

ON INVARIABILITY OF SELF-INJECTIVE ALGEBRAS OF TILTED TYPE UNDER STABLE EQUIVALENCES

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ABSTRACT. We prove that a large class of self-injective algebras of tilted type is invariant under stable equivalences of module categories.

1. INTRODUCTION AND THE MAIN RESULT

Throughout the paper, by an algebra is meant a basic, connected, Artin algebra (associative, with an identity) over a fixed commutative Artinian ring K . For an algebra A , we denote by $\text{mod } A$ the category of finitely generated right A -modules and by $\underline{\text{mod}} A$ the stable module category of A . Recall that the objects of $\underline{\text{mod}} A$ are the objects of $\text{mod } A$ without nonzero projective direct summands, and for any two objects M and N in $\underline{\text{mod}} A$ the K -module $\underline{\text{Hom}}_A(M, N)$ of morphisms from M to N is the quotient $\text{Hom}_A(M, N)/P(M, N)$, where $P(M, N)$ is the submodule of $\text{Hom}_A(M, N)$ consisting of all A -homomorphisms that factorize through projective A -modules. Two algebras A and Λ are said to be stably equivalent if their stable module categories $\underline{\text{mod}} A$ and $\underline{\text{mod}} \Lambda$ are equivalent. Recall also that $D = \text{Hom}_K(-, E)$, where E is a minimal injective cogenerator in $\text{mod } K$, defines a duality between the categories of left and right modules. An algebra A is called self-injective if $A \cong D(A)$ in $\text{mod } A$, that is, the projective A -modules are injective. An important class of self-injective algebras is formed by the algebras of the form \widehat{B}/G where \widehat{B} is the repetitive algebra [5] (locally bounded, without identity)

$$\widehat{B} = \bigoplus_{k \in \mathbb{Z}} (B_k \oplus (DB)_k)$$

of an algebra B , where $B_k = B$ and $(DB)_k = DB$ for all $k \in \mathbb{Z}$, the multiplication in \widehat{B} is defined by

$$(a_k, f_k)_k \cdot (b_k, g_k)_k = (a_k b_k, a_k g_k + f_k b_{k+1})_{k \in \mathbb{Z}}$$

for $a_k, b_k \in B_k$, $f_k, g_k \in (DB)_k$, and G is an admissible group of K -automorphisms of \widehat{B} . More precisely, for a fixed set $\mathcal{E} = \{e_i \mid 1 \leq i \leq m\}$ of primitive orthogonal idempotents of B with $1_B = e_1 + \cdots + e_m$, consider the canonical set $\widehat{\mathcal{E}} = \{e_{j,k} \mid 1 \leq j \leq m, k \in \mathbb{Z}\}$ of primitive orthogonal idempotents of \widehat{B} such that $e_{j,k} \widehat{B} = (e_j B)_k \oplus (e_j DB)_k$ for $1 \leq j \leq m$ and $k \in \mathbb{Z}$. By an automorphism of \widehat{B} we mean a K -algebra

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automorphism of \widehat{B} that fixes the chosen set $\widehat{\mathcal{E}}$ of primitive orthogonal idempotents of \widehat{B} . A group G of automorphisms of \widehat{B} is said to be admissible if the induced action of G on $\widehat{\mathcal{E}}$ is free and has finitely many orbits. Then the orbit algebra \widehat{B}/G is a self-injective algebra and the G -orbits in $\widehat{\mathcal{E}}$ form a canonical set of primitive orthogonal idempotents of \widehat{B}/G whose sum is the identity of \widehat{B}/G (see [4]). We denote by $\nu_{\widehat{B}}$ the Nakayama automorphism of \widehat{B} such that $\nu_{\widehat{B}}(e_{j,k}) = e_{j,k+1}$ for all $1 \leq j \leq m, k \in \mathbb{Z}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension $B \rtimes DB$ of B by DB . An automorphism φ of \widehat{B} is said to be positive (respectively, rigid) when $\varphi(B_k) \subseteq \sum_{i \geq k} B_i$ (respectively, $\varphi(B_k) = B_k$) for any $k \in \mathbb{Z}$. Moreover, φ is said to be strictly positive if it is positive but not rigid. We shall also consider \widehat{B} as a locally bounded K -category with the objects set as $\widehat{\mathcal{E}}$.

Let R be a tilted algebra of type Δ that is not a Dynkin quiver. Then the Auslander-Reiten quiver $\Gamma_{\widehat{R}}$ of \widehat{R} is of the form

$$\Gamma_{\widehat{R}} = \bigvee_{p \in \mathbb{Z}} (\mathcal{X}_p \vee \mathcal{R}_p)$$

where, for each $p \in \mathbb{Z}$, \mathcal{X}_p is a component with the stable part of the form $\mathbb{Z}\Delta$, \mathcal{R}_p is a family of components whose stable parts are tubes (if Δ is Euclidean) or of type $\mathbb{Z}A_\infty$ (if A is wild), and $\nu_{\widehat{R}}(\mathcal{X}_p) = \mathcal{X}_{p+2}$ and $\nu_{\widehat{R}}(\mathcal{R}_p) = \mathcal{R}_{p+2}$, for the induced action of $\nu_{\widehat{R}}$ on $\Gamma_{\widehat{R}}$ (see [1], [3], [6]). Furthermore, an automorphism φ of \widehat{R} is positive (respectively, strictly positive) if and only if there exists $q \geq 0$ (respectively, $q > 0$) such that $\varphi(\mathcal{X}_p) = \mathcal{X}_{p+q}$ and $\varphi(\mathcal{R}_p) = \mathcal{R}_{p+q}$ for all $p \in \mathbb{Z}$. We also note that the class of algebras of the form $\widehat{R}/(\psi\nu_{\widehat{R}})$ with R a tilted algebra of type Δ (not a Dynkin quiver) and ψ a strictly positive (respectively, positive) automorphism of \widehat{R} coincides in the class of all self-injective algebras of tilted type whose stable Auslander-Reiten quiver admits at least three (respectively, two) components of type $\mathbb{Z}\Delta$. We have proved in [8, Theorem 1] that a self-injective algebra is stably equivalent to an algebra $\widehat{R}/(\psi\nu_{\widehat{R}})$ with ψ a positive automorphism of \widehat{R} if and only if A is a socle equivalent to an algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ where B is a tilted algebra of type Δ and φ is a positive automorphism of \widehat{B} . Recall that two self-injective algebras A and Λ are called socle equivalent if the factor algebras $A/\text{soc } A$ and $\Lambda/\text{soc } \Lambda$ are isomorphic. Our main result shows that for ψ strictly positive we may replace “socle equivalent” by “isomorphic”.

Theorem. *Let A be a self-injective algebra stably equivalent to a self-injective algebra of the form $\widehat{R}/(\psi\nu_{\widehat{R}})$ where R is a tilted algebra of type Δ that is not a Dynkin quiver and ψ is a strictly positive automorphism of \widehat{R} . Then A is isomorphic to an algebra of the form $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some tilted algebra B of type Δ and a strictly positive automorphism φ of \widehat{B} .*

We note that the strict positivity of ψ is necessary for the validity of the theorem. Namely, if A is a nonsplittable Hochschild extension of a hereditary algebra H not of Dynkin type (see [7] and [10] for existence of such algebras), then A is stably equivalent to the trivial extension $H \rtimes D(H) \cong \widehat{H}/(\nu_{\widehat{H}})$ but is not isomorphic to an algebra $\widehat{B}/(\varphi\nu_{\widehat{B}})$ with B a tilted algebra of type Δ and φ a positive automorphism

of \widehat{B} . It would be interesting to know if the theorem also holds for self-injective algebras of Dynkin type.

For general background concerning representation theory of Artin algebras and self-injective algebras we refer to [2] and [11].

2. PRELIMINARY RESULTS

The aim of this section is to prove some preliminary results that we need in the proof of our main theorem. As an application we establish also a criterion (Proposition 2.6) for a self-injective Artin algebra A to be isomorphic to an algebra of the form \widehat{B}/G , where \widehat{B} is the repetitive algebra of an Artin algebra B of finite global dimension and G is an admissible infinite cyclic group of automorphisms of \widehat{B} .

Let A be a self-injective algebra, $\{e_i \mid 1 \leq i \leq n\}$ a fixed set of primitive orthogonal idempotents of A such that $1_A = e_1 + \cdots + e_n$, and $\nu = \nu_A$ a fixed Nakayama automorphism of A inducing an A -bimodule isomorphism $A \cong {}_\nu DA$, where ${}_\nu DA$ denotes the left A -module obtained from the canonical left A -module DA by changing the operation of A by $a \cdot f = \nu(a)f$ for $a \in A$, $f \in DA$. Hence we have $\text{soc}(e_i A) \cong \text{top}(\nu(e_i)A) (= \nu(e_i)A/\nu(e_i)\text{rad } A)$ as right A -modules for all $i \in \{1, \dots, n\}$. Since $\{\nu(e_i)A \mid 1 \leq i \leq n\}$ is a set of representatives of indecomposable projective A -modules, there is a permutation of $\{1, \dots, n\}$, denoted again by ν , such that $\nu(e_i)A \cong e_{\nu(i)}A$ for all $i \in \{1, \dots, n\}$. Let I be a (two-sided) ideal of A , $B = A/I$ and e an idempotent of A such that $e + I$ is the identity of B . We may assume that $e = e_1 + \cdots + e_m$ for some $m \leq n$, and $\{e_i \mid 1 \leq i \leq m\}$ is the subset of $\{e_i \mid 1 \leq i \leq n\}$ consisting of all idempotents e_i that are not in I . Then such an idempotent e is uniquely determined by I up to an inner automorphism of A and is called a residual identity of B . We note that $B \cong eAe/eIe$ and $1 - e \in I$.

Let A be a self-injective algebra, I an ideal of A , and $B = A/I$. For idempotents f and f' of A , f is said to be a summand of f' if $ff' = f'f = f$, and we denote this fact by $f \leq f'$. Moreover, the orthogonality $ff' = 0 = f'f$ of f and f' is denoted by $f \perp f'$. Fix a set $\{e_i \mid 1 \leq i \leq n\}$ of primitive orthogonal idempotents of A such that $1_A = \sum_{i=1}^n e_i$ and $e = \sum_{i=1}^m e_i$, for some $m \leq n$, is a residual identity of B . We denote the residue class of each idempotent e_j , $1 \leq j \leq m$, in $B = A/I$ by e_j again, for simplicity. Since $B \cong eAe/eIe$ as algebras, we often identify them. Finally, for $i \in \{1, \dots, m\}$, let $e_B^{(i)}$ (respectively, $e_I^{(i)}$) be the sum of all idempotents e_j , $1 \leq j \leq n$, such that $e_i B e_j \neq 0$ (respectively, $e_i(I/\text{soc } I)e_j \neq 0$). Following [7, (2.1)] the ideal I is said to be *deforming* if $eIe = \ell_{eAe}(I) = r_{eAe}(I)$ and the ordinary quiver $Q(B)$ of B has no oriented cycles. Here, $\ell_{eAe}(I)$ is the left annihilator of I in eAe and $r_{eAe}(I)$ is the right annihilator of I in eAe .

From now on we assume that the ideal I is deforming. For the proofs of our main results, we need several technical lemmas. We recall first the following properties of a deforming ideal proved in [7, Section 1].

Lemma 2.1. (1) $\text{Soc}(A) \subseteq I$.

(2) For e_i ($i \leq m$), the right B -modules $e_i B$ and $e_i(I/\text{soc } I)$ have no common composition factors, and similarly for $e_i(\text{rad } B)$ and $e_i I$.

(3) There are no idempotents e_{i_j} ($0 \leq j \leq t$) such that

$$e_{i_0}(\text{rad } B)e_{i_1} \neq 0, e_{i_1}(\text{rad } B)e_{i_2} \neq 0, \dots, e_{i_{t-1}}(\text{rad } B)e_{i_t} \neq 0, e_{i_t}(\text{rad } B)e_{\nu(i_0)} \neq 0.$$

- (4) $e_i I e_B^{(i)} \subseteq \text{soc}(e_i I)$ and $e_i A e_I^{(i)} \subseteq e_i I$ for $i \leq m$.
- (5) $e_B^{(i)} \perp e_I^{(i)}$.
- (6) $e^{(i)} a = e_i a e_B^{(i)} + e_i a e_I^{(i)} + e_i a e_{\nu(i)}$ for $a \in A$ and $1 \leq i \in m$ with $\nu(i) \neq i$.

Lemma 2.2. *For any $i \leq m$, the following equalities hold:*

- (1) $e_i(eAe)e_I^{(i)}I = 0$;
- (2) $e_i \text{rad}(eAe)e_{\nu(i)}I = 0$ and $Ie_{\nu^{-1}(i)} \text{rad}(eAe)e_i = 0$.

Proof. (1) Assume that $e_i a e_I^{(i)} I \neq 0$ for some $a \in eAe$. Since $e e_I^{(i)} \neq 0$ by assumption, there is some $e_j \leq e$ such that $e_j \leq e_I^{(i)}$ and $e_i a e_j I \neq 0$. This implies that $e_i a e_j \notin eIe$, because $\ell_{eAe}(I) = eIe$ by our assumption. Thus $0 \neq e_i \bar{a} e_j \in B$ so that $e_j \leq e_B^{(i)}$, where \bar{a} denotes the residue class of a in A/I . Then it follows that $e_B^{(i)} e_I^{(i)} \neq 0$, a contradiction to Lemma 2.1(2).

(2) Assume that $e_i a e_{\nu(i)} I \neq 0$ for some $a = eae \in \text{rad}(eAe)$. In particular, we have $e e_{\nu(i)} \neq 0$, so that $e_{\nu(i)} \leq e$ and $e_i a e_{\nu(i)} \notin eIe$, because $\ell_{eAe}(I) = eIe$. Thus we have $e_i(\text{rad } B)e_{\nu(i)} \neq 0$, a contradiction to Lemma 2.1(3). The proof of the second equality is similar. □

Lemma 2.3. *If $1 \leq i, \nu(i) \leq m$, then $e_i(I/\text{soc } I)e_{\nu(i)} = 0$ and $e_i(\text{rad } A)e_{\nu(i)} \subseteq \text{soc}(e_i I)$.*

Proof. (1) If $e_i(I/\text{soc } I)e_{\nu(i)} \neq 0$, then $e_{\nu(i)}(\text{rad } B)e_{\nu(i)} \neq 0$ by [7, Lemma 1.2], which implies that $\mathcal{Q}(B)$ has an oriented cycle, a contradiction. Hence we have $e_i(I/\text{soc } I)e_{\nu(i)} = 0$.

(2) By Lemma 2.1(3), $e_i(\text{rad } B)e_{\nu(i)} = 0$ and so $e_i(\text{rad } A)e_{\nu(i)} \subseteq e_i I e_{\nu(i)}$. If $e_i(\text{rad } A)e_{\nu(i)} \not\subseteq \text{soc}(e_i I)$, then $e_i(\text{rad } A/\text{soc } I)e_{\nu(i)} \neq 0$. Hence, it follows from Lemma 2.1(2) that $e_i(I/\text{soc } I)e_{\nu(i)} \neq 0$, which is impossible by the assertion proved in (1) above. Thus we conclude that $e_i(\text{rad } A)e_{\nu(i)} \subseteq \text{soc}(e_i I)$. □

Lemma 2.4. *For $a, b \in A \setminus I$, the following statements hold:*

- (1) if $ab \in \text{rad } A$, then $e_i a e b e_{\nu(i)} = 0$ for $1 \leq i, \nu(i) \leq m$;
- (2) $e_i a e b e_{\nu(i)} = 0$ for $1 \leq i, \nu(i) \leq m$ and $\nu(i) \neq i$.

Proof. (1) It suffices to prove the assertion for $a = e_i a e_j$ and $b = e_j b e_{\nu(i)}$ for $e_j \in B$. Suppose that $ab \neq 0$. We claim that both a and b belong to $\text{rad } A$. In fact, if $a \notin \text{rad } A$, then $e_i = e_j$ and there is some $a' \in e_i A e_i$ with $a' a = e_j$. Then $b = a'(ab) \in \text{soc } I$ because $ab \in \text{soc } I$ by Lemma 2.3, a contradiction to the assumption. Thus $a \in \text{rad } A$. Similarly, we have $b \in \text{rad } A$. It therefore follows that $e_i B e_j \neq 0$ and $e_j B e_{\nu(i)} \neq 0$, a contradiction to Lemma 2.1(3).

(2) Since $\nu(i) \neq i$, $e_i a e b e_{\nu(i)} \in \text{rad } A$ obviously, and the assertion follows from (1). □

Denote by $\varrho : eAe \rightarrow eAe/eIe = B$ the canonical algebra epimorphism and define the map $\phi : eAe/eIe \rightarrow eAe$ by

$$\phi(a + eIe) = \sum_{i=1}^m e_i a e_B^{(i)}$$

for all $a \in eAe$.

Proposition 2.5. *Assume $\nu(i) \neq i$ for all $i \in \{1, \dots, m\}$. Then ϕ is an algebra homomorphism such that $\varrho\phi = 1_B$.*

Proof. We first prove that ϕ is a well-defined K -homomorphism. It suffices to show that $e_i I e_B^{(i)} = 0$ for any $i \leq m$. For this, suppose that $e_i I e_B^{(i)} \neq 0$. Then $e_i I e_B^{(i)} e_{\nu(i)} \neq 0$, because $e_i I e_B^{(i)} \subseteq \text{soc}(e_i I) \cong \text{top}(e_{\nu(i)} A)$ by Lemma 2.1 (4). In particular, $e_B^{(i)} e_{\nu(i)} \neq 0$ and so $e_{\nu(i)} \leq e_B^{(i)}$. Hence $e_i B e_{\nu(i)} \neq 0$ by definition. On the other hand, $e_i(\text{rad } B) e_{\nu(i)} = 0$ by Lemma 2.1(3). Thus we have $e_{\nu(i)} = e_i$, equivalently $\nu(i) = i$, a contradiction.

The proof that ϕ is an algebra homomorphism, is divided into several steps, and involves the relations (5) and (6) of Lemma 2.1.

(1) $\phi(e + eIe) = e$.

For $a = e, e_i a e_I^{(i)} = e_i e_I^{(i)} = 0$ because $e_i \leq e_B^{(i)}$ and $e_B^{(i)} \perp e_I^{(i)}$, and $e_i a e_{\nu(i)} = e_i e_{\nu(i)} = 0$ because $\nu(i) \neq i$. Hence $\phi(e + eIe) = \sum_{i=1}^m e_i e e_B^{(i)} = e$.

(2) $\phi((a + eIe)(b + eIe)) = \phi(a + eIe)\phi(b + eIe)$ for $a, b \in eAe$.

We may assume that $a = e_i a e_{i'}$, $b = e_j b e_{j'}$ for $e_{i'} \leq e_B^{(i)}$, $e_{j'} \leq e_B^{(j)}$. Moreover, it is enough to consider the case when $a \notin I$ and $b \notin I$. Let $\alpha = \phi((a + eIe)(b + eIe))$ and $\beta = \phi(a + eIe)\phi(b + eIe)$. Then $\alpha = \phi(ab + eIe) = e_i a e_{i'} e_j b e_{j'} e_B^{(i)}$, and $\beta = e_i a e_{i'} e_B^{(i)} \cdot e_j b e_{j'} e_B^{(j)}$. Here, if $e_j \not\leq e_B^{(i)}$, then $e_B^{(i)} e_j = 0$ and so $e_j e_j = 0$. Hence we have $\alpha = 0$ and $\beta = 0$, and $\alpha = \beta$ as required. Now, assume that $e_j \leq e_B^{(i)}$. Then $\beta = e_i a e_{i'} e_j b e_{j'} e_B^{(j)}$. Since $e_B^{(i)} e_B^{(j)} = e_B^{(j)} e_B^{(i)}$, it therefore suffices to show that $\alpha = \alpha e_B^{(j)}$ and $\beta = \beta e_B^{(j)}$.

(a) First we show that $\alpha = \alpha e_B^{(j)}$. Obviously, $e_j b e_{j'} e_B^{(i)} = e_j b e_{j'} e_B^{(i)} (e_B^{(j)} + e_I^{(j)} + e_{\nu(j)})$. We claim that $e_j b e_{j'} e_B^{(i)} e_I^{(j)} = 0$ and $e_j b e_{j'} e_B^{(i)} e_{\nu(j)} = 0$. In fact, if $e_j b e_{j'} e_B^{(i)} e_I^{(j)} \neq 0$, then $0 \neq e_{j'} e_B^{(i)} e_I^{(j)} = e_{j'} \leq e_I^{(j)}$. Since $e_j b e_{j'} = b \notin I$ by assumption, we have $e_j B e_I^{(j)} \neq 0$, which contradicts the orthogonality $e_B^{(j)} \perp e_I^{(j)}$. Next, if $e_j b e_{j'} e_B^{(i)} e_{\nu(j)} \neq 0$, then $0 \neq e_{j'} e_B^{(i)} e_{\nu(j)} = e_{j'} = e_{\nu(j)}$ and $b = e_j b e_{\nu(j)}$. Hence $e_j b e_{\nu(j)} \in \text{rad } A \setminus I$ because $j \neq \nu(j)$ by assumption so that $e_j(\text{rad } B) e_{\nu(j)} \neq 0$, which contradicts Lemma 2.1 (3). Thus we have proved that $e_j b e_{j'} e_B^{(i)} = e_j b e_{j'} e_B^{(i)} e_B^{(j)}$, so that $\alpha = \alpha e_B^{(j)}$.

(b) Secondly, we show that $\beta = \beta e_B^{(j)}$. Since $\beta = e_i \beta = e_i \beta e_B^{(i)} + e_i \beta (e_I^{(i)} + e_{\nu(i)})$, it suffices to show that $\beta e_I^{(i)} = 0$ and $\beta e_{\nu(i)} = 0$. Suppose that $\beta e_I^{(i)} \neq 0$ contrarily. Then it follows from the proof of [7, Proposition 3.1] that $\beta e_I^{(i)} \in \text{soc}(e_i I)$. Hence $\beta e_I^{(i)} e_{\nu(i)} \neq 0$, and so $e_{j'} e_B^{(j)} e_I^{(i)} e_{\nu(i)} \neq 0$. It follows that $e_{j'} e_B^{(j)} e_I^{(i)} e_{\nu(i)} = e_{\nu(i)} = e_{j'} \leq e$. Therefore, $\beta = e_i a e_j b e_{\nu(i)}$, so that $\beta = 0$ by Lemma 2.4 (2), a contradiction.

Therefore, we proved that ϕ is an algebra homomorphism. Finally, for $a = eae \in eAe$, $a - \sum_{i=1}^m e_i a e_B^{(i)} = \sum_{i=1}^m (e_i a e - e_i a e_B^{(i)}) = \sum_{i=1}^m (e_i a e_I^{(i)} + e_i a e_{\nu(i)})$, which belongs to eIe . Hence we have $\rho\phi = 1_B$. □

We will now prove the announced criterion.

Proposition 2.6. *Let A be a self-injective algebra, I an ideal of A , $B = A/I$, and e a residual identity of B . Assume that the ordinary quiver $Q(B)$ of B has no oriented cycles, $IeI = 0$, Ie is an injective cogenerator in $\text{mod } B$, and $e_i \neq e_{\nu(i)}$ for any $i \in \{1, \dots, m\}$. Then A is isomorphic to an algebra \widehat{B}/G where G is an*

infinite cyclic group of automorphisms of \widehat{B} generated by $\varphi\nu_{\widehat{B}}$ for some positive automorphism φ of \widehat{B} .

Proof. It follows from [7, Proposition 2.3] that I is a deforming ideal, and by Proposition 2.5 the canonical epimorphism $\varrho : eAe \rightarrow eAe/eIe = B$ splits. Applying now [9, Theorem 3.8] we conclude that A is isomorphic to $\widehat{B}/(\varphi\nu_{\widehat{B}})$ for some positive automorphism φ of \widehat{B} . \square

3. SELF-INJECTIVE ALGEBRAS WITH DEFORMING IDEAL

Let A be a self-injective algebra, I a deforming ideal of A , $B = A/I$ and e a residual identity of B . Then I can be considered as a (eAe/eIe) -bimodule. Denote by $A[I]$ the direct sum of K -modules $(eAe/eIe) \oplus I$ with the multiplication

$$(b, x) \cdot (b', x') = (bb', bx' + xb' + xx')$$

for $b, b' \in eAe/eIe$ and $x, x' \in I$. Then $A[I]$ is an algebra with the identity $(e, 1 - e)$ and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider I as an ideal of $A[I]$. The following fact has been proved in [7, Theorem 4.1].

Proposition 3.1. *$A[I]$ is a self-injective algebra with a deforming ideal I , the Nakayama permutation of $A[I]$ is the same as the Nakayama permutation of A , and the algebras $A[I]$ and A are socle equivalent.*

Moreover, if K is a field, we have proved in [8, Theorem 3] that the algebras $A[I]$ and A are also stably equivalent. The aim of this section is to prove the following fact, needed in the proof of our main theorem.

Proposition 3.2. *Let A be a self-injective algebra with a deforming ideal I , $B = A/I$ and let e be a residual identity of B . Assume that $IeI = 0$ and $e_i \neq e_{\nu(i)}$ for any primitive summand e_i of e . Then the algebras $A[I]$ and A are isomorphic.*

Proof. We assume as before that $e = \sum_{i=1}^m e_i$ for $m \leq n$ and a set $\{e_i \mid 1 \leq i \leq n\}$ of primitive idempotents of A with $1_A = \sum_{i=1}^n e_i$. Denote by \bar{a} the residue class of $a \in eAe$ in eAe/eIe . Since by our assumption $\nu(i) \neq i$ for $i \in \{1, \dots, m\}$, invoking Proposition 2.5 we have an algebra monomorphism $\phi : eAe/eIe \rightarrow eAe$ such that $\phi(\bar{a}) = \sum_{i=1}^m e_i a e_B^{(i)}$. We define a homomorphism $\Phi : A[I] \rightarrow A$ of K -modules by

$$\Phi(\bar{a}, x) = \phi(\bar{a}) + x$$

for $\bar{a} \in eAe/eIe$ and $x \in I$. We claim that Φ is an algebra isomorphism. Consider the following commutative diagram of canonical short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & A[I] & \longrightarrow & eAe/eIe & \longrightarrow & 0 \\ & & & & \downarrow 1 & & \downarrow \Phi & & \downarrow \Phi' \\ 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0, \end{array}$$

where 1 is the identity map and Φ' is an isomorphism given by $\Phi'(\bar{a}) = \phi(\bar{a}) + I$, for $\bar{a} \in eAe/eIe$. In order to prove that Φ is an algebra isomorphism, it suffices to show that Φ is an algebra homomorphism. Observe that

$$\Phi(1_{A[I]}) = \Phi(\bar{e}, 1 - e) = \phi(\bar{e}) + (1 - e) = e + (1 - e) = 1 = 1_A.$$

Take $\bar{a}, \bar{b} \in eAe/eIe$ and $x, y \in I$. We shall prove that

$$\Phi((\bar{a}, x)(\bar{b}, y)) = \Phi(\bar{a}, x)\Phi(\bar{b}, y).$$

It follows from the definition of multiplication in $A[I]$ that

$$\Phi((\bar{a}, x)(\bar{b}, y)) = \Phi(\bar{a}\bar{b}, ay + xb + xy) = \phi(\bar{a}\bar{b}) + (ay + xb) + xy$$

and

$$\Phi(\bar{a}, x)\Phi(\bar{b}, y) = (\phi(\bar{a}) + x)(\phi(\bar{b}) + y) = \phi(\bar{a})\phi(\bar{b}) + (\phi(\bar{a})y + x\phi(\bar{b})) + xy.$$

Hence, it suffices to show that $\phi(\bar{a})y = ay$ and $x\phi(\bar{b}) = xb$.

(a) $\phi(\bar{a})y = ay$ for $a \in eAe$ and $y \in I$:

In fact, $\phi(\bar{a})y = \sum_{i=1}^m e_i a e_B^{(i)} y$, and

$$ay = \left(\sum_{i=1}^m e_i a \right) y = \sum_i e_i a e_B^{(i)} y + \sum_i e_i a e_I^{(i)} y + \sum_i e_i a e_{\nu(i)} y.$$

Here, $e_i a e_I^{(i)} y = 0$ and $e_i a e_{\nu(i)} y = 0$ by Lemma 2.2, because $e_i a e_{\nu(i)} \in \text{rad } A$ by $\nu(i) \neq i$, and therefore $\phi(\bar{a})y = ay$.

(b) $x\phi(\bar{b}) = xb$ for $b \in eAe$ and $x \in I$:

Since $e_i A e_I^{(i)} \subseteq e_i I$ by Lemma 2.1, we have $x e_i b e_I^{(i)} \in I e_i I \subseteq I e I$. Then $x e_i b e_I^{(i)} = 0$ because $I e I = 0$ by assumption. On the other hand, $e_i A e_{\nu(i)} = e_i (\text{rad } A) e_{\nu(i)}$ by $\nu(i) \neq i$, and hence $e_i b e_{\nu(i)} \in I$ by (1) and (2) of Lemma 2.1. Hence $x e_i b e_{\nu(i)} \in I e_i I \subseteq I e I = 0$, and $x e_i b e_{\nu(i)} = 0$. Thus,

$$\begin{aligned} xb &= \sum_{i=1}^m x e_i b e_B^{(i)} + \sum_i x e_i b e_I^{(i)} + \sum_i x e_i b e_{\nu(i)} \\ &= \sum_i x e_i b e_B^{(i)} = x \sum_i e_i b e_B^{(i)} \\ &= x\phi(\bar{b}). \end{aligned}$$

Therefore, the algebras $A[I]$ and A are isomorphic. □

4. PROOF OF THE THEOREM

We first prove the following general facts.

Lemma 4.1. *Let B be an algebra and φ a positive automorphism of \widehat{B} that fixes a finite subset of the object set of \widehat{B} . Then φ is a rigid automorphism of \widehat{B} .*

Proof. We write as before $\widehat{B} = \bigoplus_{k \in \mathbb{Z}} (B_k \oplus (DB)_k)$ with $B_k = B$ and $(DB)_k = DB$ for all $k \in \mathbb{Z}$, and identify the objects of \widehat{B} (respectively, B_k) with the vertices of the ordinary quivers $Q(\widehat{B})$ (respectively, $Q(B_k)$). Since φ is a positive automorphism of \widehat{B} , we may assume (without loss of generality) that φ fixes a subset of $Q(B_0)$. In order to prove that φ is rigid, it is enough to show that $\varphi(Q(B_0)) = Q(B_0)$. For this, since the quiver $Q(B_0)$ is connected, it suffices to show that $\varphi(\rightarrow S) = (\rightarrow S)$ and $\varphi(S \rightarrow) = (S \rightarrow)$ for any subset S of $Q(B_0)$ with $\varphi(S) = S$, where $(\rightarrow S)$ and $(S \rightarrow)$ denote the sets of all predecessors of the vertices of S and of all successors of the vertices of S in $Q(B_0)$, respectively. Therefore, let S be a subset of $Q(B_0)$ with $\varphi(S) = S$.

Now, let $x \rightarrow \cdots \rightarrow s$ be a path in $Q(B_0)$ with $s \in S$. Then $\varphi(x) \rightarrow \cdots \rightarrow \varphi(s)$ is a path in $Q(\widehat{B})$ and $\varphi(s) \in Q(B_0)$, because $\varphi(S) = S$ by assumption. Since φ is positive, $\varphi(x) \in \sum_{k \geq 0} Q(B_k)$ and it follows that $\varphi(x)$ belongs to $Q(B_0)$, because there is no path from $Q(B_k)$ to $Q(B_0)$ for any $k > 0$. Thus $\varphi(\rightarrow S) = (\rightarrow S)$.

Next, suppose that there is a path $s \rightarrow \cdots \rightarrow x$ in $\mathcal{Q}(B_0)$ for $s \in S$ with $\varphi(x) \notin \mathcal{Q}(B_0)$, that is, $\varphi(x) \in \mathcal{Q}(B_k)$ for some $k > 0$. Since S is finite and $\mathcal{Q}(\widehat{B})$ is locally finite, the φ -orbit of x is finite also. This implies that there is an integer ℓ such that $\varphi^\ell(x) \in \mathcal{Q}(B_{k'})$ for some $k' > 0$ and $\varphi^{\ell+1}(x) \in \mathcal{Q}(B_0)$. This, however, implies that $\varphi(\mathcal{Q}(B_{k'})) \cap \mathcal{Q}(B_0) \neq \emptyset$, which contradicts the positivity of φ . Thus we conclude that $\varphi(S \rightarrow) = (S \rightarrow)$. \square

Corollary 4.2. *Let B be an algebra and φ a strictly positive automorphism of \widehat{B} . Then φ acts freely on the objects of \widehat{B} .*

Proof of the Theorem. Let A be a self-injective algebra that is stably equivalent to a self-injective algebra of the form $\widehat{R}/(\psi\nu_{\widehat{R}})$, where R is a tilted algebra of type Δ (which is not a Dynkin quiver) and ψ is a strictly positive automorphism of \widehat{R} . It follows from the proof of Theorem 1 in [8] that there are an ideal I of A and a residual identity $e = \sum_{i=1}^m e_i$ of $B = A/I$ such that $IeI = 0$, Ie is an injective cogenerator in $\text{mod } B$, B is a tilted algebra of type Δ , and $A[I] \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$ for a positive automorphism φ of \widehat{B} . Moreover, by [8, Theorem 3], the algebras A and $A[I]$ are stably equivalent. Since by assumption A is stably equivalent to $\widehat{R}/(\psi\nu_{\widehat{R}})$, we conclude that $A[I]$ is stably equivalent to $\widehat{R}/(\psi\nu_{\widehat{R}})$. Furthermore, since ψ is a strictly positive automorphism of \widehat{R} , the number of simple modules of $\widehat{R}/(\psi\nu_{\widehat{R}})$ is greater than the number of simple modules of R , which is the same as the number of vertices of Δ . Applying now [8, Corollary] we conclude that the automorphism φ is not rigid, and so is strictly positive. Hence, invoking Corollary 4.2 and Proposition 3.1, we obtain that $\nu(i) \neq i$ for the Nakayama permutation ν of A and $i \in \{1, \dots, m\}$. Since B is a tilted algebra, the ordinary quiver $Q(B)$ of B has no oriented cycles, and then the properties of I stated above imply that I is a deforming ideal of A . Therefore, applying Proposition 3.2, we obtain that A and $A[I]$ are isomorphic, and so A and $\widehat{B}/(\varphi\nu_{\widehat{B}})$ are also isomorphic. This finishes the proof. \square

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