\in\mathbb{Z}} (B_k \oplus (DB)_k)$$

of an algebra B, where $B_k = B$ and $(DB)_k = DB$ for all $k \in \mathbb{Z}$, the multiplication in $\widehat{B}$ is defined by

$$(a_k, f_k)_k \cdot (b_k, g_k)_k = (a_k b_k, a_k g_k + f_k b_{k+1})_{k\in\mathbb{Z}}$$

for $a_k, b_k \in B_k$, $f_k, g_k \in (DB)_k$, and G is an admissible group of K-automorphisms of $\widehat{B}$. More precisely, for a fixed set $\mathcal{E} = \{e_i \mid 1 \le i \le m\}$ of primitive orthogonal idempotents of B with $1_B = e_1 + \cdots + e_m$, consider the canonical set $\widehat{\mathcal{E}} = \{e_{j,k} \mid 1 \le j \le m,\ k \in \mathbb{Z}\}$ of primitive orthogonal idempotents of $\widehat{B}$ such that $e_{j,k}\widehat{B} = (e_jB)_k \oplus (e_jDB)_k$ for $1 \le j \le m$ and $k \in \mathbb{Z}$. By an automorphism of $\widehat{B}$ we mean a K-algebra

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automorphism of $\widehat{B}$ that fixes the chosen set $\widehat{\mathcal{E}}$ of primitive orthogonal idempotents of $\widehat{B}$. A group G of automorphisms of $\widehat{B}$ is said to be admissible if the induced action of G on $\widehat{\mathcal{E}}$ is free and has finitely many orbits. Then the orbit algebra $\widehat{B}/G$ is a self-injective algebra and the G-orbits in $\widehat{\mathcal{E}}$ form a canonical set of primitive orthogonal idempotents of $\widehat{B}/G$ whose sum is the identity of $\widehat{B}/G$ (see [4]). We denote by $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$ such that $\nu_{\widehat{B}}(e_{j,k}) = e_{j,k+1}$ for all $1 \leq j \leq m$, $k \in \mathbb{Z}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension $B \ltimes DB$ of B by DB. An automorphism φ of $\widehat{B}$ is said to be positive (respectively, rigid) when $\varphi(B_k) \subseteq \sum_{i \geq k} B_i$ (respectively, $\varphi(B_k) = B_k$) for any $k \in \mathbb{Z}$. Moreover, φ is said to be strictly positive if it is positive but not rigid. We shall also consider $\widehat{B}$ as a locally bounded K-category with the objects set as $\widehat{\mathcal{E}}$.

Let R be a tilted algebra of type Δ that is not a Dynkin quiver. Then the Auslander-Reiten quiver $\Gamma_{\widehat{R}}$ of $\widehat{R}$ is of the form

$$\Gamma_{\widehat{R}} = \bigvee_{p \in \mathbb{Z}} (\mathcal{X}_p \vee \mathcal{R}_p)$$

where, for each $p \in \mathbb{Z}$, $\mathcal{X}_p$ is a component with the stable part of the form $\mathbb{Z}\Delta$, $\mathcal{R}_p$ is a family of components whose stable parts are tubes (if Δ is Euclidean) or of type $\mathbb{Z}\mathbb{A}_\infty$ (if A is wild), and $\nu_{\widehat{R}}(\mathcal{X}_p) = \mathcal{X}_{p+2}$ and $\nu_{\widehat{R}}(\mathcal{R}_p) = \mathcal{R}_{p+2}$, for the induced action of $\nu_{\widehat{R}}$ on $\Gamma_{\widehat{R}}$ (see [1], [3], [6]). Furthermore, an automorphism φ of $\widehat{R}$ is positive (respectively, strictly positive) if and only if there exists $q \geq 0$ (respectively, $q > 0$) such that $\varphi(\mathcal{X}_p) = \mathcal{X}_{p+q}$ and $\varphi(\mathcal{R}_p) = \mathcal{R}_{p+q}$ for all $p \in \mathbb{Z}$. We also note that the class of algebras of the form $\widehat{R}/(\psi\nu_{\widehat{R}})$ with R a tilted algebra of type Δ (not a Dynkin quiver) and ψ a strictly positive (respectively, positive) automorphism of $\widehat{R}$ coincides in the class of all self-injective algebras of tilted type whose stable Auslander-Reiten quiver admits at least three (respectively, two) components of type $\mathbb{Z}\Delta$. We have proved in [8, Theorem 1] that a self-injective algebra is stably equivalent to an algebra $\widehat{R}/(\psi\nu_{\widehat{R}})$ with ψ a positive automorphism of $\widehat{R}$ if and only if A is a socle equivalent to an algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ where B is a tilted algebra of type Δ and φ is a positive automorphism of $\widehat{B}$. Recall that two self-injective algebras A and Λ are called socle equivalent if the factor algebras $A/\operatorname{soc} A$ and $\Lambda/\operatorname{soc}\Lambda$ are isomorphic. Our main result shows that for ψ strictly positive we may replace "socle equivalent" by "isomorphic".

Theorem. *Let A be a self-injective algebra stably equivalent to a self-injective algebra of the form $\widehat{R}/(\psi\nu_{\widehat{R}})$ where R is a tilted algebra of type Δ that is not a Dynkin quiver and ψ is a strictly positive automorphism of $\widehat{R}$. Then A is isomorphic to an algebra of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some tilted algebra B of type Δ and a strictly positive automorphism φ of $\widehat{B}$.*

We note that the strict positivity of ψ is necessary for the validity of the theorem. Namely, if A is a nonsplittable Hochschild extension of a hereditary algebra H not of Dynkin type (see [7] and [10] for existence of such algebras), then A is stably equivalent to the trivial extension $H \ltimes D(H) \cong \widehat{H}/(\nu_{\widehat{H}})$ but is not isomorphic to an algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ with B a tilted algebra of type Δ and φ a positive automorphism

of $\widehat{B}$. It would be interesting to know if the theorem also holds for self-injective algebras of Dynkin type.

For general background concerning representation theory of Artin algebras and self-injective algebras we refer to [2] and [11].

2. Preliminary results

The aim of this section is to prove some preliminary results that we need in the proof of our main theorem. As an application we establish also a criterion (Proposition 2.6) for a self-injective Artin algebra A to be isomorphic to an algebra of the form $\widehat{B}/G$, where $\widehat{B}$ is the repetitive algebra of an Artin algebra B of finite global dimension and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$.

Let A be a self-injective algebra, $\{e_i \,|\, 1 \leq i \leq n\}$ a fixed set of primitive orthogonal idempotents of A such that $1_A = e_1 + \cdots + e_n$, and $\nu = \nu_A$ a fixed Nakayama automorphism of A inducing an A-bimodule isomorphism $A \cong {}_\nu DA$, where ${}_\nu DA$ denotes the left A-module obtained from the canonical left A-module DA by changing the operation of A by $a \cdot f = \nu(a)f$ for $a \in A$, $f \in DA$. Hence we have $\operatorname{soc}(e_iA) \cong \operatorname{top}(\nu(e_i)A)$ $(= \nu(e_i)A/\nu(e_i)\operatorname{rad} A)$ as right A-modules for all $i \in \{1, \ldots, n\}$. Since $\{\nu(e_i)A \,|\, 1 \leq i \leq n\}$ is a set of representatives of indecomposable projective A-modules, there is a permutation of $\{1, \ldots, n\}$, denoted again by ν, such that $\nu(e_i)A \cong e_{\nu(i)}A$ for all $i \in \{1, \ldots, n\}$. Let I be a (two-sided) ideal of A, $B = A/I$ and e an idempotent of A such that $e + I$ is the identity of B. We may assume that $e = e_1 + \cdots + e_m$ for some $m \leq n$, and $\{e_i \,|\, 1 \leq i \leq m\}$ is the subset of $\{e_i \,|\, 1 \leq i \leq n\}$ consisting of all idempotents e_i that are not in I. Then such an idempotent e is uniquely determined by I up to an inner automorphism of A and is called a residual identity of B. We note that $B \cong eAe/eIe$ and $1 - e \in I$.

Let A be a self-injective algebra, I an ideal of A, and $B = A/I$. For idempotents f and f' of A, f is said to be a summand of f' if $ff' = f'f = f$, and we denote this fact by $f \leq f'$. Moreover, the orthogonality $ff' = 0 = f'f$ of f and f' is denoted by $f \perp f'$. Fix a set $\{e_i \,|\, 1 \leq i \leq n\}$ of primitive orthogonal idempotents of A such that $1_A = \sum_{i=1}^n e_i$ and $e = \sum_{i=1}^m e_i$, for some $m \leq n$, is a residual identity of B. We denote the residue class of each idempotent e_j, $1 \leq j \leq m$, in $B = A/I$ by e_j again, for simplicity. Since $B \cong eAe/eIe$ as algebras, we often identify them. Finally, for $i \in \{1, \ldots, m\}$, let $e_B^{(i)}$ (respectively, $e_I^{(i)}$) be the sum of all idempotents e_j, $1 \leq j \leq n$, such that $e_iBe_j \neq 0$ (respectively, $e_i(I/\operatorname{soc} I)e_j \neq 0$). Following [7, (2.1)] the ideal I is said to be *deforming* if $eIe = \ell_{eAe}(I) = r_{eAe}(I)$ and the ordinary quiver $Q(B)$ of B has no oriented cycles. Here, $\ell_{eAe}(I)$ is the left annihilator of I in eAe and $r_{eAe}(I)$ is the right annihilator of I in eAe.

From now on we assume that the ideal I is deforming. For the proofs of our main results, we need several technical lemmas. We recall first the following properties of a deforming ideal proved in [7, Section 1].

Lemma 2.1. (1) $\operatorname{Soc}(A) \subseteq I$.

(2) *For e_i $(i \leq m)$, the right B-modules e_iB and $e_i(I/\operatorname{soc} I)$ have no common composition factors, and similarly for $e_i(\operatorname{rad} B)$ and e_iI.*

(3) *There are no idempotents e_{i_j} $(0 \leq j \leq t)$ such that*

$$e_{i_0}(\operatorname{rad} B)e_{i_1} \neq 0, e_{i_1}(\operatorname{rad} B)e_{i_2} \neq 0, \ldots, e_{i_{t-1}}(\operatorname{rad} B)e_{i_t} \neq 0, e_{i_t}(\operatorname{rad} B)e_{\nu(i_0)} \neq 0.$$

(4) $e_iIe_B^{(i)} \subseteq \operatorname{soc}(e_iI)$ *and* $e_iAe_I^{(i)} \subseteq e_iI$ *for* $i \leq m$.
(5) $e_B^{(i)} \perp e_I^{(i)}$.
(6) $e^{(i)}a = e_iae_B^{(i)} + e_iae_I^{(i)} + e_iae_{\nu(i)}$ *for* $a \in A$ *and* $1 \leq i \in m$ *with* $\nu(i) \neq i$.

Lemma 2.2. *For any* $i \leq m$*, the following equalities hold:*
(1) $e_i(eAe)e_I^{(i)}I = 0$;
(2) $e_i \operatorname{rad}(eAe)e_{\nu(i)}I = 0$ *and* $Ie_{\nu^{-1}(i)} \operatorname{rad}(eAe)e_i = 0$.

Proof. (1) Assume that $e_iae_I^{(i)}I \neq 0$ for some $a \in eAe$. Since $ee_I^{(i)} \neq 0$ by assumption, there is some $e_j \leq e$ such that $e_j \leq e_I^{(i)}$ and $e_iae_jI \neq 0$. This implies that $e_iae_j \notin eIe$, because $\ell_{eAe}(I) = eIe$ by our assumption. Thus $0 \neq e_i\bar{a}e_j \in B$ so that $e_j \leq e_B^{(i)}$, where $\bar{a}$ denotes the residue class of a in A/I. Then it follows that $e_B^{(i)}e_I^{(i)} \neq 0$, a contradiction to Lemma 2.1(2).

(2) Assume that $e_iae_{\nu(i)}I \neq 0$ for some $a = eae \in \operatorname{rad}(eAe)$. In particular, we have $ee_{\nu(i)} \neq 0$, so that $e_{\nu(i)} \leq e$ and $e_iae_{\nu(i)} \notin eIe$, because $\ell_{eAe}(I) = eIe$. Thus we have $e_i(\operatorname{rad} B)e_{\nu(i)} \neq 0$, a contradiction to Lemma 2.1(3). The proof of the second equality is similar. □

Lemma 2.3. *If* $1 \leq i$, $\nu(i) \leq m$*, then* $e_i(I/\operatorname{soc} I)e_{\nu(i)} = 0$ *and* $e_i(\operatorname{rad} A)e_{\nu(i)} \subseteq \operatorname{soc}(e_iI)$.

Proof. (1) If $e_i(I/\operatorname{soc} I)e_{\nu(i)} \neq 0$, then $e_{\nu(i)}(\operatorname{rad} B)e_{\nu(i)} \neq 0$ by [7, Lemma 1.2], which implies that $\mathcal{Q}(B)$ has an oriented cycle, a contradiction. Hence we have $e_i(I/\operatorname{soc} I)e_{\nu(i)} = 0$.

(2) By Lemma 2.1(3), $e_i(\operatorname{rad} B)e_{\nu(i)} = 0$ and so $e_i(\operatorname{rad} A)e_{\nu(i)} \subseteq e_iIe_{\nu(i)}$. If $e_i(\operatorname{rad} A)e_{\nu(i)} \nsubseteq \operatorname{soc}(e_iI)$, then $e_i(\operatorname{rad} A/\operatorname{soc} I)e_{\nu(i)} \neq 0$. Hence, it follows from Lemma 2.1(2) that $e_i(I/\operatorname{soc} I)e_{\nu(i)} \neq 0$, which is impossible by the assertion proved in (1) above. Thus we conclude that $e_i(\operatorname{rad} A)e_{\nu(i)} \subseteq \operatorname{soc}(e_iI)$. □

Lemma 2.4. *For* $a, b \in A \setminus I$*, the following statements hold:*
(1) *if* $ab \in \operatorname{rad} A$*, then* $e_iaebe_{\nu(i)} = 0$ *for* $1 \leq i$, $\nu(i) \leq m$;
(2) $e_iaebe_{\nu(i)} = 0$ *for* $1 \leq i$, $\nu(i) \leq m$ *and* $\nu(i) \neq i$.

Proof. (1) It suffices to prove the assertion for $a = e_iae_j$ and $b = e_jbe_{\nu(i)}$ for $e_j \in B$. Suppose that $ab \neq 0$. We claim that both a and b belong to $\operatorname{rad} A$. In fact, if $a \notin \operatorname{rad} A$, then $e_i = e_j$ and there is some $a' \in e_iAe_i$ with $a'a = e_i$. Then $b = a'(ab) \in \operatorname{soc} I$ because $ab \in \operatorname{soc} I$ by Lemma 2.3, a contradiction to the assumption. Thus $a \in \operatorname{rad} A$. Similarly, we have $b \in \operatorname{rad} A$. It therefore follows that $e_iBe_j \neq 0$ and $e_jBe_{\nu(i)} \neq 0$, a contradiction to Lemma 2.1(3).

(2) Since $\nu(i) \neq i$, $e_iaebe_{\nu(i)} \in \operatorname{rad} A$ obviously, and the assertion follows from (1). □

Denote by $\varrho : eAe \longrightarrow eAe/eIe = B$ the canonical algebra epimorphism and define the map $\phi : eAe/eIe \longrightarrow eAe$ by

$$\phi(a + eIe) = \sum_{i=1}^{m} e_iae_B^{(i)}$$

for all $a \in eAe$.

Proposition 2.5. *Assume* $\nu(i) \neq i$ *for all* $i \in \{1, \ldots, m\}$*. Then* ϕ *is an algebra homomorphism such that* $\varrho\phi = 1_B$.

Proof. We first prove that ϕ is a well-defined K-homomorphism. It suffices to show that $e_iIe_B^{(i)} = 0$ for any $i \leq m$. For this, suppose that $e_iIe_B^{(i)} \neq 0$. Then $e_iIe_B^{(i)}e_{\nu(i)} \neq 0$, because $e_iIe_B^{(i)} \subseteq \operatorname{soc}(e_iI) \cong \operatorname{top}(e_{\nu(i)}A)$ by Lemma 2.1 (4). In particular, $e_B^{(i)}e_{\nu(i)} \neq 0$ and so $e_{\nu(i)} \leq e_B^{(i)}$. Hence $e_iBe_{\nu(i)} \neq 0$ by definition. On the other hand, $e_i(\operatorname{rad} B)e_{\nu(i)} = 0$ by Lemma 2.1(3). Thus we have $e_{\nu(i)} = e_i$, equivalently $\nu(i) = i$, a contradiction.

The proof that ϕ is an algebra homomorphism, is divided into several steps, and involves the relations (5) and (6) of Lemma 2.1.

(1) $\phi(e + eIe) = e$.

For $a = e, e_iae_I^{(i)} = e_ie_I^{(i)} = 0$ because $e_i \leq e_B^{(i)}$ and $e_B^{(i)} \perp e_I^{(i)}$, and $e_iae_{\nu(i)} = e_ie_{\nu(i)} = 0$ because $\nu(i) \neq i$. Hence $\phi(e + eIe) = \sum_{i=1}^m e_iee_B^{(i)} = e$.

(2) $\phi\big((a + eIe)(b + eIe)\big) = \phi(a + eIe)\phi(b + eIe)$ for $a, b \in eAe$.

We may assume that $a = e_iae_{i'}$, $b = e_jbe_{j'}$ for $e_{i'} \leq e_B^{(i)}$, $e_{j'} \leq e_B^{(j)}$. Moreover, it is enough to consider the case when $a \notin I$ and $b \notin I$. Let $\alpha = \phi\big((a + eIe)(b + eIe)\big)$ and $\beta = \phi(a + eIe)\phi(b + eIe)$. Then $\alpha = \phi(ab + eIe) = e_iae_{i'}e_jbe_{j'}e_B^{(i)}$, and $\beta = e_iae_{i'}e_B^{(i)} \cdot e_jbe_{j'}e_B^{(j)}$. Here, if $e_j \not\leq e_B^{(i)}$, then $e_B^{(i)}e_j = 0$ and so $e_{j'}e_j = 0$. Hence we have $\alpha = 0$ and $\beta = 0$, and $\alpha = \beta$ as required. Now, assume that $e_j \leq e_B^{(i)}$. Then $\beta = e_iae_{i'}e_jbe_{j'}e_B^{(j)}$. Since $e_B^{(i)}e_B^{(j)} = e_B^{(j)}e_B^{(i)}$, it therefore suffices to show that $\alpha = \alpha e_B^{(j)}$ and $\beta = \beta e_B^{(i)}$.

(a) First we show that $\alpha = \alpha e_B^{(j)}$. Obviously, $e_jbe_{j'}e_B^{(i)} = e_jbe_{j'}e_B^{(i)}(e_B^{(j)} + e_I^{(j)} + e_{\nu(j)})$. We claim that $e_jbe_{j'}e_B^{(i)}e_I^{(j)} = 0$ and $e_jbe_{j'}e_B^{(i)}e_{\nu(j)} = 0$. In fact, if $e_jbe_{j'}e_B^{(i)}e_I^{(j)} \neq 0$, then $0 \neq e_{j'}e_B^{(i)}e_I^{(j)} = e_{j'} \leq e_I^{(j)}$. Since $e_jbe_{j'} = b \notin I$ by assumption, we have $e_jBe_I^{(j)} \neq 0$, which contradicts the orthogonality $e_B^{(j)} \perp e_I^{(j)}$. Next, if $e_jbe_{j'}e_B^{(i)}e_{\nu(j)} \neq 0$, then $0 \neq e_{j'}e_B^{(i)}e_{\nu(j)} = e_{j'} = e_{\nu(j)}$ and $b = e_jbe_{\nu(j)}$. Hence $e_jbe_{\nu(j)} \in \operatorname{rad} A \backslash I$ because $j \neq \nu(j)$ by assumption so that $e_j(\operatorname{rad} B)e_{\nu(j)} \neq 0$, which contradicts Lemma 2.1 (3). Thus we have proved that $e_jbe_{j'}e_B^{(i)} = e_jbe_{j'}e_B^{(i)}e_B^{(j)}$, so that $\alpha = \alpha e_B^{(j)}$.

(b) Secondly, we show that $\beta = \beta e_B^{(i)}$. Since $\beta = e_i\beta = e_i\beta e_B^{(i)} + e_i\beta(e_I^{(i)} + e_{\nu(i)})$, it suffices to show that $\beta e_I^{(i)} = 0$ and $\beta e_{\nu(i)} = 0$. Suppose that $\beta e_I^{(i)} \neq 0$ contrarily. Then it follows from the proof of [7, Proposition 3.1] that $\beta e_I^{(i)} \in \operatorname{soc}(e_iI)$. Hence $\beta e_I^{(i)}e_{\nu(i)} \neq 0$, and so $e_{j'}e_B^{(j)}e_I^{(i)}e_{\nu(i)} \neq 0$. It follows that $e_{j'}e_B^{(j)}e_I^{(i)}e_{\nu(i)} = e_{\nu(i)} = e_{j'} \leq e$. Therefore, $\beta = e_iae_jbe_{\nu(i)}$, so that $\beta = 0$ by Lemma 2.4 (2), a contradiction.

Therefore, we proved that ϕ is an algebra homomorphism. Finally, for $a = eae \in eAe$, $a - \sum_{i=1}^m e_iae_B^{(i)} = \sum_{i=1}^m(e_iae - e_iae_B^{(i)}) = \sum_{i=1}^m(e_iae_I^{(i)} + e_iae_{\nu(i)})$, which belongs to eIe. Hence we have $\varrho\phi = 1_B$. □

We will now prove the announced criterion.

Proposition 2.6. *Let A be a self-injective algebra, I an ideal of A, $B = A/I$, and e a residual identity of B. Assume that the ordinary quiver $Q(B)$ of B has no oriented cycles, $IeI = 0$, Ie is an injective cogenerator in $\operatorname{mod} B$, and $e_i \neq e_{\nu(i)}$ for any $i \in \{1, \ldots, m\}$. Then A is isomorphic to an algebra $\widehat{B}/G$ where G is an*

infinite cyclic group of automorphisms of $\widehat{B}$ generated by $\varphi\nu_{\widehat{B}}$ for some positive automorphism φ of $\widehat{B}$.

Proof. It follows from [7, Proposition 2.3] that I is a deforming ideal, and by Proposition 2.5 the canonical epimorphism $\varrho : eAe \longrightarrow eAe/eIe = B$ splits. Applying now [9, Theorem 3.8] we conclude that A is isomorphic to $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of $\widehat{B}$. □

3. Self-injective algebras with deforming ideal

Let A be a self-injective algebra, I a deforming ideal of A, $B = A/I$ and e a residual identity of B. Then I can be considered as a (eAe/eIe)-bimodule. Denote by $A[I]$ the direct sum of K-modules $(eAe/eIe) \oplus I$ with the multiplication

$$(b, x) \cdot (b', x') = (bb', bx' + xb' + xx')$$

for $b, b' \in eAe/eIe$ and $x, x' \in I$. Then $A[I]$ is an algebra with the identity $(e, 1-e)$ and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider I as an ideal of $A[I]$. The following fact has been proved in [7, Theorem 4.1].

Proposition 3.1. *$A[I]$ is a self-injective algebra with a deforming ideal I, the Nakayama permutation of $A[I]$ is the same as the Nakayama permutation of A, and the algebras $A[I]$ and A are socle equivalent.*

Moreover, if K is a field, we have proved in [8, Theorem 3] that the algebras $A[I]$ and A are also stably equivalent. The aim of this section is to prove the following fact, needed in the proof of our main theorem.

Proposition 3.2. *Let A be a self-injective algebra with a deforming ideal I, $B = A/I$ and let e be a residual identity of B. Assume that $IeI = 0$ and $e_i \neq e_{\nu(i)}$ for any primitive summand e_i of e. Then the algebras $A[I]$ and A are isomorphic.*

Proof. We assume as before that $e = \sum_{i=1}^{m} e_i$ for $m \leq n$ and a set $\{e_i \mid 1 \leq i \leq n\}$ of primitive idempotents of A with $1_A = \sum_{i=1}^{n} e_i$. Denote by $\overline{a}$ the residue class of $a \in eAe$ in eAe/eIe. Since by our assumption $\nu(i) \neq i$ for $i \in \{1, \ldots, m\}$, invoking Proposition 2.5 we have an algebra monomorphism $\phi : eAe/eIe \longrightarrow eAe$ such that $\phi(\overline{a}) = \sum_{i=1}^{m} e_i a e_B^{(i)}$. We define a homomorphism $\Phi : A[I] \longrightarrow A$ of K-modules by

$$\Phi(\overline{a}, x) = \phi(\overline{a}) + x$$

for $\overline{a} \in eAe/eIe$ and $x \in I$. We claim that Φ is an algebra isomorphism. Consider the following commutative diagram of canonical short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & A[I] & \longrightarrow & eAe/eIe & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle \Phi'} & & \\
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0,
\end{array}$$

where 1 is the identity map and Φ' is an isomorphism given by $\Phi'(\overline{a}) = \phi(\overline{a}) + I$, for $\overline{a} \in eAe/eIe$. In order to prove that Φ is an algebra isomorphism, it suffices to show that Φ is an algebra homomorphism. Observe that

$$\Phi(1_{A[I]}) = \Phi(\overline{e}, 1-e) = \phi(\overline{e}) + (1-e) = e + (1-e) = 1 = 1_A\,.$$

Take $\overline{a}, \overline{b} \in eAe/eIe$ and $x, y \in I$. We shall prove that

$$\Phi((\overline{a}, x)(\overline{b}, y)) = \Phi(\overline{a}, x)\Phi(\overline{b}, y)\,.$$

It follows from the definition of multiplication in $A[I]$ that

$$\Phi\big((\bar{a},x)(\bar{b},y)\big) = \Phi(\bar{a}\bar{b}, ay + xb + xy) = \phi(\bar{a}\bar{b}) + (ay + xb) + xy$$

and

$$\Phi(\bar{a},x)\Phi(\bar{b},y) = \big(\phi(\bar{a}) + x\big)\big(\phi(\bar{b}) + y\big) = \phi(\bar{a})\phi(\bar{b}) + (\phi(\bar{a})y + x\phi(\bar{b})) + xy.$$

Hence, it suffices to show that $\phi(\bar{a})y = ay$ and $x\phi(\bar{b}) = xb$.

(a) *$\phi(\bar{a})y = ay$ for $a \in eAe$ and $y \in I$:*

In fact, $\phi(\bar{a})y = \sum_{i=1}^{m} e_i a e_B^{(i)} y$, and

$$ay = \Big(\sum_{i=1}^{m} e_i a\Big)y = \sum_i e_i a e_B^{(i)} y + \sum_i e_i a e_I^{(i)} y + \sum_i e_i a e_{\nu(i)} y.$$

Here, $e_i a e_I^{(i)} y = 0$ and $e_i a e_{\nu(i)} y = 0$ by Lemma 2.2, because $e_i a e_{\nu(i)} \in \operatorname{rad} A$ by $\nu(i) \neq i$, and therefore $\phi(\bar{a})y = ay$.

(b) *$x\phi(\bar{b}) = xb$ for $b \in eAe$ and $x \in I$:*

Since $e_i A e_I^{(i)} \subseteq e_i I$ by Lemma 2.1, we have $x e_i b e_I^{(i)} \in I e_i I \subseteq IeI$. Then $x e_i b e_I^{(i)} = 0$ because $IeI = 0$ by assumption. On the other hand, $e_i A e_{\nu(i)} = e_i(\operatorname{rad} A)e_{\nu(i)}$ by $\nu(i) \neq i$, and hence $e_i b e_{\nu(i)} \in I$ by (1) and (2) of Lemma 2.1. Hence $x e_i b e_{\nu(i)} \in I e_i I \subseteq IeI = 0$, and $x e_i b e_{\nu(i)} = 0$. Thus,

$$\begin{aligned} xb &= \sum_{i=1}^{m} x e_i b e_B^{(i)} + \sum_i x e_i b e_I^{(i)} + \sum_i x e_i b e_{\nu(i)} \\ &= \sum_i x e_i b e_B^{(i)} = x \sum_i e_i b e_B^{(i)} \\ &= x\phi(\bar{b}). \end{aligned}$$

Therefore, the algebras $A[I]$ and A are isomorphic. □

4. Proof of the theorem

We first prove the following general facts.

Lemma 4.1. *Let B be an algebra and φ a positive automorphism of $\widehat{B}$ that fixes a finite subset of the object set of $\widehat{B}$. Then φ is a rigid automorphism of $\widehat{B}$.*

Proof. We write as before $\widehat{B} = \bigoplus_{k\in\mathbb{Z}}(B_k \oplus (DB)_k)$ with $B_k = B$ and $(DB)_k = DB$ for all $k \in \mathbb{Z}$, and identify the objects of $\widehat{B}$ (respectively, B_k) with the vertices of the ordinary quivers $Q(\widehat{B})$ (respectively, $Q(B_k)$). Since φ is a positive automorphism of $\widehat{B}$, we may assume (without loss of generality) that φ fixes a subset of $Q(B_0)$. In order to prove that φ is rigid, it is enough to show that $\varphi(Q(B_0)) = Q(B_0)$. For this, since the quiver $Q(B_0)$ is connected, it suffices to show that $\varphi(\to S) = (\to S)$ and $\varphi(S \to) = (S \to)$ for any subset S of $Q(B_0)$ with $\varphi(S) = S$, where $(\to S)$ and $(S \to)$ denote the sets of all predecessors of the vertices of S and of all successors of the vertices of S in $Q(B_0)$, respectively. Therefore, let S be a subset of $Q(B_0)$ with $\varphi(S) = S$.

Now, let $x \to \cdots \to s$ be a path in $\mathcal{Q}(B_0)$ with $s \in S$. Then $\varphi(x) \to \cdots \to \varphi(s)$ is a path in $\mathcal{Q}(\widehat{B})$ and $\varphi(s) \in \mathcal{Q}(B_0)$, because $\varphi(S) = S$ by assumption. Since φ is positive, $\varphi(x) \in \sum_{k\geq 0} \mathcal{Q}(B_k)$ and it follows that $\varphi(x)$ belongs to $\mathcal{Q}(B_0)$, because there is no path from $\mathcal{Q}(B_k)$ to $\mathcal{Q}(B_0)$ for any $k > 0$. Thus $\varphi(\to S) = (\to S)$.

Next, suppose that there is a path $s \to \cdots \to x$ in $\mathcal{Q}(B_0)$ for $s \in S$ with $\varphi(x) \notin \mathcal{Q}(B_0)$, that is, $\varphi(x) \in \mathcal{Q}(B_k)$ for some $k > 0$. Since S is finite and $Q(\widehat{B})$ is locally finite, the φ-orbit of x is finite also. This implies that there is an integer ℓ such that $\varphi^\ell(x) \in \mathcal{Q}(B_{k'})$ for some $k' > 0$ and $\varphi^{\ell+1}(x) \in \mathcal{Q}(B_0)$. This, however, implies that $\varphi(\mathcal{Q}(B_{k'})) \cap \mathcal{Q}(B_0) \neq \emptyset$, which contradicts the positivity of φ. Thus we conclude that $\varphi(S \to) = (S \to)$. □

Corollary 4.2. *Let B be an algebra and φ a strictly positive automorphism of $\widehat{B}$. Then φ acts freely on the objects of $\widehat{B}$.*

Proof of the Theorem. Let A be a self-injective algebra that is stably equivalent to a self-injective algebra of the form $\widehat{R}/(\psi\nu_{\widehat{R}})$, where R is a tilted algebra of type Δ (which is not a Dynkin quiver) and ψ is a strictly positive automorphism of $\widehat{R}$. It follows from the proof of Theorem 1 in [8] that there are an ideal I of A and a residual identity $e = \sum_{i=1}^m e_i$ of $B = A/I$ such that $IeI = 0$, Ie is an injective cogenerator in $\operatorname{mod} B$, B is a tilted algebra of type Δ, and $A[I] \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$ for a positive automorphism φ of $\widehat{B}$. Moreover, by [8, Theorem 3], the algebras A and $A[I]$ are stably equivalent. Since by assumption A is stably equivalent to $\widehat{R}/(\psi\nu_{\widehat{R}})$, we conclude that $A[I]$ is stably equivalent to $\widehat{R}/(\psi\nu_{\widehat{R}})$. Furthermore, since ψ is a strictly positive automorphism of $\widehat{R}$, the number of simple modules of $\widehat{R}/(\psi\nu_{\widehat{R}})$ is greater than the number of simple modules of R, which is the same as the number of vertices of Δ. Applying now [8, Corollary] we conclude that the automorphism φ is not rigid, and so is strictly positive. Hence, invoking Corollary 4.2 and Proposition 3.1, we obtain that $\nu(i) \neq i$ for the Nakayama permutation ν of A and $i \in \{1, \ldots, m\}$. Since B is a tilted algebra, the ordinary quiver $Q(B)$ of B has no oriented cycles, and then the properties of I stated above imply that I is a deforming ideal of A. Therefore, applying Proposition 3.2, we obtain that A and $A[I]$ are isomorphic, and so A and $\widehat{B}/(\varphi\nu_{\widehat{B}})$ are also isomorphic. This finishes the proof. □

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