REFINING THE CONSTANT IN A MAXIMUM PRINCIPLE FOR THE BERGMAN SPACE

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Abstract. Let $A^2(D)$ be the Bergman space over the open unit disk $D$ in the complex plane. Korenblum conjectured that there is an absolute constant $c$, $0 < c < 1$, such that whenever $|f(z)| \leq |g(z)|$ ($f, g \in A^2(D)$) in the annulus $c < |z| < 1$, then $\|f\| \leq \|g\|$. In this note we give an example to show that $c < 0.69472$.

Let $D$ be the open unit disk in the complex plane $C$. The Bergman space $A^2(D)$ consists of analytic functions $f$ in $D$ such that

$$\|f\| = \left[ \int_D |f(z)|^2 dA(z) \right]^{\frac{1}{2}} < +\infty,$$

where

$$dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta, \quad z = x + iy = re^{i\theta}$$

is the normalized Lebesgue area measure on $D$. Korenblum [1] conjectured that there is an absolute constant $c$, $0 < c < 1$, such that whenever $|f(z)| \leq |g(z)|$ in the annulus $c < |z| < 1$ ($f, g \in A^2(D)$), then $\|f\| \leq \|g\|$.


On the other hand, the example of $f(z) = \frac{1}{\sqrt{2}}$, $g(z) = z$ shows that $c \leq \frac{1}{\sqrt{2}}$. However, Martin (see [1]) gave the following example to show that $c = \frac{1}{\sqrt{2}}$ is not sharp.

Example. Let

$$f(z) = \frac{1 + (\sqrt{2} - 1)z^{10}}{1 + (\sqrt{2} - 1)z^{-10}}, \quad g(z) = \sqrt{2}z.$$ 

Then $|f(z)| \leq |g(z)|$ for $\frac{1}{\sqrt{2}} < |z| < 1$ but $\|f\| > \|g\| = 1$.

In fact, an upper bound on $c$ can be found from Martin’s example. Namely, if $f$ and $g$ are as in Martin’s example, consider instead the pair $h$ and $g$, where $h = \frac{1}{\|f\|} f$. Then $\|h\| = \|g\| = 1$ and $|h(z)| \leq |g(z)|$ in an annulus $c' < |z| < 1$. Using Mathematica and Lemma 1 below, we can easily obtain that $c' = 0.70450 \cdots < \frac{1}{\sqrt{2}}$.

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Lemma 1 (see [4]). If \( f(z) = \sum_{k=0}^{+\infty} a_k z^k \in A^2(\mathbb{D}) \), then

\[
\|f\| = \left( \sum_{k=0}^{+\infty} \frac{|a_k|^2}{k+1} \right)^{1/2}.
\]

Before stating our example, we recall that the singular inner functions are defined as

\[ S_a(z) = \exp \left( \frac{1+az}{1-z} \right), \]

which play an important role in Bergman spaces [5], where \( a \) is any positive constant. Our main result is the following.

Theorem. Let

\[ f(z) = e^{-a} S_a(z^n) = e^{-\frac{2a}{1+c^n}}, \]
\[ g(z) = e^{-\frac{2a}{1+c^n}} z, \]

where \( 0 < c < 1 \), \( a = -\frac{1+c^n}{c^n} \log c > 0, \quad n \in \mathbb{N} \). Then \( |f(z)| \leq |g(z)| \) in \( c < |z| < 1 \). Moreover, when \( n = 14 \) and \( c = 0.69472 \), we have \( \|f\| > \|g\| \).

Proof. It is easy to see that

\[ \varphi(r) = \max_{|z|=r} \left| \frac{f(z)}{g(z)} \right| = \max_{|z|=r} \left| \frac{|f(z)|}{e^{-\frac{2a}{1+c^n}} r} \right| = e^{-\frac{2a}{1+c^n}}. \]

Hence, we have

\[ \varphi(c) = 1, \quad \varphi(1) = \lim_{r \to 1^-} \varphi(r) = 1. \]

Since \( \frac{f(z)}{g(z)} \) is analytic in \( c \leq |z| < 1 \), the maximum modulus theorem implies that \( |f(z)| \leq |g(z)| \) in \( c < |z| < 1 \).

A direct calculation shows that the Taylor expansion of \( f(z) \) at 0 is

\[ f(z) = e^{-2a} \left[ 1 - 2az^n + 2(a^2 - a)z^{2n} - \frac{4a^3 - 12a^2 + 6a}{3} z^{3n} + \cdots \right]. \]

It follows from Lemma 1 that

\[
\int_{\mathbb{D}} |f(z)|^2 dA(z) - \int_{\mathbb{D}} |g(z)|^2 dA(z) > e^{-4a} \left[ 1 + \frac{4a^2}{n+1} + \frac{4(a^2 - a)^2}{2n+1} + \frac{(4a^3 - 12a^2 + 6a)^2}{9(3n+1)} - \frac{e^{2a}}{2} \right] \triangleq I(a).
\]

Using Mathematica, we obtain that when \( n = 14 \) and \( c = 0.69472 \),
\[ e^{4a} I(a) = 0.0000214904 > 0. \]

So we have \( \|f\| > \|g\| \). \( \square \)

Remark. It is likely that for all functions \( f(z) \) and \( g(z) \) (which depend on \( n \) and \( a > 0 \)) defined in the theorem, \( c = 0.6947116 \cdots \) is the best one.
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