ON INVARIABILITY OF SELF-INJECTIVE ALGEBRAS OF TILTED TYPE UNDER STABLE EQUIVALENCES

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Abstract. We prove that a large class of self-injective algebras of tilted type is invariant under stable equivalences of module categories.

1. Introduction and the Main Result

Throughout the paper, by an algebra is meant a basic, connected, Artin algebra (associative, with an identity) over a fixed commutative Artinian ring $K$. For an algebra $A$, we denote by $\text{mod}
A$ the category of finitely generated right $A$-modules and by $\text{mod}^A$ the stable module category of $A$. Recall that the objects of $\text{mod}^A$ are the objects of $\text{mod}
A$ without nonzero projective direct summands, and for any two objects $M$ and $N$ in $\text{mod}^A$ the $K$-module $\text{Hom}^A(M, N)$ is the quotient $\text{Hom}^A(M, N)/P(M, N)$, where $P(M, N)$ is the submodule of $\text{Hom}^A(M, N)$ consisting of all $A$-homomorphisms that factorize through projective $A$-modules. Two algebras $A$ and $\Lambda$ are said to be stably equivalent if their stable module categories $\text{mod}^A$ and $\text{mod}^\Lambda$ are equivalent. Recall also that $D = \text{Hom}_K(-, E)$, where $E$ is a minimal injective cogenerator in $\text{mod}
K$, defines a duality between the categories of left and right modules. An algebra $A$ is called self-injective if $A = D(A)$ in $\text{mod}
A$, that is, the projective $A$-modules are injective. An important class of self-injective algebras is formed by the algebras of the form $\bar{B}/G$ where $\bar{B}$ is the repetitive algebra $\mathcal{B}$ (locally bounded, without identity)

$$\bar{B} = \bigoplus_{k \in \mathbb{Z}} (B_k \oplus (DB)_k)$$

of an algebra $B$, where $B_k = B$ and $(DB)_k = DB$ for all $k \in \mathbb{Z}$, the multiplication in $\bar{B}$ is defined by

$$(a_k, f_k)_k \cdot (b_k, g_k)_k = (a_kb_k, a_kg_k + f_kb_{k+1})_{k \in \mathbb{Z}}$$

for $a_k, b_k \in B_k$, $f_k, g_k \in (DB)_k$, and $G$ is an admissible group of $K$-automorphisms of $\bar{B}$. More precisely, for a fixed set $E = \{e_i \mid 1 \leq i \leq m\}$ of primitive orthogonal idempotents of $B$ with $1_B = e_1 + \cdots + e_m$, consider the canonical set $E = \{e_{j,k} \mid 1 \leq j \leq m, k \in \mathbb{Z}\}$ of primitive orthogonal idempotents of $\bar{B}$ such that $e_{j,k} \bar{B} = (e_jB)_k \oplus (e_jDB)_k$ for $1 \leq j \leq m$ and $k \in \mathbb{Z}$. By an automorphism of $\bar{B}$ we mean a $K$-algebra

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automorphism of $\hat{B}$ that fixes the chosen set $\widehat{E}$ of primitive orthogonal idempotents of $\hat{B}$. A group $G$ of automorphisms of $\hat{B}$ is said to be admissible if the induced action of $G$ on $\widehat{E}$ is free and has finitely many orbits. Then the orbit algebra $\hat{B}/G$ is a self-injective algebra and the $G$-orbits in $\widehat{E}$ form a canonical set of primitive orthogonal idempotents of $\hat{B}/G$ whose sum is the identity of $\hat{B}/G$ (see [1]). We denote by $\nu_{\hat{B}}$ the Nakayama automorphism of $\hat{B}$ such that $\nu_{\hat{B}}(e_{i,k}) = e_{i,k+1}$ for all $1 \leq j \leq m$, $k \in \mathbb{Z}$. Then the infinite cyclic group $(\nu_{\hat{B}})$ generated by $\nu_{\hat{B}}$ is admissible and $\hat{B}/(\nu_{\hat{B}})$ is the trivial extension $B \times DB$ of $B$ by $DB$. An automorphism $\varphi$ of $\hat{B}$ is said to be positive (respectively, rigid) when $\varphi(B_k) \subseteq \sum_{i \geq k} B_i$ (respectively, $\varphi(B_k) = B_k$) for any $k \in \mathbb{Z}$. Moreover, $\varphi$ is said to be strictly positive if it is positive but not rigid. We shall also consider $\hat{B}$ as a locally bounded $K$-category with the objects set as $\widehat{E}$.

Let $R$ be a tilted algebra of type $\Delta$ that is not a Dynkin quiver. Then the Auslander-Reiten quiver $\Gamma_{\hat{R}}$ of $\hat{R}$ is of the form

$$\Gamma_{\hat{R}} = \bigcup_{p \in \mathbb{Z}} (X_p \lor R_p)$$

where, for each $p \in \mathbb{Z}$, $X_p$ is a component with the stable part of the form $\mathbb{Z}\Delta$, $R_p$ is a family of components whose stable parts are tubes (if $\Delta$ is Euclidean) or of type $\mathbb{Z}\Lambda_{\infty}$ (if $A$ is wild), and $\nu_{\hat{R}}(X_p) = X_{p+2}$ and $\nu_{\hat{R}}(R_p) = R_{p+2}$, for the induced action of $\nu_{\hat{R}}$ on $\Gamma_{\hat{R}}$ (see [1], [3], [2]). Furthermore, an automorphism $\varphi$ of $\hat{R}$ is positive (respectively, strictly positive) if and only if there exists $q \geq 0$ (respectively, $q > 0$) such that $\varphi(X_p) = X_{p+q}$ and $\varphi(R_p) = R_{p+q}$ for all $p \in \mathbb{Z}$. We also note that the class of algebras of the form $\hat{R}/(\psi/\nu_{\hat{R}})$ with $R$ a tilted algebra of type $\Delta$ (not a Dynkin quiver) and $\psi$ a strictly positive (respectively, positive) automorphism of $\hat{R}$ coincides in the class of all self-injective algebras of tilted type whose stable Auslander-Reiten quiver admits at least three (respectively, two) components of type $\mathbb{Z}\Delta$. We have proved in [3] Theorem 1] that a self-injective algebra is stably equivalent to an algebra $\hat{R}/(\psi/\nu_{\hat{R}})$ with $\psi$ a positive automorphism of $\hat{R}$ if and only if $A$ is a socle equivalent to an algebra $\widehat{B}/(\varphi/\nu_{\hat{B}})$ where $B$ is a tilted algebra of type $\Delta$ and $\varphi$ is a positive automorphism of $\hat{B}$. Recall that two self-injective algebras $A$ and $\Lambda$ are called socle equivalent if the factor algebras $A/\text{soc} A$ and $\Lambda/\text{soc} \Lambda$ are isomorphic. Our main result shows that for $\psi$ strictly positive we may replace “socle equivalent” by “isomorphic”.

**Theorem.** Let $A$ be a self-injective algebra stably equivalent to a self-injective algebra of the form $\hat{R}/(\psi/\nu_{\hat{R}})$ where $R$ is a tilted algebra of type $\Delta$ that is not a Dynkin quiver and $\psi$ is a strictly positive automorphism of $\hat{R}$. Then $A$ is isomorphic to an algebra of the form $\hat{B}/(\varphi/\nu_{\hat{B}})$ for some tilted algebra $B$ of type $\Delta$ and a strictly positive automorphism $\varphi$ of $\hat{B}$.

We note that the strict positivity of $\psi$ is necessary for the validity of the theorem. Namely, if $A$ is a nonsplittable Hochschild extension of a hereditary algebra $H$ not of Dynkin type (see [7] and [10] for existence of such algebras), then $A$ is stably equivalent to the trivial extension $H \ltimes D(H) \cong \hat{H}/(\nu_{\hat{R}})$ but is not isomorphic to an algebra $\hat{B}/(\varphi_{\hat{B}})$ with $B$ a tilted algebra of type $\Delta$ and $\varphi$ a positive automorphism.
of \( \bar{B} \). It would be interesting to know if the theorem also holds for self-injective algebras of Dynkin type.


2. Preliminary results

The aim of this section is to prove some preliminary results that we need in the proof of our main theorem. As an application we establish also a criterion (Proposition 2.4) for a self-injective Artin algebra \( A \) to be isomorphic to an algebra of the form \( \bar{B}/G \), where \( \bar{B} \) is the repetitive algebra of an Artin algebra \( B \) of finite global dimension and \( G \) is an admissible infinite cyclic group of automorphisms of \( \bar{B} \).

Let \( A \) be a self-injective algebra, \( \{e_i \mid 1 \leq i \leq n\} \) a fixed set of primitive orthogonal idempotents of \( A \) such that \( 1_A = e_1 + \cdots + e_n \), and \( \nu = \nu_A \) a fixed Nakayama automorphism of \( A \) inducing an \( A \)-bimodule isomorphism \( A \cong \nu DA \), where \( \nu DA \) denotes the left \( A \)-module obtained from the canonical left \( A \)-module \( DA \) by changing the operation of \( A \) by \( a \cdot f = \nu(a)f \) for \( a \in A \), \( f \in DA \). Hence we have \( \text{soc}(e_iA) \cong \text{soc}(\nu(e_i)A) = \nu(e_i)A/\nu(e_i) \text{rad} A \) as right \( A \)-modules for all \( i \in \{1, \ldots, n\} \). Since \( \{\nu(e_i)A \mid 1 \leq i \leq n\} \) is a set of representatives of indecomposable projective \( A \)-modules, there is a permutation of \( \{1, \ldots, n\} \), denoted again by \( \nu \), such that \( \nu(e_i)A \cong e_{\nu(i)}A \) for all \( i \in \{1, \ldots, n\} \). Let \( I \) be a (two-sided) ideal of \( A \), \( B = A/I \) and \( e \) an idempotent of \( A \) such that \( e + I \) is the identity of \( B \). We may assume that \( e = e_1 + \cdots + e_m \) for some \( m \leq n \), and \( \{e_i \mid 1 \leq i \leq m\} \) is the subset of \( \{e_i \mid 1 \leq i \leq n\} \) consisting of all idempotents \( e_i \) that are not in \( I \). Then such an idempotent \( e \) is uniquely determined by \( I \) up to an inner automorphism of \( A \) and is called a residual identity of \( B \). We note that \( B \cong eAe/eIe \) and \( 1 - e \in I \).

Let \( A \) be a self-injective algebra, \( I \) an ideal of \( A \), and \( B = A/I \). For idempotents \( f \) and \( f' \) of \( A \), \( f \) is said to be a summand of \( f' \) if \( ff' = f'f = f \), and we denote this fact by \( f \leq f' \). Moreover, the orthogonality \( ff' = 0 = f'f \) of \( f \) and \( f' \) is denoted by \( f \perp f' \). Fix a set \( \{e_i \mid 1 \leq i \leq n\} \) of primitive orthogonal idempotents of \( A \) such that \( 1_A = \sum_{i=1}^{n} e_i \) and \( e = \sum_{i=1}^{m} e_i \), for some \( m \leq n \), is a residual identity of \( B \). We denote the residue class of each idempotent \( e_j \), \( 1 \leq j \leq m \), in \( B = A/I \) by \( e_j \) again, for simplicity. Since \( B \cong eAe/eIe \) as algebras, we often identify them. Finally, for \( i \in \{1, \ldots, m\} \), let \( e_B^{(i)} \) (respectively, \( e_I^{(i)} \)) be the sum of all idempotents \( e_j \), \( 1 \leq j \leq n \), such that \( e_jBe_j \neq 0 \) (respectively, \( e_j(I/\text{soc} I)e_j \neq 0 \)). Following [7 (2.1)] the ideal \( I \) is said to be deforming if \( eIe = \ell_{eAe}(I) = r_{eAe}(I) \) and the ordinary quiver \( Q(B) \) of \( B \) has no oriented cycles. Here, \( \ell_{eAe}(I) \) is the left annihilator of \( I \) in \( eAe \) and \( r_{eAe}(I) \) is the right annihilator of \( I \) in \( eAe \).

From now on we assume that the ideal \( I \) is deforming. For the proofs of our main results, we need several technical lemmas. We recall first the following properties of a deforming ideal proved in [7 Section 1].

**Lemma 2.1.**

1. \( \text{Soc}(A) \subseteq I \).

2. For \( e_i \) (\( i \leq m \)), the right \( B \)-modules \( e_iB \) and \( e_i(I/\text{soc} I) \) have no common composition factors, and similarly for \( e_i(\text{rad} B) \) and \( e_iI \).

3. There are no idempotents \( e_{i_j} \) (\( 0 \leq j \leq t \)) such that

\[
e_{i_0}(\text{rad} B)e_{i_1} \neq 0, e_{i_1}(\text{rad} B)e_{i_2} \neq 0, \ldots, e_{i_{t-1}}(\text{rad} B)e_{i_t} \neq 0, e_{i_t}(\text{rad} B)e_{\nu(i_0)} \neq 0.
\]
(4) \( e_i I e_i^{(i)} \subseteq \text{soc}(e_i I) \) and \( e_i A e_i^{(i)} \subseteq e_i I \) for \( i \leq m \).

(5) \( e_i^{(i)} \perp e_j^{(j)} \).

(6) \( e_i^{(i)} a = e_i a e_i^{(i)} + e_i e_i^{(i)} + e_i a e_{\nu(i)} \) for \( a \in A \) and \( 1 \leq i \leq m \) with \( \nu(i) \neq i \).

Lemma 2.2. For any \( i \leq m \), the following equalities hold:

(1) \( e_i(eAe)e_i^{(i)} I = 0 \);

(2) \( e_i \text{rad}(eAe)e_{\nu(i)} I = 0 \) and \( I e_{\nu^{-1}(i)} \text{rad}(eAe)e_i = 0 \).

Proof. (1) Assume that \( e_i^{(i)} I \neq 0 \) for some \( a \in eAe \). Since \( e e_i^{(i)} \neq 0 \) by assumption, there is some \( e_j \leq e \) such that \( e_j \leq e_i^{(i)} \) and \( e_i a e_i^{(i)} I \neq 0 \). This implies that \( e_i a \notin eIe \), because \( \ell_{eAe}(I) = eIe \) by our assumption. Thus \( 0 \neq e_i a e_i^{(i)} \in B \) so that \( e_j \leq e_i^{(i)} \), where \( a \) denotes the residue class of \( a \) in \( A/I \). Then it follows that \( e_i^{(i)} e_j^{(j)} \neq 0 \), a contradiction to Lemma 2.1(2).

(2) Assume that \( e_i a e_{\nu(i)} I \neq 0 \) for some \( a = eae \in \text{rad}(eAe) \). In particular, we have \( e_{\nu(i)} \neq 0 \), so that \( e_i^{(i)} e_{\nu(i)} \leq e \) and \( e_i a e_{\nu(i)} \notin eIe \), because \( \ell_{eAe}(I) = eIe \). Thus we have \( e_i \text{rad}(B)e_{\nu(i)} I \neq 0 \), a contradiction to Lemma 2.3(3). The proof of the second equality is similar.

Lemma 2.3. If \( 1 \leq i, \nu(i) \leq m \), then \( e_i(I/\text{soc} I)e_{\nu(i)} = 0 \) and \( e_i(\text{rad} A)e_{\nu(i)} \subseteq \text{soc}(e_i I) \).

Proof. (1) If \( e_i(I/\text{soc} I)e_{\nu(i)} \neq 0 \), then \( e_{\nu(i)} \text{rad}(B)e_{\nu(i)} \neq 0 \) by [7, Lemma 1.2], which implies that \( Q(B) \) has an oriented cycle, a contradiction. Hence we have \( e_i(I/\text{soc} I)e_{\nu(i)} = 0 \).

(2) By Lemma 2.1(3), \( e_i \text{rad}(B)e_{\nu(i)} = 0 \) and so \( e_i \text{rad}(A)e_{\nu(i)} \subseteq e_i I e_{\nu(i)} \). If \( e_i \text{rad}(A)e_{\nu(i)} \notin \text{soc}(e_i I) \), then \( e_i \text{rad}(A/\text{soc} I)e_{\nu(i)} \neq 0 \). Hence, it follows from Lemma 2.1(2) that \( e_i(I/\text{soc} I)e_{\nu(i)} \neq 0 \), which is impossible by the assertion proved in (1) above. Thus we conclude that \( e_i \text{rad}(A)e_{\nu(i)} \subseteq \text{soc}(e_i I) \).

Lemma 2.4. For \( a, b \in A \setminus I \), the following statements hold:

(1) if \( ab \in \text{rad} A \), then \( e_i a e_i b e_{\nu(i)} = 0 \) for \( 1 \leq i, \nu(i) \leq m \);

(2) \( e_i a e_i b e_{\nu(i)} = 0 \) for \( 1 \leq i, \nu(i) \leq m \) and \( \nu(i) \neq i \).

Proof. (1) It suffices to prove the assertion for \( a = e_i a e_j \) and \( b = e_j b e_{\nu(i)} \) for \( e_j \in B \). Suppose that \( ab \neq 0 \). We claim that both \( a \) and \( b \) belong to \( \text{rad} A \). In fact, if \( a \notin \text{rad} A \), then \( e_i = e_j \) and there is some \( a' \in e_i A e_i \) with \( a'a = e_i \). Then \( b = a'(ab) \in \text{soc} I \) because \( ab \in \text{soc} I \) by Lemma 2.3, contradicting the assumption. Thus \( a \in \text{rad} A \). Similarly, we have \( b \in \text{rad} A \). It therefore follows that \( e_i B e_j \neq 0 \) and \( e_j B e_{\nu(i)} \neq 0 \), a contradiction to Lemma 2.1(3).

(2) Since \( \nu(i) \neq i \), \( e_i a e_i b e_{\nu(i)} \in \text{rad} A \) obviously, and the assertion follows from (1).

Denote by \( \varrho : eAe \rightarrow eAe/eIe = B \) the canonical algebra epimorphism and define the map \( \phi : eAe/eIe \rightarrow eAe \) by

\[
\phi(a + eIe) = \sum_{i=1}^{m} e_i a e_i^{(i)}
\]

for all \( a \in eAe \).

Proposition 2.5. Assume \( \nu(i) \neq i \) for all \( i \in \{1, \ldots, m\} \). Then \( \phi \) is an algebra homomorphism such that \( \varrho \phi = 1_B \).
Proof. We first prove that $\phi$ is a well-defined $K$-homomorphism. It suffices to show that $e_i P e_i^{(i)} = 0$ for any $i \leq m$. For this, suppose that $e_i P e_i^{(i)} \neq 0$. Then $e_i P e_i^{(i)} e_{\nu(i)} \neq 0$, because $e_i P e_i^{(i)} \subseteq \text{soc}(e_i I) \cong \text{top}(e_{\nu(i)} A)$ by Lemma 2.1 (4). In particular, $e_i P e_i^{(i)} \neq 0$ and so $e_{\nu(i)} \neq e_i^{(i)}$. Hence $e_i P e_{\nu(i)} \neq 0$ by definition. On the other hand, $e_i (\text{rad } B) e_{\nu(i)} = 0$ by Lemma 2.1 (3). Thus we have $e_{\nu(i)} = e_i$, equivalently $\nu(i) = i$, a contradiction.

The proof that $\phi$ is an algebra homomorphism, is divided into several steps, and involves the relations (5) and (6) of Lemma 2.1.

(1) $\phi(e + e I e) = e$.

For $a = e, e a e^{(i)}_j = e_j e^{(i)}_j = 0$ because $e_i \leq e^{(i)}_B$ and $e^{(i)}_B \perp e^{(i)}_I$, and $e_i a e_{\nu(i)} = e_i P e_{\nu(i)} = 0$ because $\nu(i) \neq i$. Hence $\phi(e + e I e) = \sum_{i=1}^m e_i P e_i^{(i)} = e$.

(2) $\phi((a + e I e)(b + e I e)) = \phi(a + e I e) \phi(b + e I e)$ for $a, b \in e A e$.

We may assume that $a = e_a e, b = e_b e_{(i)}$. for $e_i \leq e_j, e_j \leq e_{(i)}$. Moreover, it is enough to consider the case when $a \notin I$ and $b \notin I$. Let $\alpha = \phi((a + e I e)(b + e I e))$ and $\beta = \phi(a + e I e) \phi(b + e I e)$ Then $\alpha = \phi(ab + e I e) = e_a e \cdot e_b e_{(i)}$, and $\beta = e_a e \cdot e_b e_{(i)}$. Here, if $e_j \notin e_{(i)}$, then $e_j e_j = 0$ and so $e_j e_j = 0$. Hence, if we have $\alpha = 0$ and $\beta = 0$, and $\alpha = \beta$ as required. Now, assume that $e_j \notin e_{(i)}$. Then $\beta = e_a e \cdot e_b e_{(i)}$. Since $e_{(i)} e_{(i)} = e_{(i)} e_{(i)}$, it therefore suffices to show that $\alpha = \alpha e_{(i)}$ and $\beta = \beta e_{(i)}$.

(a) First we show that $\alpha = \alpha e_{(i)}$. Obviously, $e_j e_{(i)} e_{(i)} = e_j e_{(i)} e_{(i)} = e_j e_{(i)} + e_{(i)} e_{(i)}$. We claim that $e_j e_{(i)} e_{(i)} = 0$ and $e_j e_{(i)} e_{(i)} = 0$. In fact, if $e_j e_{(i)} e_{(i)} e_{(i)} \neq 0$, then $0 \neq e_j e_{(i)} e_{(i)} = e_j e_{(i)} \leq e_{(i)}$. Since $e_j e_{(i)} = b \notin I$ by assumption, we have $e_j e_{(i)} \perp e_{(i)}$, which contradicts the orthogonality $e_{(i)} \perp e_{(i)}$. Next, if $e_j e_{(i)} e_{(i)} \neq 0$, then $0 \neq e_j e_{(i)} e_{(i)} = e_j e_{(i)} = e_{(i)}$. Hence $e_j e_{(i)} \in \text{rad } A\backslash I$ because $j \notin e_{(i)}$ by assumption so that $e_j (\text{rad } B) e_{(i)} = 0$, which contradicts Lemma 2.1 (3). Thus we have proved that $e_j e_{(i)} e_{(i)} = e_j e_{(i)} e_{(i)}$, so that $\alpha = \alpha e_{(i)}$.

(b) Secondly, we show that $\beta = \beta e_{(i)}$. Since $\beta = e_i \beta = e_i \beta e_{(i)} + e_i \beta e_{(i)}$, it suffices to show that $\beta e_{(i)} = 0$ and $\beta e_{(i)} = 0$. Suppose that $\beta e_{(i)} \neq 0$ contrarily. Then it follows from the proof of 2.4 Proposition 3.1 that $e_i e_{(i)} \in \text{soc}(e_i I)$. Hence $\beta e_{(i)} e_{(i)} \neq 0$, and so $e_j e_{(i)} e_{(i)} \neq 0$. It follows that $e_j e_{(i)} e_{(i)} e_{(i)} = e_{(i)} = e_{(i)} \leq e$. Therefore, $\beta = e_a e \cdot e_{(i)}$, so that $\beta = 0$ by Lemma 2.1 (2), a contradiction.

Therefore, we proved that $\phi$ is an algebra homomorphism. Finally, for $a = e a e \in e A e$, $a - \sum_{i=1}^m e_a e^{(i)} = \sum_{i=1}^m (e_a e - e_a e^{(i)}) = \sum_{i=1}^m (e_a e^{(i)} + e_a e_{(i)})$, which belongs to $e I e$. Hence we have $\phi = 1_B$. 

We will now prove the announced criterion.

**Proposition 2.6.** Let $A$ be a self-injective algebra, $I$ an ideal of $A$, $B = A/I$, and $e$ a residual identity of $B$. Assume that the ordinary quiver $Q(B)$ of $B$ has no oriented cycles, $I e I = 0$, $I e$ is an injective cogenerator in $\text{mod } B$, and $e_i \neq e_{\nu(i)}$ for any $i \in \{1, \ldots, m\}$. Then $A$ is isomorphic to an algebra $\tilde{B}/G$ where $G$ is an
infinite cyclic group of automorphisms of $\widehat{B}$ generated by $\varphi_{\nu_B}$ for some positive automorphism $\varphi$ of $\widehat{B}$.

Proof. It follows from [7, Proposition 2.3] that $I$ is a deforming ideal, and by Proposition 2.5 the canonical epimorphism $\varphi : eAe \rightarrow eAe/eIe = B$ splits. Applying now [9, Theorem 3.8] we conclude that $A$ is isomorphic to $\widehat{B}/(\varphi_{\nu_B})$ for some positive automorphism $\varphi$ of $\widehat{B}$. \qed

3. Self-injective algebras with deforming ideal

Let $A$ be a self-injective algebra, $I$ a deforming ideal of $A$, $B = A/I$ and $e$ a residual identity of $B$. Then $I$ can be considered as a $(eAe/eIe)$-bimodule. Denote by $A[I]$ the direct sum of $K$-modules $(eAe/eIe) \oplus I$ with the multiplication

$$(b, x) \cdot (b', x') = (bb', bx' + xb' + xx')$$

for $b, b' \in eAe/eIe$ and $x, x' \in I$. Then $A[I]$ is an algebra with the identity $(e, 1 - e)$ and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider $I$ as an ideal of $A[I]$. The following fact has been proved in [7, Theorem 4.1].

Proposition 3.1. $A[I]$ is a self-injective algebra with a deforming ideal $I$, the Nakayama permutation of $A[I]$ is the same as the Nakayama permutation of $A$, and the algebras $A[I]$ and $A$ are socle equivalent.

Moreover, if $K$ is a field, we have proved in [8, Theorem 3] that the algebras $A[I]$ and $A$ are also stably equivalent. The aim of this section is to prove the following fact, needed in the proof of our main theorem.

Proposition 3.2. Let $A$ be a self-injective algebra with a deforming ideal $I$, $B = A/I$ and let $e$ be a residual identity of $B$. Assume that $IeI = 0$ and $e_i \neq e_{\nu(i)}$ for any primitive summand $e_i$ of $e$. Then the algebras $A[I]$ and $A$ are isomorphic.

Proof. We assume as before that $e = \sum_{i=1}^{n} e_i$ for $m \leq n$ and a set $\{ e_i | 1 \leq i \leq n \}$ of primitive idempotents of $A$ with $1_A = \sum_{i=1}^{n} e_i$. Denote by $\overline{a}$ the residue class of $a \in eAe$ in $eAe/eIe$. Since by our assumption $\nu(i) \neq i$ for $i \in \{1, \ldots, m\}$, invoking Proposition 2.5, we have an algebra monomorphism $\phi : eAe/eIe \rightarrow eAe$ such that $\phi(\overline{a}) = \sum_{i=1}^{n} e_i \overline{a}$. We define a homomorphism $\Phi : A[I] \rightarrow A$ of $K$-modules by

$$\Phi(\overline{a}, x) = \phi(\overline{a}) + x$$

for $\overline{a} \in eAe/eIe$ and $x \in I$. We claim that $\Phi$ is an algebra isomorphism. Consider the following commutative diagram of canonical short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & I & \rightarrow & A[I] & \rightarrow & eAe/eIe & \rightarrow & 0 \\
& & \downarrow 1 & & \downarrow \Phi & & \downarrow \phi' & & \downarrow 0, \\
0 & \rightarrow & I & \rightarrow & A & \rightarrow & A/I & \rightarrow & 0,
\end{array}
$$

where $1$ is the identity map and $\Phi'$ is an isomorphism given by $\Phi'(\overline{a}) = \phi(\overline{a}) + I$, for $\overline{a} \in eAe/eIe$. In order to prove that $\Phi$ is an algebra isomorphism, it suffices to show that $\Phi$ is an algebra homomorphism. Observe that $\Phi(1_{A[I]}) = \Phi(1 - e) = \phi(1) + (1 - e) = e + (1 - e) = 1 = 1_A$.

Take $\overline{a}, \overline{b} \in eAe/eIe$ and $x, y \in I$. We shall prove that

$$\Phi((\overline{a}, x)(\overline{b}, y)) = \Phi(\overline{a}, x)\Phi(\overline{b}, y).$$
In order to prove that $I e_i$ because for all $i$ with $I e_i$, there is no path from $Q$ to $a$ of the vertices of $S$.

Proof. Here, $e_i a e_{\nu(i)} = 0$ and $e_i a e_{\nu(i)} y = 0$ by Lemma 2.2 because $e_i a e_{\nu(i)} \in \text{rad} A$ by $\nu(i) \neq i$, and therefore $\phi(\bar{a}) y = ay$.

(b) $x \phi(\overline{b}) = xb$ for $b \in e.A e$ and $x \in I$:

Since $e_i A e_{\nu(i)} \subseteq e_i I$ by Lemma 2.1, we have $xe_i b e_{\nu(i)} \in I e_i I \subseteq I e I$. Then $xe_i b e_{\nu(i)} = 0$ because $I e I = 0$ by assumption. On the other hand, $e_i A e_{\nu(i)} = e_i (\text{rad} A) e_{\nu(i)}$ by $\nu(i) \neq i$, and hence $e_i b e_{\nu(i)} \in I$ by (1) and (2) of Lemma 2.1. Hence $xe_i b e_{\nu(i)} \in I e_i I \subseteq I e I = 0$, and $xe_i b e_{\nu(i)} = 0$. Thus, $xb = \sum_{i=1}^{m} xe_i b e_{\nu(i)} + \sum_{i} xe_i b e_{\nu(i)} + \sum_{i} xe_i b e_{\nu(i)}$

Hence, $xb = \sum_{i=1}^{m} xe_i b e_{\nu(i)} = \sum_{i} xe_i b e_{\nu(i)} = x \sum_{i} e_i b e_{\nu(i)} = x \phi(\overline{b})$.

Therefore, the algebras $A[I]$ and $A$ are isomorphic. 

4. Proof of the theorem

We first prove the following general facts.

Lemma 4.1. Let $B$ be an algebra and $\varphi$ a positive automorphism of $\hat{B}$ that fixes a finite subset of the object set of $\hat{B}$. Then $\varphi$ is a rigid automorphism of $\hat{B}$.

Proof. We write as before $\hat{B} = \bigoplus_{k \in \mathbb{Z}} (B_k \oplus (DB)_k)$ with $B_k = B$ and $(DB)_k = DB$ for all $k \in \mathbb{Z}$, and identify the objects of $\hat{B}$ (respectively, $B_k$) with the vertices of the ordinary quivers $Q(\hat{B})$ (respectively, $Q(B_k)$). Since $\varphi$ is a positive automorphism of $B$, we may assume (without loss of generality) that $\varphi$ fixes a subset of $Q(B_0)$. In order to prove that $\varphi$ is rigid, it is enough to show that $\varphi(Q(B_0)) = Q(B_0)$. For this, since the quiver $Q(B_0)$ is connected, it suffices to show that $\varphi(-S) = (-S)$ and $\varphi(S \rightarrow) = (S \rightarrow)$ for any subset $S$ of $Q(B_0)$ with $\varphi(S) = S$, where $(-S)$ and $(S \rightarrow)$ denote the sets of all predecessors of the vertices of $S$ and of all successors of the vertices of $S$ in $Q(B_0)$, respectively. Therefore, let $S$ be a subset of $Q(B_0)$ with $\varphi(S) = S$.

Now, let $x \rightarrow \cdots \rightarrow s$ be a path in $Q(B_0)$ with $s \in S$. Then $\varphi(x) \rightarrow \cdots \rightarrow \varphi(s)$ is a path in $Q(\hat{B})$ and $\varphi(s) \in Q(B_0)$, because $\varphi(S) = S$ by assumption. Since $\varphi$ is positive, $\varphi(x) \in \sum_{k \geq 0} Q(B_k)$ and it follows that $\varphi(x)$ belongs to $Q(B_k)$, because there is no path from $Q(B_k)$ to $Q(B_0)$ for any $k > 0$. Thus $\varphi(-S) = (-S)$.
Next, suppose that there is a path $s \to \cdots \to x$ in $Q(B_0)$ for $s \in S$ with $\varphi(x) \notin Q(B_0)$, that is, $\varphi(x) \in Q(B_k)$ for some $k > 0$. Since $S$ is finite and $Q(B)$ is locally finite, the $\varphi$-orbit of $x$ is finite also. This implies that there is an integer $\ell$ such that $\varphi^{\ell}(x) \in Q(B_{k'})$ for some $k' > 0$ and $\varphi^{\ell+1}(x) \in Q(B_0)$. This, however, implies that $\varphi(Q(B_{k'})) \cap Q(B_0) \neq \emptyset$, which contradicts the positivity of $\varphi$. Thus we conclude that $\varphi(S \to) = (S \to)$. 

\textbf{Corollary 4.2.} Let $B$ be an algebra and $\varphi$ a strictly positive automorphism of $\tilde{B}$. Then $\varphi$ acts freely on the objects of $\tilde{B}$.

\textit{Proof of the Theorem.} Let $A$ be a self-injective algebra that is stably equivalent to a self-injective algebra of the form $R/(\psi v_R)$, where $R$ is a tilted algebra of type $\Delta$ (which is not a Dynkin quiver) and $\psi$ is a strictly positive automorphism of $R$. It follows from the proof of Theorem 1 in [8] that there are an ideal $I$ of $A$ and a residual identity $e = \sum_{i=1}^{m} e_i$ of $B = A/I$ such that $IeI = 0$, $Ie$ is a injective cogenerator in $\text{mod} B$, $B$ is a tilted algebra of type $\Delta$, and $A[I] \cong \tilde{B}/(\varphi v_{\tilde{B}})$ for a positive automorphism $\varphi$ of $\tilde{B}$. Moreover, by [8] Theorem 31, the algebras $A$ and $A[I]$ are stably equivalent. Since by assumption $A$ is stably equivalent to $R/(\psi v_R)$, we conclude that $A[I]$ is stably equivalent to $\tilde{R}/(\psi v_{\tilde{R}})$. Furthermore, since $\psi$ is a strictly positive automorphism of $\tilde{R}$, the number of simple modules of $\tilde{R}/(\psi v_{\tilde{R}})$ is greater than the number of simple modules of $R$, which is the same as the number of vertices of $\Delta$. Applying now [8] Corollary we conclude that the automorphism $\varphi$ is not rigid, and so is strictly positive. Hence, invoking Corollary 4.2 and Proposition 3.1, we obtain that $\nu(i) \neq i$ for the Nakayama permutation $\nu$ of $A$ and $i \in \{1, \ldots, m\}$. Since $B$ is a tilted algebra, the ordinary quiver $Q(B)$ of $B$ has no oriented cycles, and then the properties of $I$ stated above imply that $I$ is a deforming ideal of $A$. Therefore, applying Proposition 4.2 we obtain that $A$ and $A[I]$ are isomorphic, and so $A$ and $\tilde{B}/(\varphi v_{\tilde{B}})$ are also isomorphic. This finishes the proof. 

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