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A NEGATIVE ANSWER TO NEVANLINNA'S TYPE QUESTION AND A PARABOLIC SURFACE WITH A LOT OF NEGATIVE CURVATURE

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In memory of Bob Brooks

Abstract. Consider a simply-connected Riemann surface represented by a Speiser graph. Nevanlinna asked if the type of the surface is determined by the mean excess of the graph: whether mean excess zero implies that the surface is parabolic, and negative mean excess implies that the surface is hyperbolic. Teichmüller gave an example of a hyperbolic simply-connected Riemann surface whose mean excess is zero, disproving the first of these implications. We give an example of a simply-connected parabolic Riemann surface with negative mean excess, thus disproving the other part. We also construct an example of a complete, simply-connected, parabolic surface with nowhere positive curvature such that the integral of curvature in any disk about a fixed basepoint is less than $-\epsilon$ times the area of disk, where $\epsilon > 0$ is some constant.

1. Introduction

The uniformization theorem states that for every simply-connected Riemann surface X there exists a conformal map $\varphi : X_0 \to X$, where X_0 is either the complex plane $\mathbb{C}$, the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, or the extended complex plane (Riemann sphere) $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and these possibilities are mutually exclusive [1]. The map φ is called the *uniformizing map*. A simply-connected Riemann surface X is said to have *hyperbolic*, *parabolic*, or *elliptic type*, according to whether it is conformally equivalent to $\mathbb{D}$, $\mathbb{C}$, or $\overline{\mathbb{C}}$, respectively. In what follows, we assume that X is not compact, thus excluding the elliptic case from consideration.

We are interested in the application of the Uniformization Theorem to the following construction. A *surface spread over the sphere* is a pair (X, ψ), where X is a topological surface and $\psi : X \to \overline{\mathbb{C}}$ a continuous, open and discrete map. The map ψ is called a *projection*. Two such surfaces (X_1, ψ_1), (X_2, ψ_2) are *equivalent*, if there exists a homeomorphism $\phi : X_1 \to X_2$, such that $\psi_1 = \psi_2 \circ \phi$. According to a theorem of Stoïlow [10], a continuous open and discrete map ψ near each point z_0 is homeomorphically equivalent to a map $z \mapsto z^k$. The number $k = k(z_0)$ is called the *local degree* of ψ at z_0. If $k \neq 1$, z_0 is called a *critical point* and $\psi(z_0)$ a *critical value*. The set of critical points is a discrete subset of X. The theorem

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of Stoïlow implies that there exists a unique conformal structure on X that makes ψ into a meromorphic function. If X is simply-connected, what is the type of the Riemann surface so obtained? This is one version of the type problem. Equivalent surfaces have the same type.

Rolf Nevanlinna's problem concerns a particular class of surfaces spread over the sphere, denoted by F_q. Let $\{a_1, \dots, a_q\}$ be distinct points in $\overline{\mathbb{C}}$.

Definition. A surface (X, ψ) belongs to the class $F_q = F(a_1, \dots, a_q)$ if ψ restricted to the complement of $\psi^{-1}\big(\{a_1, \dots, a_q\}\big)$ is a covering map onto its image $\overline{\mathbb{C}} \setminus \{a_1, \dots, a_q\}$.

Assume that $(X, \psi) \in F_q$ and X is noncompact. We fix a Jordan curve L, visiting the points $a_1, \dots, a_q$ in cyclic order. The curve L is usually called a *base curve*. It decomposes the sphere into two simply-connected regions H_1, the region to the left of L, and H_2, the region to the right of L. Let L_i, $i = 1, 2, \dots, q$, be the arc of L from a_i to a_{i+1} (with indices taken modulo q). Let us fix points p_1 in H_1 and p_2 in H_2, and choose q Jordan arcs $\gamma_1, \dots, \gamma_q$ in $\overline{\mathbb{C}}$, such that each arc γ_i has p_1 and p_2 as its endpoints, and has a unique point of intersection with L, which is in L_i. We take these arcs to be interiorwise disjoint, that is, $\gamma_i \cap \gamma_j = \{p_1, p_2\}$ when $i \neq j$. Let Γ' denote the graph embedded in $\overline{\mathbb{C}}$, whose vertices are $p_1,\ p_2$, and whose edges are $\gamma_i,\ i = 1, \dots, q$, and let $\Gamma = \psi^{-1}(\Gamma')$. We identify Γ with its image in $\mathbb{R}^2$ under an orientation-preserving homeomorphism of X onto $\mathbb{R}^2$. The graph Γ has the following properties:

(1) Γ is infinite, connected;
(2) Γ is homogeneous of degree q;
(3) Γ is bipartite.

A graph, properly embedded in the plane and satisfying these properties is called a *Speiser graph*, also known as a *line complex*. The vertices of a Speiser graph Γ are traditionally marked by $\times$ and $\circ$, such that each edge of Γ connects a vertex marked $\times$ with a vertex marked $\circ$. Such a marking exists, since Γ is bipartite. Each *face* of Γ, i.e., connected component of $\mathbb{R}^2 \setminus \Gamma$, has either a finite even number of edges along its boundary, in which case it is called an *algebraic elementary region*, or infinitely many edges, in which case it is called a *logarithmic elementary region*. Two Speiser graphs $\Gamma_1,\ \Gamma_2$ are said to be *equivalent*, if there is a sense-preserving homeomorphism of the plane that takes Γ_1 to Γ_2.

The above construction is reversible. Suppose that the faces of a Speiser graph Γ are labelled by $a_1, \dots, a_q$, so that when going counterclockwise around a vertex $\times$, the indices are encountered in their cyclic order, and around $\circ$ in the reversed cyclic order. We fix a simple closed curve $L \subset \overline{\mathbb{C}}$ passing through $a_1, \dots, a_q$. Let $H_1, H_2, L_1, \dots, L_q$ be as before. Let Γ^* be the planar dual of Γ. If e is an edge of Γ^* from a face of Γ marked a_j to a face of Γ marked a_{j+1}, let ψ map e homeomorphically onto the corresponding arc L_j of L. This defines ψ on the edges and vertices of Γ^*. We then extend ψ to the faces of Γ^* in the obvious way. This defines a surface spread over the sphere $(\mathbb{R}^2, \psi) \in F(a_1, \dots, a_q)$. See [6] for further details.

For a Speiser graph Γ, Nevanlinna introduces the following characteristics. Let $V(\Gamma)$ denote the set of vertices of the graph Γ. To each vertex $v \in V(\Gamma)$ we assign the number

$$E(v) = 2 - \sum_{f \in F(v)} (1 - 1/k_f),$$

where $F(v)$ denotes the set of faces containing v on their boundary and $2\,k_f$ is the number of edges on the boundary of f, $k_f \in \{1, 2, \ldots, \infty\}$. The function $E: V(\Gamma) \to \mathbb{R}$ is called the *excess*. This definition is motivated by considering the curvature, as follows. The ψ-pullback of the spherical metric $2|dw|/(1+|w|^2)$ is generally singular, i.e., it may degenerate on $\psi^{-1}(\{a_1, \ldots, a_q\})$. The surface X, endowed with the pullback metric, is a spherical polyhedral surface, which is a particular kind of orbifold. The *integral curvature* ω on X is a signed Borel measure, so that for each Borel subset $B \subset X$, $\omega(B)$ is the area of B with respect to the pullback metric minus $2\pi \sum_z (k_z - 1)$, where the sum is over all critical points $z \in B$ and k_z is the local degree of ψ at z.

Each vertex of Γ represents a hemisphere, and each face of Γ with $2k$ edges represents a critical point, where k is the local degree of f at this point. Therefore, each vertex of Γ has positive integral curvature 2π, and each face with $2k$ edges has negative integral curvature $-2\pi(k-1)$. We assign the negative curvature evenly to all the vertices of the face. A face with infinitely many edges contributes -2π to each vertex on its boundary. The curvature assigned to every $v \in V(\Gamma)$ is exactly $2\pi E(v)$.

Nevanlinna also defines the mean excess of a Speiser graph Γ. We fix a base vertex $v_0 \in V(\Gamma)$, and consider an exhaustion of Γ by a sequence of finite graphs $\Gamma_{(i)}$, where $\Gamma_{(i)}$ is the ball of combinatorial radius i, centered at v. By averaging E over all the vertices of $\Gamma_{(i)}$, and taking the limit, we obtain the *mean excess*, denoted $E_m = E_m(\Gamma)$, provided that the limit exists. If the limit does not exist, we consider the *upper mean excess* $\overline{E}_m$ and *lower mean excess* $\underline{E}_m$, which are the upper ($\limsup$) and lower ($\liminf$) limits, respectively.

Nevanlinna's Problem [5]**.** Does $\underline{E}_m \geq 0$ imply that the surface X with the pullback complex structure is parabolic? Conversely, does $\overline{E}_m < 0$ imply that it is hyperbolic?

Teichmüller [11] constructed an example of a surface with hyperbolic type, for which the mean excess is zero, thus giving a negative answer to the first question.

We will shortly prove that the answer to the other question is negative as well, by constructing a parabolic surface $(\mathbb{R}^2, \psi) \in F_3$ with $E_m < 0$.

In Section 3, we shall construct an example of a nonpositively curved, simply-connected, complete, parabolic surface, whose curvature in any ball about a fixed basepoint is less than a negative constant times the area of the ball.

2. Counterexample

P. Doyle [3] proved that the surface (X, ψ) is parabolic if and only if a certain modification of the Speiser graph is recurrent. (See [4] and [9] for background on recurrence and transience of infinite graphs.) In the particular case where k_f is bounded, the recurrence of the Speiser graph itself is equivalent to (X, ψ) being parabolic. Though we will not really need this fact, it is not too hard to see that in a Speiser graph satisfying $E_m < 0$ the number of vertices in a ball grows exponentially with the radius. Thus, we may begin searching for a counterexample by considering recurrent graphs with exponential growth. A very simple standard example of this sort is a tree constructed as follows. In an infinite 3-regular tree T_3, let $v_0, v_1, \ldots$ be an infinite simple path. Let T be the set of vertices u in T_3 such that $d(u, v_n) \leq n$ for all sufficiently large n. Note that there is a unique infinite

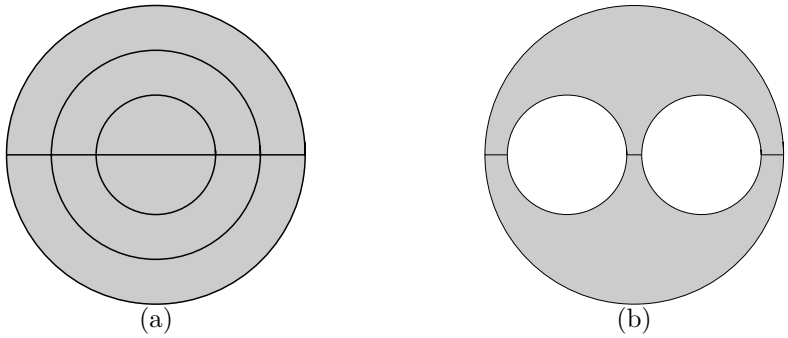

FIGURE 1. The surfaces $S(v)$. (a) shows $S(v)$ for a leaf v, where $s = 2$. (b) shows $S(v)$ for a degree 3 vertex.

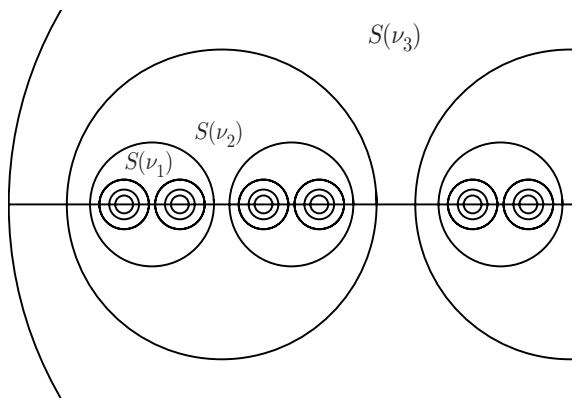

FIGURE 2. The Speiser graph with $s = 2$. The disk $S(v_0)$ is the left "eye" in $S(v_1)$.

simple path in T starting from any vertex u. This implies that T is recurrent. It is straightforward to check that the number of vertices of T in the ball $B(v_0, r)$ grows exponentially with r.

Our Speiser graph counterexample is a simple construction based on the tree T. Fix a parameter $s \in \{1, 2, \dots\}$, whose choice will be discussed later. To every leaf (degree-one vertex) v of T associate a closed disk $S(v)$ and on it draw the graph indicated in Figure 1(a), where the number of concentric circles, excluding $\partial S(v)$, is s. If v is not a leaf, then it has degree 3. We then associate to it the graph indicated in Figure 1(b), drawn on a triply connected domain $S(v)$. We combine these to form the Speiser graph Γ as indicated Figure 2 by pasting the outer boundary of the surface corresponding to each vertex into the appropriate inner boundary component of its parent. Here, the parent of v is the vertex v' such that $d(v', v_n) = d(v, v_n) - 1$ for all sufficiently large n.

Every vertex of Γ has degree 4 and every face has 2, 4 or 6 edges on its boundary. Therefore, Γ is a Speiser graph. Consequently, as discussed above, there is a surface spread over the sphere $X = (\mathbb{R}^2, \psi)$ whose Speiser graph is Γ. It is immediate to verify that Γ is recurrent, for example, by the Nash-Williams criterion. Doyle's Theorem [3] then implies that X is parabolic. Alternatively, one can arrive at the

same conclusion by noting that there is an infinite sequence of disjoint isomorphic annuli on $(\mathbb{R}^2, \Gamma)$ separating any fixed point from ∞, and applying extremal length. (See [1], [8] for the basic properties of extremal length.)

We now show that $\overline{E}_m < 0$ for Γ. Note that the excess is positive only on vertices on the boundary of 2-gons, which arise from leaves in T. On the other hand, every vertex of degree 3 in T gives rise to vertices in Γ with negative excess. Take as a basepoint for Γ a vertex $w_0 \in S(v_1)$ with negative excess, say. It is easy to see that there are constants, $a > 1, c > 0$ such that the number n_r^- of negative excess vertices in the combinatorial ball $B(w_0, r)$ about w_0 satisfies $c\,a^r \le n_r^- \le a^r/c$.

If w is a vertex with positive excess, then there is a unique vertex $\sigma(w)$ with negative excess closest to v; in fact, if $w \in S(v)$, then $\sigma(w)$ is the closest vertex to w on $\partial S(v)$, and the (combinatorial) distance from w to $\sigma(w)$ is our parameter s. The map $w \mapsto \sigma(w)$ is clearly injective. This implies that the number n_r^+ of positive excess vertices in $B(w_0, r)$ satisfies $n_{r+s}^+ \le n_r^-$, $r \in \{0, 1, 2, \dots\}$. By choosing s sufficiently large, we may therefore arrange to have the total excess in $B(w_0, r)$ to be less than $-\epsilon\, a^r$, for some $\epsilon > 0$ and every $r \in \{0, 1, 2, \dots\}$. It is clear that the number of vertices with zero excess in $B(w_0, r)$ is bounded by a constant (which may depend on s) times n_r^-. Hence, $\overline{E}_m < 0$ for Γ.

By allowing s to depend on the vertex in T, if necessary, we may arrange to have $\underline{E}_m = \overline{E}_m$; that is, E_m exists, while maintaining $E_m < 0$. We have thus demonstrated that the resulting surface is a counterexample in F_4 to the second implication in Nevanlinna's problem.

3. A nonpositive curvature example

We now construct an example of a simply-connected, complete, parabolic surface Y of nowhere positive curvature, with the property

$$\int_{D(a,r)} \text{curvature} < -\epsilon \text{ area}\big(D(a,r)\big), \tag{1}$$

for some fixed $a \in Y$ and every $r > 0$, where $D(a, r)$ denotes the open disc centered at a of radius r, and $\epsilon > 0$ is some fixed constant.

Consider the surface $\mathbb{C} = \mathbb{R}^2$ with the metric $|dz|/y$ in $P = \{z = x + iy : \ y \ge 1\}$, and $\exp(1 - y)|dz|$ in $Q = \{y < 1\}$. We denote this surface by Y. Let β denote the curve $\{y = 1\}$ in Y, i.e., the common boundary of P and Q.

Let Q' denote the universal cover of $\{z \in \mathbb{C} : |z| > 1\}$. Note that Q is isometric to Q' via the map $z \mapsto \exp(iz + 1)$. Hence, the curvature is zero on Q, and the geodesic curvature of ∂Q is -1. The geodesic curvature of ∂P is 1. Consequently, Y has no concentrated curvature on β. The surface Y is thus a "surface of bounded curvature", also known as an Aleksandrov surface (see [2], [7]). The curvature measure of Y is absolutely continuous with respect to area; the curvature of Y is -1 (times area measure) on P and 0 on Q.

The surface Y is parabolic, and the uniformizing map is the identity map onto $\mathbb{R}^2$ with the standard metric.

We will now prove (1) with $a = i$. Set $\beta_r = D(a, r) \cap \beta$. Note that the shortest path in Y between any two points on β is contained in P, and is the arc of a circle orthogonal to $\{y = 0\}$. Using the Poincaré disc model, it is easy to see that there exists a constant $c > 0$, such that

$$c\, e^{r/2} \le \text{length}\, \beta_r \le e^{r/2}/c, \tag{2}$$

where the right inequality holds for all r, and the left for all sufficiently large r. By considering the intersection of $D(a,r)$ with the strip $1 < y < 2$ it is clear that

$$O(1)\,\mathrm{area}\big(P \cap D(a,r)\big) \geq \mathrm{length}\,\beta_r, \tag{3}$$

for all sufficiently large r.

Consider some point $p \in Q$, and let p' be the point on β closest to p. It follows easily (for example, by using the isometry of Q and Q') that if q is any point in β, then $d_Q(p,q) = d_Q(p,p') + d_Q(p',q) + O(1)$. Consequently, if $d(p,a) \leq r$, then there is an $s \in [0,r]$ such that $p' \in \beta_s$ and $d_Q(p,p') \leq r - s + O(1)$. Furthermore, it is clear that the set of points p in Q such that $p' \in \beta_s$ and $d_Q(p,p') \leq t$ has area $O(t^2+t) \times \mathrm{length}\,\beta_s$. Consequently,

$$\mathrm{area}\big(Q \cap D(a,r)\big) \leq O(1) \sum_{j=0}^{r} (j+1)^2\,\mathrm{length}\,\beta_{r-j}\,.$$

Using (2), we have

$$\mathrm{area}\big(Q \cap D(a,r)\big) \leq O(1)\ \mathrm{length}\,\beta_r, \tag{4}$$

for all sufficiently large r.

Now, combining (3) and (4), we obtain (1) for all sufficiently large r. It therefore holds for all r.

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