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ON RAKHMANOV'S THEOREM FOR JACOBI MATRICES

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ABSTRACT. We prove Rakhmanov's theorem for Jacobi matrices without the additional assumption that the number of bound states is finite. This result solves one of Nevai's open problems.

Consider a measure $d\mu$ with bounded support in $\mathbb{R}$. Assume that it has infinitely many growth points. Let $\{p_k\}$ $(k = 0, 1, \ldots)$ be the system of polynomials orthonormal with respect to that measure, i.e.,

$$\int_{-\infty}^{\infty} p_k(x)p_m(x)d\mu(x) = \delta_{k,m}, \quad \deg p_l(x) = l.$$

It is known [1] that the following recurrence relations hold:

$$\begin{aligned} &b_{-1}p_0(x) = 1, \quad v_0p_0(x) + b_0p_1(x) = xp_0(x), \\ &b_{k-1}p_{k-1}(x) + v_kp_k(x) + b_kp_{k+1}(x) = xp_k(x), \quad k = 1, 2, \ldots, \end{aligned} \tag{1}$$

with $b_k > 0, k = -1, 0, \ldots$, and v_k real. Thus, for any fixed x, p_k are generalized eigenvectors of the following Jacobi matrix [1]:

$$J = \begin{bmatrix} v_0 & b_0 & 0 & 0 & \ldots \\ b_0 & v_1 & b_1 & 0 & \ldots \\ 0 & b_1 & v_2 & b_2 & \ldots \\ 0 & 0 & b_2 & v_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}. \tag{2}$$

The corresponding operator J is bounded and self-adjoint in the Hilbert space $\ell^2(\mathbb{Z}^+)$. And, vice-versa, we can associate to any real Jacobi matrix with positive off-diagonal entries a certain measure on the line (spectral measure). In a general situation, a limit circle case might occur such that this measure is non-unique. But in the limit point case, this measure is uniquely defined. If the corresponding self-adjoint operator J is bounded, then the measure has bounded support. In this paper, we consider only that case. We introduce the following notation: $\sigma(A)$ denotes the spectrum of the self-adjoint bounded operator A, $\sigma_{ess}(A)$ its essential spectrum, and $\|A\|$ the norm of the operator. The following theorem was proved by Nevai.

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Theorem 1 (Nevai [7], [6]). *If* $\sigma_{ess}(J)=[-1,1]$, *where* J *has a finite number of eigenvalues outside the segment* $[-1,1]$ *and, for the corresponding spectral measure* μ, $\mu'(x)>0$ *for a.e.* $x\in[-1,1]$, *then* $v_k\to 0$, $b_k\to 1/2$ *as* $k\to\infty$.

The following open question was posed in [6] (Conjecture 2.7):

Is the statement of the theorem true without the second assumption on J, *i.e., for the case when one has an infinite number of eigenvalues* (*bound states*) *outside the segment* $[-1,1]$*?*

The main result of our short note is the affirmative answer to this question. To prove that, we will need one auxiliary lemma.

Lemma 1. *Consider a bounded self-adjoint operator* A *in the Hilbert space, such that* $\sigma_{ess}(A)=[a,b]$. *Let* $\{e_j\}, j=1,2,\ldots$, *be the orthonormal basis, and* A_n *the self-adjoint bounded operator, generated by the quadratic form* (Af,f), *where* f *is orthogonal to* $\{e_1,\ldots,e_n\}$. *Then* $\inf\sigma(A_n)\to a$, *and* $\sup\sigma(A_n)\to b$ *as* $n\to\infty$.

Proof. We will only show that $\inf\sigma(A_n)\to a$ as $n\to\infty$. The other statement can be proved similarly. Let λ_j,ψ_j $(j=1,2,\ldots)$ denote eigenvalues and the corresponding normed eigenfunctions, such that $\lambda_1\le\lambda_2\le\cdots<a$, $\|\psi_j\|=1$. Consider any vector f from the Hilbert space. By the spectral theorem, $f=\sum_{j=1}^{\infty}(f,\psi_j)\psi_j+f_c$ and

$$\begin{aligned}(Af,f)&\ge a(f_c,f_c)+\sum_{j=1}^{\infty}\lambda_j|(f,\psi_j)|^2=a\left[(f_c,f_c)+\sum_{j=1}^{\infty}|(f,\psi_j)|^2\right]\\&\quad+\sum_{j=1}^{\infty}(\lambda_j-a)|(f,\psi_j)|^2=a\|f\|^2+\sum_{j=1}^{\infty}(\lambda_j-a)|(f,\psi_j)|^2.\end{aligned}\tag{3}$$

Notice that for fixed j, $(f,\psi_j)\to 0$ uniformly in $f\in\{\|f\|\le 1$ and f is orthogonal to $\{e_1,\ldots,e_n\}\}$ as $n\to\infty$. Indeed,

$$(f,\psi_j)=\sum_{k=1}^{\infty}(f,e_k)\overline{(\psi_j,e_k)}=\sum_{k=n+1}^{\infty}(f,e_k)\overline{(\psi_j,e_k)}.$$

Consequently, by the Cauchy inequality,

$$|(f,\psi_j)|\le\|f\|\left(\sum_{k=n+1}^{\infty}|(\psi_j,e_k)|^2\right)^{1/2}\to 0$$

as $n\to\infty$. By the variational principle,

$$\inf\sigma(A_n)=\inf_{\|f\|=1,(f,e_j)=0,j=1,\ldots,n}(Af,f).$$

By the Courant-Hilbert variational principle [11], $\inf\sigma(A_n)\le\lambda_{n+1}$. We know that $\lambda_j\to a$ as $j\to\infty$. Consequently, from (3), we have $\inf\sigma(A_n)\to a$ as $n\to\infty$. □

Let ψ $(x\in[-1,1],\psi(-1)=0)$ be a non-decreasing bounded function with an infinite number of growth points. Consider the system of polynomials p_k $(k=0,1,\ldots)$ orthonormal with respect to the measure $d\psi$ on the segment $[-1,1]$. Introduce the function

$$\sigma(\theta)=\begin{cases}-\psi(\cos\theta), & 0\le\theta\le\pi,\\ \psi(\cos\theta), & \pi\le\theta\le 2\pi,\end{cases}\tag{4}$$

which is bounded and non-decreasing on $[0, 2\pi]$. Consider the polynomials $\phi_k(z) = \alpha_k z^k + \cdots$ $(\alpha_k > 0)$ orthonormal on the circle with respect to the measure $d\sigma$, i.e.,

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \phi_k(z)\overline{\phi_m(z)}\, d\sigma(\theta) = \delta_{k,m}, \quad z = e^{i\theta}.$$

ϕ_n are related to p_k by the formula

$$p_k(x) = \frac{\phi_{2k}(z) + \phi^*_{2k}(z)}{\sqrt{2\pi\left[1 + \alpha_{2k}^{-1}\phi_{2k}(0)\right]}}\, z^{-k}, \quad k = 0, 1, \ldots, \tag{5}$$

where $x = (z + z^{-1})/2$ and $\phi^*_k(z) = z^k\overline{\phi_k(\bar{z}^{-1})}$ [12], [3]. Calculations yield the following identities [3], [4]:

$$b_k = \tfrac{1}{2}\sqrt{(1 - a_{2k-1})(1 - a_{2k}^2)(1 + a_{2k+1})}, \tag{6}$$

$$v_k = -\tfrac{1}{2}\left[a_{2k-2}(1 + a_{2k-1}) - a_{2k}(1 - a_{2k-1})\right], \quad k \geq 1, \tag{7}$$

where $a_k = -\alpha_{k+1}^{-1}\phi_{k+1}(0)$ are the so-called reflection coefficients for $d\sigma$ (parameters of the system [3]). In our particular case, a_k are real numbers. Notice that multiplication of the function ψ by the positive constant leads to the multiplication of $p_n(x), \sigma(\theta), \phi_n(z)$ by certain constants, but it does not change b_n, v_n, a_n, i.e., parameters of the orthogonal systems.

The following remarkable theorem was first proved by Rakhmanov [10]. Later, various proofs were given by other authors.

Theorem 2 (Rakhmanov [10], [5])**.** *If* $\sigma'(\theta) > 0$ *for a.e.* $\theta \in [0, 2\pi]$*, then* $a_k \to 0$ *as* $k \to \infty$*.*

We will need the following generalization of this result.

Theorem 3. *If* $\mathbb{K} \equiv \{\theta \in [0, 2\pi) : \sigma'(\theta) > 0\}$*, then*

$$\limsup_{n\to\infty} |a_n|^2 \leq 8\left(1 - \left(\frac{|\mathbb{K}|}{2\pi}\right)^{3/2}\right), \tag{8}$$

where $|\mathbb{K}|$ *means the Lebesgue measure of a set* $\mathbb{K}$*.*

The proof can be obtained by a slight modification of the arguments given in [5]. We will also need the following well-known result (see, e.g., [5]).

Lemma 2 ([5])**.** *If* ν *is a measure on* $\mathbb{R}$ *singular with respect to Lebesgue measure, then there exists a sequence of continuous functions* $\{h_n\}$ *such that* $0 < h_n(x) \leq 1$ *for all* $x \in \mathbb{R}$*,*

$$\lim_{n\to\infty} h_n(x) = 1$$

for a.e. x*, and*

$$\lim_{n\to\infty}\int\limits_{-\infty}^{\infty} h_n(x)d\nu(x) = 0.$$

Proof of Theorem 3. From Theorem 3 of [8],

$$|a_n| \leq \frac{1}{2\pi}\int\limits_0^{2\pi}\left|\frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2} - 1\right| d\theta, \quad z = e^{i\theta}.$$

The Cauchy inequality yields

$$\left[\int_0^{2\pi}\left|\frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2}-1\right|d\theta\right]^2 \le \int_0^{2\pi}\left(\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|}+1\right)^2 d\theta \int_0^{2\pi}\left(\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|}-1\right)^2 d\theta.$$

Due to the well-known identity [3]

$$\int_0^{2\pi}\frac{|\phi_n(z)|^2}{|\phi_{n+1}(z)|^2}d\theta=\int_0^{2\pi}|\phi_n(z)|^2 d\sigma(\theta)=2\pi,\ z=e^{i\theta},$$

we have an estimate

$$|a_n|^2 \le 8\left[1-\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|}d\theta\right]. \tag{9}$$

We show that

$$\left(\frac{|\mathbb{K}|}{2\pi}\right)^{3/2} \le \liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|}d\theta. \tag{10}$$

Then, the statement of the theorem will be immediate due to (9). Estimate (10), in turn, can be proved by the argument used in Theorem 3 of [5]. One needs to start with a fixed non-negative, continuous, 2π periodic function $f(x)$. Then, Hölder's inequality yields

$$\begin{aligned}&\left[\frac{1}{2\pi}\int_0^{2\pi}(\sigma'(\theta)f(\theta))^{1/4}d\theta\right]^4\\ &\le\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|}d\theta\right]^2\left[\frac{1}{2\pi}\int_0^{2\pi}|\phi_{n+1}(z)|^2\sigma'(\theta)d\theta\right]\left[\frac{1}{2\pi}\int_0^{2\pi}f(\theta)|\phi_n(z)|^{-2}d\theta\right].\end{aligned}$$

The second factor is not greater than 1; measure $|\phi_n(\theta)|^{-2}d\theta$ converges weakly to $d\sigma$ [3]. Therefore,

$$\begin{aligned}&\left[\frac{1}{2\pi}\int_0^{2\pi}(\sigma'(\theta)f(\theta))^{1/4}d\theta\right]^4\\ &\le\left[\liminf_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{|\phi_n(z)|}{|\phi_{n+1}(z)|}d\theta\right]^2\left[\frac{1}{2\pi}\int_0^{2\pi}f(\theta)d\sigma\right].\end{aligned} \tag{11}$$

Assume that σ has a singular component σ_s. Then, by Lemma 2, we can choose the sequence of 2π periodic functions $\{h_k\}$ with the properties $0<h_k\le 1$, $h_k(\theta)\to 1$ a.e.,

$$\int_0^{2\pi}h_k(\theta)d\sigma_s(\theta)\to 0$$

as $k\to\infty$. If σ is purely a.c., let $h_k(\theta)\equiv 1$ for all k. Fix any $\varepsilon>0$, then we can find the sequence of continuous, non-negative, 2π periodic functions $g_l(\theta)$ such that $g_l \le \varepsilon^{-1}$, $g_l \to (\sigma' + \varepsilon)^{-1}$ a.e. For fixed $\varepsilon > 0$, choose f in (11) equal to $h_k g_l$. Taking $k \to \infty$, then $l \to \infty$, and, finally, $\varepsilon \to 0$, we get estimate (10). ☐

Remark. That is not clear even for small values of $(2\pi - |\mathbb{K}|)$ whether the estimate (8) is sharp or not.

Consider a bounded operator in $\ell^2(\mathbb{Z}^+)$ given by a Jacobi matrix

$$J = \begin{bmatrix} v_0 & b_0 & 0 & 0 & \dots \\ b_0 & v_1 & b_1 & 0 & \dots \\ 0 & b_1 & v_2 & b_2 & \dots \\ 0 & 0 & b_2 & v_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{12}$$

Theorem 4. *Assume that $\sigma_{ess}(J) = [-1,1]$, and for the corresponding spectral measure $\mu(x)$, we have $\mu'(x) > 0$ for a.e. $x \in (-1,1)$. Then, $v_k \to 0$ and $b_k \to 1/2$ as $k \to \infty$.*

Proof. Consider the sequence of matrices

$$J_n = \begin{bmatrix} v_n & b_n & 0 & 0 & \dots \\ b_n & v_{n+1} & b_{n+1} & 0 & \dots \\ 0 & b_{n+1} & v_{n+2} & b_{n+2} & \dots \\ 0 & 0 & b_{n+2} & v_{n+3} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{13}$$

These are matrices of the operators generated by the quadratic forms (Jx, x), restricted to $x \in \{(x, e_j) = 0, j = 0, 1, \dots, n-1\}$, where $e_j = (0, \dots, 0, 1, 0, \dots)$ is the standard basis in $\ell^2(\mathbb{Z}^+)$. Matrix J can be represented as the rank-two perturbation of the following orthogonal sum $J_n \oplus Q_n$, where Q_n is the finite Jacobi matrix

$$Q_n = \begin{bmatrix} v_0 & b_0 & 0 & \dots & \dots \\ b_0 & v_1 & b_1 & \dots & \dots \\ 0 & b_1 & v_2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{n-2} & v_{n-1} \end{bmatrix}. \tag{14}$$

From Weyl's theorem, we infer that $\sigma_{ess}(J_n) = [-1,1]$. The support of the a.c. component of the spectral measure is invariant under finite-rank perturbations. So, it is equal to $[-1,1]$ for any J_n. Due to Lemma 1, we have $\inf \sigma(J_n) \to -1$, $\sup \sigma(J_n) \to 1$ as $n \to \infty$. That means $\|J_n\| \to 1$. Consider the matrix (operator) $\hat{J}_n = \|J_n\|^{-1} J_n$. Now, the spectral measure $d\psi^{(n)}$ of this operator is supported on the interval $[-1,1]$. Therefore, we can consider the corresponding system of polynomials orthogonal on the unit circle with measure $d\sigma^{(n)}$, given by (4), and the sequence of reflection parameters $\{a_k^{(n)}\}$. The complement in $[0, 2\pi]$ to the support of the a.c. part of the measure $d\sigma^{(n)}$ is the union of three intervals: $(0, \kappa_1^{(n)})$, $(\pi - \kappa_2^{(n)}, \pi + \kappa_2^{(n)})$ and $(2\pi - \kappa_1^{(n)}, 2\pi)$, where $\kappa_{1(2)}^{(n)} \to 0$ as $n \to \infty$. Therefore, Theorem 3 yields $\limsup_{k\to\infty} |a_k^{(n)}| \to 0$ as $n \to \infty$. Bearing in mind formulas (6), (7), and identity $\lim_{n\to\infty} \|J_n\| = 1$, we have

$$\limsup_{k\to\infty} |v_k| = 0 \quad \text{and} \quad \limsup_{k\to\infty} |b_k - 1/2| = 0.$$

☐

The idea of a truncation process was suggested by Yuditskii. Meanwhile, this approach is standard and was used before (see, e.g., [9]). The important analogs of Theorem 4 can be obtained for "continuous orthogonal systems" such as Krein systems, Dirac operators, one-dimensional Schrödinger operators. This is done in a separate publication [2].

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