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# CONTINUOUS-TRACE GROUPOID CROSSED PRODUCTS

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ABSTRACT. Let $G$ be a second countable, locally compact groupoid with Haar system, and let $\mathcal{A}$ be a bundle of $C^*$-algebras defined over the unit space of $G$ on which $G$ acts continuously. We determine conditions under which the associated crossed product $C^*(G;\mathcal{A})$ is a continuous trace $C^*$-algebra.

## 1. INTRODUCTION

Throughout this note, $G$ will denote a second countable, locally compact groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$. Also, we shall fix a bundle[1] $\mathcal{A}$ of (separable) $C^*$-algebras over the unit space $G^{(0)}$ of $G$. We shall write $p$ for the projection of $\mathcal{A}$ onto $G^{(0)}$. We shall assume that there is a continuous action, denoted $\sigma$, of $G$ on $\mathcal{A}$. This means the following: First of all, $\sigma$ must be a homomorphism from $G$ into the isomorphism groupoid of $\mathcal{A}$, $\operatorname{Iso}(\mathcal{A})$, so that, in particular, $\sigma_\gamma : A(s(\gamma)) \to A(r(\gamma))$ is a $C^*$-isomorphism for each $\gamma \in G$. Second, let $s^*(\mathcal{A})$ and $r^*(\mathcal{A})$ be the bundles on $G$ obtained by pulling back $\mathcal{A}$ via $s$ and $r$, so that $s^*(\mathcal{A}) = \{(\gamma, a) \mid a \in A(s(\gamma))\}$ and similarly for $r^*(\mathcal{A})$. Then $\sigma$ determines a bundle map $\sigma^* : s^*(\mathcal{A}) \to r^*(\mathcal{A})$ by the formula $\sigma^*(\gamma, a) = (\gamma, \sigma_\gamma(a))$. The continuity assumption that we make is that for each continuous section $f$ of $s^*(\mathcal{A})$, $\sigma^* \circ f$ is a continuous section of $r^*(\mathcal{A})$.

Let $C_c(G, r^*(\mathcal{A}))$ denote the space of continuous sections of $r^*(\mathcal{A})$ with compact support and, for $f, g \in C_c(G, r^*(\mathcal{A}))$, set

$$f * g(\gamma) := \int f(\eta)\sigma_\eta(g(\eta^{-1}\gamma))\, d\lambda^{r(\gamma)}(\eta)$$

and

$$f^*(\gamma) = \sigma_\gamma(f(\gamma^{-1})^*).$$

Then, with respect to these operations and pointwise addition and scalar multiplication, $C_c(G, r^*(\mathcal{A}))$ becomes a topological $*$-algebra in the inductive limit topology to which Renault's disintegration theorem [16] applies. The enveloping $C^*$-algebra of $C_c(G, r^*(\mathcal{A}))$ is called the *crossed product of $G$ acting on $\mathcal{A}$* and is denoted $C^*(G;\mathcal{A})$ [16]. The basic problem we study in this note is

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[1]We follow [4] for the general theory of Banach- and $C^*$-bundles. However, we adopt the increasingly popular convention that bundles are to be denoted by calligraphic letters. The fibres in a given bundle are then denoted by the corresponding roman letter. Thus, if $\mathcal{A}$ is a bundle of $C^*$-algebras, say, over a space $X$, then the fibre over $x \in X$ will be denoted $A(x)$.

**Question 1.** Under what circumstances on $G$ and $\mathcal{A}$ is $C^*(G;\mathcal{A})$ a continuous trace $C^*$-algebra?

The first systematic investigation into questions of this type that we know of is Green's pioneering study [5]. There, Green deals with the case where $G$ is the transformation group groupoid obtained by letting a locally compact *group* $H$ act on a locally compact Hausdorff space $X$ and where $\mathcal{A}$ is the trivial line bundle over $G^{(0)} = X$. His principal result asserts that if $H$ acts freely on $X$, then $C^*(G;\mathcal{A})$ is continuous trace if and only if the action of $H$ is proper. In this event, in fact, $C^*(G;\mathcal{A})$ is strongly Morita equivalent to $C_0(X/H)$.

Another precedent to this investigation is the work of the second two authors in [10]. Here, the hypothesis is that $G$ is a *principal* groupoid and the bundle $\mathcal{A}$ is again the trivial line bundle. It was shown that $C^*(G;\mathcal{A}) = C^*(G)$ has continuous trace if and only if the action of $G$ on $G^{(0)}$ is proper. In this event, again, $C^*(G)$ is strongly Morita equivalent to $C_0(G^{(0)}/G)$.

The first example where the bundle $\mathcal{A}$ is non-trivial was considered by Raeburn and Rosenberg in [14]. They considered a locally compact *group* acting on a continuous trace $C^*$-algebra $A$ and showed that if the natural action of $G$ on the spectrum of $A$, $\hat{A}$, is free and proper, then the cross product $C^*$-algebra $A \rtimes G$ has continuous trace. In [13], Olesen and Raeburn proved a conditioned converse: if the group $G$ is abelian and acts freely on $\hat{A}$, then if $A \rtimes G$ is continuous trace, the action of $G$ on $\hat{A}$ must be proper. Quite recently, Deicke [2] used non-abelian duality theory to remove the hypothesis that $G$ is abelian. Thus, the best result in this direction is: If $G$ acts on $A$ yielding a free action on $\hat{A}$, then $A \rtimes G$ is continuous trace if and only if the action of $G$ on $\hat{A}$ is proper.

Our objective in this note, Theorem 1, is to prove a result that contains all of these examples as special cases — and much more, as well. It is based on two hypotheses and some ancillary considerations that we will elaborate. The first hypothesis is

**Hypothesis 1.** *The $C^*$-algebra $C_0(G^{(0)}, \mathcal{A})$ is continuous trace.*

Here, $C_0(G^{(0)}, \mathcal{A})$ denotes the $C^*$-algebra of continuous sections of the bundle $\mathcal{A}$ that vanish at infinity on $G^{(0)}$. An hypothesis on the bundle $\mathcal{A}$ of this nature is natural and reasonable, in view of the fact that in the trivial case $G = G^{(0)}$, we have $C^*(G;\mathcal{A}) \cong C_0(G^{(0)}, \mathcal{A})$. Furthermore, we note that a compact group can act on an antiliminal $C^*$-algebra in such a way that the crossed product is continuous trace [17].

To state our second hypothesis, we need a couple of remarks about the spectrum of $C_0(G^{(0)}, \mathcal{A})$. We shall denote it by $X$ throughout this note. Observe that $X$ may be expressed as the disjoint union of spectra $\coprod_{u\in G^{(0)}} A(u)^\wedge$ and the natural projection $\hat{p}$ from $X$ to $G^{(0)}$ is continuous and open [8], [12]. The groupoid $G$ acts on $X$ (using the map $\hat{p}$) as follows. If $x \in X$, we shall write $x = [\pi_x]$ in order to specify a particular irreducible representation in the equivalence class represented by $x$. Thus, if $X * G$ denotes the space $\{(x,\gamma) \in X \times G \mid \hat{p}(x) = r(x)\}$ and if $(x,\gamma) \in X * G$, then $x \cdot \gamma$ is defined to be $[\pi_x \circ \sigma_\gamma]$. The fact that the action of $G$ on $X$ is well defined and is continuous is easily checked (as in [15, Lemma 7.1] for example). Our second hypothesis is

**Hypothesis 2.** *The action of $G$ on $X$ is free.*

For $x \in X$, we shall write $\mathcal{K}(x)$ for the quotient $\mathcal{A}(\hat{p}(x))/\ker(\pi_x)$. Then $\mathcal{K}(x)$ is well defined (i.e., it is independent of the choice of $\pi_x$) and is an elementary $C^*$-algebra. Hypothesis 1 guarantees that the collection $\{\mathcal{K}(x)\}_{x\in X}$ may be given the structure of an elementary $C^*$-algebra bundle over $X$ satisfying Fell's condition [3, Proposition 10.5.8]. As a result, we find that $C_0(G^{(0)}, \mathcal{A})$ is naturally isomorphic to $C_0(X, \mathcal{K})$. The action of $G$ on $X$ induces one on $\mathcal{K}$ that we shall use. We find it preferable to express this in terms of the *action groupoid* $X * G$ defined as follows.

As a set, $X * G := \{(x, \gamma) \mid \hat{p}(x) = r(\gamma)\}$ and the groupoid operations are defined by the formulae

$$(x, \alpha)(x\alpha, \beta) := (x, \alpha\beta) \text{ and}$$
$$(x, \alpha)^{-1} := (x\alpha, \alpha^{-1}),$$

$(x, \alpha)$, $(x, \beta) \in X * G$. Note, in particular, that the unit space of $X * G$ may be identified with $X$ via: $(x, \hat{p}(x)) \longleftrightarrow x$. Note, too, that the range and source maps on $X * G$, denoted $\tilde{r}$ and $\tilde{s}$, are given by the equations $\tilde{r}(x, \alpha) = (x, r(\alpha))$ and $\tilde{s}(x, \alpha) = (x\alpha, s(\alpha))$. The groupoid $X * G$ in the product topology is clearly locally compact, Hausdorff, and separable. It has a Haar system $\{\tilde{\lambda}^x\}_{x\in X}$ given by the formula $\tilde{\lambda}^x = \delta_x \times \lambda^{\hat{p}(x)}$. Observe that the action of $G$ on $X$ is free (resp. proper) iff $X * G$ is principal (resp. proper).

The groupoid $X * G$ acts on $\mathcal{K}$ via the formula

$$\tilde{\sigma}_{(x,\gamma)}(k) := \sigma_\gamma(a) + \ker(\pi_x),$$

where $k = a + \ker \pi_{x\gamma}$ lies in $K(\tilde{s}(x, \gamma))$. Note that this action is well defined since $\pi_x \circ \sigma_\gamma = \pi_{x\cdot\gamma}$. We promote this action of $X * G$ on $\mathcal{K}$ to one on $X * \mathcal{K} := \{(x, k) \mid k \in K(x)\}$: $(x, k) \cdot (x, \gamma) := (x \cdot \gamma, \tilde{\sigma}^{-1}_{(x,\gamma)}(k))$. If the action of $G$ on $X$ is free and proper, then the action of $X * G$ on $X * \mathcal{K}$ is also free and proper. In this case, we write $\mathcal{K}^X$ for the quotient space $X * \mathcal{K}/X * G$. Then $\mathcal{K}^X$ is naturally a bundle of elementary $C^*$-algebras over $X/G$. In fact, using the methods of Theorem 1.1 of [14], it is easy to see that $\mathcal{K}^X$ satisfies Fell's condition. Thus, in particular, $C_0(X/G, \mathcal{K}^X)$ has continuous trace, if $G$ acts on $X$ freely and properly.

With these preliminaries at our disposal, we are able to state the main result of this paper as

**Theorem 1.** *Under Hypotheses 1 and 2, $C^*(G; \mathcal{A})$ has continuous trace if and only if the action of $G$ on $X$ is proper. In this event, $C^*(G; \mathcal{A})$ is strongly Morita equivalent to $C_0(X/G, \mathcal{K}^X)$, where $\mathcal{K}^X$ is the elementary $C^*$-bundle over $X/G$, satisfying Fell's condition, that was just defined.*

## 2. Sufficiency in Theorem 1

First we reduce the proof of Theorem 1 to the case when $G$ is a principal groupoid. This reduction is accomplished with the aid of

**Theorem 2.** *In the notation established above, $C^*(G; \mathcal{A})$ is isomorphic to $C^*(X * G; \mathcal{K})$.*

*Proof.* For $u \in G^{(0)}$, identify $A(u)$ with $C_0(\hat{p}^{-1}(u), \mathcal{K})$. Also, set $C_{cc}(G, r^*(\mathcal{A})) := \{f \in C_c(G, r^*(\mathcal{A})) \mid (x, \gamma) \to \|f(\gamma)(x)\| \text{ has compact support in } X * G\}$. Recall that $r^*(\mathcal{A}) = \{(\gamma, a) \mid a \in A(r(\gamma))\}$ and, likewise, $\tilde{r}^*(\mathcal{K}) = \{((x, \gamma), a) \mid a \in K(x)\}$. So, in the definition of $C_{cc}(G, r^*(\mathcal{A}))$, $f(\gamma)$, which nominally is in $A(r(\gamma))$, is to be viewed in $C_0(\hat{p}^{-1}(u), \mathcal{K})$. In particular, $f(\gamma)(x)$ lies in $K(x)$, when $\hat{p}(x) =$

$r(\gamma)$. Now it is straightforward to check that $C_{cc}(G, r^*(\mathcal{A}))$ is a subalgebra of $C_c(G, r^*(\mathcal{A}))$ that is dense in $C_c(G, r^*(\mathcal{A}))$ in the inductive limit topology. Further, if we define $\Psi : C_c(X * G, \tilde{r}^*(\mathcal{K})) \to C_{cc}(G, r^*(\mathcal{A}))$ by the formula

$$[\Psi(f)(\gamma)](x) = f(x, \gamma),$$

$f \in C_c(X * G, \tilde{r}^*(\mathcal{K}))$, then $\Psi$ is an algebra $*$-homomorphism that is continuous in the inductive limit topologies on $C_c(X*G, \tilde{r}^*(\mathcal{K}))$ and $C_{cc}(G, r^*(\mathcal{A}))$. Hence by Renault's disintegration theorem [16, Theorem 4.1], $\Psi$ extends to a $C^*$-homomorphism from $C^*(X * G; \mathcal{K})$ into $C^*(G; \mathcal{A})$.

The inverse $\Phi : C_{cc}(G, r^*(\mathcal{A})) \to C_c(X * G, \tilde{r}^*(\mathcal{K}))$ to $\Psi$ is given formally by the formula

$$\Phi(f)(x, \gamma) = f(\gamma)(x),$$

$f \in C_{cc}(G, r^*(\mathcal{A}))$; i.e., at the level of $C_c$, $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity maps on the appropriate algebras. The problem is that *a priori* $\Phi$ is not continuous in the inductive limit topologies. However, $\Phi$ *is* manifestly continuous with respect to the so-called $L^I$-norms on $C_{cc}(G, r^*(\mathcal{A}))$ and $C_c(X * G, \tilde{r}^*(\mathcal{K}))$, where for $f \in C_{cc}(G, r^*(\mathcal{A}))$, the $L^I$-norm of $f$ is

$$\max \left\{ \sup_{u \in G^{(0)}} \int \|f(\gamma)\|_{A(u)} \; d\lambda^u, \sup_{u \in G^{(0)}} \int \|f^*(\gamma)\|_{A(u)} \; d\lambda^u \right\},$$

and similarly for $C_c(X * G, \tilde{r}^*(\mathcal{K}))$. Since every representation of one of these algebras is continuous in the $L^I$-norm by Renault's disintegration theorem [16, Theorem 4.1], we conclude that $\Phi$ extends to a $C^*$-homomorphism from $C^*(G; \mathcal{A})$ to $C^*(X * G; \mathcal{K})$ that is the inverse of $\Psi$. □

*On the basis of Theorem 2 we may and shall assume from now on that* $X = G^{(0)}$ *and that* $A(u)$ *is an elementary* $C^*$*-algebra for every* $u \in G^{(0)}$.

*Proof of the sufficiency.* If $G$ acts freely and properly on $G^{(0)}$, then $G^{(0)}$ is an equivalence between $G$ and the quotient space $G^{(0)}/G$ (viewed as a cotrivial groupoid) in the sense of [9]. Further, if $G^{(0)} * \mathcal{A} := \{(u, a) \mid a \in A(u)\}$, then $G^{(0)} * \mathcal{A}$ serves as a bundle of Morita equivalences between $\mathcal{A}$ and $\mathcal{A}^{G^{(0)}}$ in the sense of [7]. (See [16], too.) Indeed, recall that $\mathcal{A}^{G^{(0)}}$ is the quotient $(G^{(0)} * \mathcal{A})/G$, where $(r(\gamma), a) \cdot \gamma = (s(\gamma), \sigma_\gamma^{-1}(a))$. Then the $\mathcal{A}$-valued inner product on $G^{(0)} * \mathcal{A}$ is given by the formula

$$\langle (u, a), (u, b) \rangle_{\mathcal{A}} = a^* b,$$

while the $\mathcal{A}^{G^{(0)}}$-valued inner product on $G^{(0)} * \mathcal{A}$ is given by the formula

$$_{\mathcal{A}^{G^{(0)}}}\langle (u, a), (u, b) \rangle = [u, ab^*],$$

where $[u, ab^*]$ denotes the image of $(u, ab^*)$ in $\mathcal{A}^{G^{(0)}}$. By Corollaire 5.4 of [16],

$$C^*(G; \mathcal{A}) \quad \text{is Morita equivalent to} \quad C_0(G^{(0)}/G, \mathcal{A}^{G^{(0)}}).$$

This second algebra is a continuous-trace $C^*$-algebra since $\mathcal{A}^{G^{(0)}}$ is a bundle of elementary $C^*$-algebras satisfying Fell's condition, as we noted earlier. □

## 3. Necessity in Theorem 1

The proof of the necessity in Theorem 1 is modeled on the proofs in [10] and [11], which in turn are inspired by ideas in [5]. There are, however, a number of new difficulties that must be overcome.

For each $u \in G^{(0)}$, we fix an irreducible representation $\pi_u$ of $A(u)$ on a Hilbert space $H_u$. Since each $A(u)$ is elementary, the $\pi_u$'s are unique up to unitary equivalence. We define $L^u$ to be the representation of $C^*(G;\mathcal{A})$ on the Hilbert space $L^2(\lambda_u) \otimes H_u$ according to the formula

$$L^u(f)\xi(\gamma) = \int \pi_u \circ \sigma_\gamma^{-1}(f(\gamma\alpha))\xi(\alpha^{-1})\, d\lambda^u(\alpha), \tag{1}$$

where $f \in C_c(G, r^*(\mathcal{A}))$ and $\xi \in L^2(\lambda_u) \otimes H_u$. (Recall that $\lambda_u$ is the image of $\lambda^u$ under inversion.) Thus, $L^u$ is the representation of $C^*(G;\mathcal{A})$ *induced* by the irreducible representation $\pi_u$ viewed as a representation of $C_0(G^{(0)}, \mathcal{A})$. This implies, in particular, that replacing $\pi_u$ by a unitarily equivalent representation does not affect the unitary equivalence class of $L^u$. The following lemma and corollary capture the salient features of the $L^u$ that we shall use.

**Lemma 3.** *Under the hypothesis that the action of $G$ on $G^{(0)}$ is free (and Hypothesis 1 on $\mathcal{A}$), the following assertions hold:*

1. *Each representation $L^u$ is irreducible.*
2. *$L^u$ is unitarily equivalent to $L^v$ if and only if $u$ and $v$ lie in the same orbit.*
3. *The map $u \to L^u$ is continuous.*

*Proof.* The proof follows the lines of the arguments in [10, Lemma 2.4 and Proposition 2.5]. Only minor changes need to be made to accommodate the presence of $\mathcal{A}$. The key point is that $L^u$ is unitarily equivalent to the representation $R^u$ of $C^*(G;\mathcal{A})$ defined by the formula

$$R^u(f)\xi(\gamma \cdot u) = \int \pi_u \circ \sigma_\gamma^{-1}(f(\gamma\alpha))\xi(\alpha^{-1} \cdot u)\, d\lambda^u(\alpha),$$

$f \in C_c(G, r^*(\mathcal{A}))$, $\xi \in L^2([u], \mu_{[u]}) \otimes H_u$, where $[u]$ denotes the orbit of $u$ and $\mu_{[u]}$ is the image of $\lambda_u$ under the map $r|s^{-1}(u)$. The fact that the action of $G$ on $G^{(0)}$ is free (i.e., $G$ is a principal groupoid) implies that $r|s^{-1}(u)$ is a bijection between $s^{-1}(u)$ and $[u]$. It is a Borel isomorphism, of course, because of our separability hypotheses and the fact that $r|s^{-1}(u)$ is continuous.

The value of $R^u$ for us lies in the fact that it is evident how to express $R^u$ as the integrated form of a representation of $(G, \mathcal{A})$ in the sense of [16, Definition 3.4]. The measure class on $G^{(0)}$ is, of course, that determined by $\mu_{[u]}$ and the Hilbert bundle $\mathcal{H}$ is the constant bundle determined by $H_u$ over the orbit of $u$, i.e.,

$$H(v) = \begin{cases} \{v\} \times H_u, & v \in [u], \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\int^{\oplus} H(v)\, d\mu_{[u]}$ is identified with $L^2([u], \mu_{[u]}) \otimes H_u$ in the standard fashion. The groupoid $G$ is represented on $\mathcal{H}$ according to the formula

$$U_\gamma((s(\gamma), \xi)) = (r(\gamma), \xi),$$

$\xi \in H_u$, $s(\gamma) \in [u]$, i.e., $\{U_\gamma\}_{\gamma \in G}$ is just the translation representation, and $\mathcal{A}$ is represented on $\mathcal{H}$ according to the formula

$$a \cdot (v, \xi) = (v, \pi_u \circ \sigma_\gamma(a)\xi),$$

$a \in A(v)$, $(v, \xi) \in H(v)$, where $\gamma$ is the unique element in $G$ with source $v$ and range $u$.

Observe that the $C_0(G^{(0)}, \mathcal{A})$ acts as multipliers on $C^*(G; \mathcal{A})$ according to the formula $\Phi \cdot f(\gamma) = \Phi(r(\gamma))f(\gamma)$ for $\Phi \in C_0(G^{(0)}, \mathcal{A})$ and $f \in C_c(G, r^*(\mathcal{A}))$. The extension $\tilde{R}^u$ of $R^u$ to the multiplier algebra $C^*(G; \mathcal{A})$ represents $C_0(G^{(0)}, \mathcal{A})$ on $L^2([u], \mu_{[u]}) \otimes H_u$ via the equation

$$\tilde{R}^u(\Phi)\xi(v) = \pi_u \circ \sigma_\gamma(\Phi(v))\xi(v),$$

again, where $\gamma$ is the unique element in $G$ with source $v$ and range $u$. It is clear from this that the weak closure of the algebra $\tilde{R}^u(C_0(G^{(0)}, \mathcal{A}))$ is the full algebra of decomposable operators on $L^2([u], \mu_{[u]}) \otimes H_u$. It follows that any projection that commutes with $R^u(C^*(G; \mathcal{A}))$ must be diagonal. On the other hand, it follows from the definition of the representation of $G$, $\{U_\gamma\}_{\gamma \in G}$, that a diagonal operator commuting with $R^u(C^*(G; \mathcal{A}))$ must commute with $\{U_\gamma\}_{\gamma \in G}$, and therefore must be constant a.e. $\mu_{[u]}$. This proves that $R^u$, and hence $L^u$, is irreducible.

If $u$ and $v$ lie in the same orbit, it is clear that translation by the (unique) $\gamma$ with source $v$ and range $u$ implements an equivalence between $L^u$ and $L^v$. On the other hand, if $u$ and $v$ lie in different orbits, then $L^u$ and $L^v$ are disjoint. Indeed, the representations $N_u$ and $N_v$ of $C_0(G^{(0)})$ obtained by restricting $\tilde{R}^u$ and $\tilde{R}^v$ to $C_0(G^{(0)})$, viewed as a subalgebra of $M(C^*(G; \mathcal{A}))$, are supported on the disjoint sets $[u]$ and $[v]$. Arguing just as we did in the proof of [10, Proposition 2.5], using [18, Lemma 4.15], we conclude $L^u$ and $L^v$ are disjoint.

Finally, to see that the map $u \mapsto L^u$ is continuous, observe that Hypothesis 1 guarantees that for each point $u \in G^{(0)}$ we can find a neighborhood $\mathcal{V}_u$ of $u$ on which the $H_v$'s can be chosen to be the fibres of a (topological) Hilbert bundle $\tilde{\mathcal{H}}$ and on which we can choose the $\pi_v$'s so that for any section $\Phi \in C_0(G^{(0)}, \mathcal{A})$ that is supported on $\mathcal{V}_u$ and any two $C_0$-sections of $\tilde{\mathcal{H}}$ over $\mathcal{V}_u$, $\xi$ and $\eta$, the function $v \mapsto (\pi_v(\Phi(v))\xi(v)|\eta(v))_{H_v}$ is continuous. It follows from the continuity of the Haar system that given such sections $\xi$ and $\eta$ of $\tilde{\mathcal{H}}$ and any two functions $g$ and $h$ in $C_c(G)$, the function $v \mapsto (L^v(f)(g \otimes \xi)|(h \otimes \eta))$ (where the inner products are taken in $L^2(\lambda_v) \otimes H_v$) is continuous for all $f \in C_c(G; r^*(\mathcal{A}))$. This shows that the map $u \mapsto L^u$ is continuous. □

**Corollary 4.** *Assume that $G$ is principal and that $\mathcal{A}$ is an elementary $C^*$-bundle over $G^{(0)}$, satisfying Fell's condition, on which $G$ acts. If $C^*(G; \mathcal{A})$ has continuous trace, then the map that sends $u \in G^{(0)}$ to the unitary equivalence class of $L^u$ defines a continuous open surjection of $G^{(0)}$ onto $C^*(G; \mathcal{A})^\wedge$ that is constant on $G$-orbits. In particular, orbits are closed and $G^{(0)}/G$ is homeomorphic to $C^*(G; \mathcal{A})^\wedge$.*

*Proof.* The proof is also essentially the same as the proof in [10, Proposition 2.5]. Here is an outline. Write $\Psi$ for the map $u \mapsto [L^u]$. Then by Lemma 3, $\Psi$ is continuous and constant on $G$-orbits. Thus $\Psi$ passes to a continuous map on $G^{(0)}/G$ with the quotient topology (no matter how bad that might be). Since, however, $C^*(G; \mathcal{A})^\wedge$ is Hausdorff by hypothesis, we conclude that $G^{(0)}/G$ is Hausdorff.

Suppose that $L$ is an irreducible representation of $C^*(G; \mathcal{A})$ and let $M$ be the representation of $C_0(G^{(0)})$ obtained by extending $L$ to the multiplier algebra of $C^*(G; \mathcal{A})$ and then restricting to $C_0(G^{(0)})$. The kernel $J$ of $M$ is the set of functions in $C_0(G^{(0)})$ that vanish on a closed set $F$ in $G^{(0)}$. Then $F$ is easily seen to be invariant. Indeed, one may do this directly or use the fact that it supports the quasi-invariant measure associated to the disintegrated form of $L$ guaranteed by [16]. Further, since $L$ is irreducible, $F$ cannot be expressed as the union of two disjoint, closed, $G$-invariant sets. Since the quotient map from $G^{(0)}$ to $G^{(0)}/G$ is continuous and open, we may apply the lemma on page 222 of [6] to conclude that $F$ is an orbit closure. Since we now know that orbits are closed, $F$ is, in fact, an orbit. Thus $L$ factors through $C^*(G|_{[u]};\mathcal{A})$. However, $G|_{[u]}$ is a *transitive* principal groupoid. A little reflection, using Theorem 3.1 of [9], reveals that every irreducible representation of $C^*(G|_{[u]};\mathcal{A})$ is unitarily equivalent to $L^u$. $\square$

We now assume the action of $G$ on $G^{(0)}$ is not proper and use Lemma 2.6 of [10] to choose a sequence $\{\gamma_n\} \subseteq G$ such that $\gamma_n \to \infty$ in the sense that $\{\gamma_n\}$ eventually escapes each compact subset of $G$, and such that $r(\gamma_n), s(\gamma_n) \to z$, for some $z \in G^{(0)}$. We shall fix this sequence for the remainder of the proof. We also choose a relatively compact neighborhood $U$ of $z \in G^{(0)}$ and a section $g$ in the Pedersen ideal of $C_0(G^{(0)},\mathcal{A})$ such that $g$ is non-negative, compactly supported, and satisfies $\operatorname{tr}(\pi_u(g(u))) \equiv 1$ on $U$. The fact that $\mathcal{A}$ satisfies Fell's condition guarantees that such choices are possible.

With these ingredients fixed, we want to build a special neighborhood $E$ of $z$ in $G$, following the analysis on pages 236–238 of [10]. First observe, as we have above, that since $G$ is principal, $r$ maps $G_z$ bijectively onto $[z]$ while $s$ maps $G^z$ bijectively onto $[z]$, where, recall, $[z]$ denotes the orbit of $z$. Since $[z]$ is closed by Corollary 4 while $r$ is continuous and open on $G$, we see that $r$ maps $G_z$ homeomorphically onto $[z]$. Likewise, $s$ maps $G^z$ homeomorphically onto $[z]$. Also, since $G$ is principal, multiplication induces a homeomorphism between $G_z \times G^z$ and $G|_{[z]}$. Let $N$ be the closed support of $g$, a compact subset of $G^{(0)}$, and set $F_z := G_z \cap r^{-1}([z]\cap N)$ and $F^z := G^z \cap s^{-1}([z]\cap N)$, obtaining compact subsets of $G_z$ and $G^z$, respectively. Then we see that if $\gamma \in G|_{[z]}$ and if $g(s(\gamma)) \neq 0$ and $g(r(\gamma)) \neq 0$, then $\gamma \in F_z F^z$.

According to Lemma 2.7 of [10], we may select symmetric, conditionally compact open neighborhoods $W_0$ and $W_1$ of $G^{(0)}$ such that $\overline{W_0} \subseteq W_1$. (Recall that a neighborhood $W$ of $G^{(0)}$ is conditionally compact in case $VW$ and $WV$ are relatively compact subsets of $G$ for each relatively compact subset $V$ in $G$.) We select such a pair, as we may, with the additional property that $F_z F^z \subseteq W_0 z W_0$. Then from the preceding paragraph, we see that if $\gamma \notin W_0 z W_0$, then either $g(s(\gamma)) = 0$ or $g(r(\gamma)) = 0$.

By construction,

$$\overline{W_1}^{7} z \backslash W_0 z \subseteq r^{-1}(G^{(0)}\backslash N).$$

So we may find relatively compact open neighborhoods $V_0$ and $V_1$ of $z$ in $G$ so that $V_0 \subseteq W_0$, $\overline{V_0} \subseteq V_1$, and so that

$$\overline{W_1}^{7}\overline{V_1}\backslash W_0 V_0 \subseteq r^{-1}(G^{(0)}\backslash N).$$

With these $V_0$ and $V_1$ so chosen, the special open neighborhood $E$ of $z$ in $G$ that we want is defined to be $E := W_0 V_0 W_0$.

Observe that we have

$$\overline{W_1}^{7}\overline{V_1}\,\overline{W_1}^{7}\backslash E = \overline{W_1}^{7}\overline{V_1}\,\overline{W_1}^{7}\backslash W_0 V_0 W_0 \subseteq r^{-1}(G^{(0)}\backslash N).$$

Set

$$g^1(\gamma) := \begin{cases} g(r(\gamma)), & \gamma \in \overline{W_1}^{7}\overline{V_1}\,\overline{W_1}^{7}, \\ 0, & \gamma \notin E. \end{cases}$$

Then, since $\overline{W_1}^7\overline{V_1}\,\overline{W_1}^7\backslash E \subseteq r^{-1}(G^{(0)}\backslash N)$, $g^1$ is a continuous section of $r^*(\mathcal{A})$ on $G$ that vanishes outside $E$.

Observe the following containment relations among relatively compact sets: $E^2 = W_0V_0W_0^2V_0W_0 \subseteq W_0^4V_0W_0^4 \subseteq \overline{W_0}^4\overline{V_0}\,\overline{W_0}^4 \subseteq W_1^4V_1W_1^4$. Hence, we may find a compactly supported function $b$ on $G$ such that $0 \le b(\gamma) \le 1$ for all $\gamma$, $b \equiv 1$ on $E^2$ and $b \equiv 0$ off $W_1^4V_1W_1^4$. Replacing $b$ by $\frac{b+b^*}{2}$, if necessary, we may assume that $b$ is a selfadjoint element of the convolution algebra of scalar-valued functions $C_c(G)$.

Define $F(\gamma) := g(r(\gamma))\sigma_\gamma(g(s(\gamma)))b(\gamma)$. By our choices of $g$ and $b$, $F$ belongs to $C_c(G, r^*(\mathcal{A}))$, $F$ is selfadjoint and

$$L^u(F)\xi(\gamma) = \pi_u(\sigma_\gamma^{-1}(g(r(\gamma)))) \int \pi_u(\sigma_\alpha^{-1}(g(r(\alpha))))b(\gamma\alpha^{-1})\xi(\alpha)\,d\lambda_u(\alpha)$$

for all $u$ by the definition of $L^u$ (cf. (1)). Let $P_{u,1}$ be the projection onto $\mathcal{E}_{u,1} := L^2(G_u \cap E) \otimes H_u$ and let $P_{u,2}$ be the projection onto the orthocomplement, $\mathcal{E}_{u,2} := L^2(G_u\backslash E) \otimes H_u$. Then, if $P_{u,1}\xi = \xi$, we see that

$$\begin{aligned} L^u(F)\xi(\gamma) &= \pi_u(\sigma_\gamma^{-1}(g(r(\gamma)))) \int_{G_u\cap E} \pi_u(\sigma_\alpha^{-1}(g(r(\alpha))))\xi(\alpha)\,d\lambda_u(\alpha) \\ &= \pi_u \circ \sigma_\gamma^{-1}(g^1(\gamma)) \int_{G_u\cap E} \pi_u \circ \sigma_\alpha^{-1}(g^1(\alpha))\xi(\alpha) d\lambda_u(\alpha) \end{aligned}$$

for all $\gamma \in G_u \cap E$ because $b$ is identically 1 on $E^2$. However, by definitions of $E$ and $g^1$, the equation persists when $\gamma \in G_u\backslash E$, yielding 0. Thus, $P_{u,1}$ commutes with $L^u(F)$. Moreover, when $u = z$, these formulas show that $L^z(F)P_{z,1} = L^z(F)$.

We now want to show that $L^u(F)P_{u,1} \ge 0$ and we want to analyze the trace, $\mathrm{tr}(L^u(F)P_{u,1})$. However, when $\xi$ is in the range of $P_{u,1}$, the formula for $L^u(F)\xi$ shows that

$$\begin{aligned} (L^u(F)\xi|\xi) &= \iint (\pi_u \circ \sigma_\gamma^{-1}(g^1(\gamma))\pi_u \circ \sigma_\alpha^{-1}(g^1(\alpha))\xi(\alpha), \xi(\gamma))d\lambda_u(\alpha)d\lambda_u(\gamma) \\ &= \iint (\pi_u \circ \sigma_\alpha^{-1}(g^1(\alpha))\xi(\alpha), \pi_u \circ \sigma_\gamma^{-1}(g^1(\gamma))\xi(\gamma))d\lambda_u(\alpha)d\lambda_u(\gamma) \\ &\ge 0. \end{aligned}$$

As for the trace, observe that if $K_u$ is defined by the formula

$$K_u(\gamma, \eta) = \pi_u(\sigma_\gamma^{-1}(g^1(\gamma))\sigma_\eta^{-1}(g^1(\eta)))$$

on $G_u \times G_u$, then our calculations show that $K_u$ is continuous, positive semidefinite and supported on $(G_u \cap E) \times (G_u \cap E)$, and that

$$(L^u(F)P_{u,1}\xi|\zeta) = \iint (K_u(\gamma, \eta)\xi(\eta), \zeta(\gamma))d\lambda_u(\eta)d\lambda_u(\gamma).$$

Consequently, we may use Duflo's generalization of Mercer's theorem, [1, Proposition 3.1.1], and the fact that $K_u(\gamma, \gamma) = \pi_u \circ \sigma_\gamma^{-1}(g(r(\gamma)))^2$ to conclude that

$$\mathrm{tr}(L^u(F)P_{u,1}) = \int_{G_u\cap E} \mathrm{tr}(\pi_u \circ \sigma_\gamma^{-1}(g(r(\gamma)))^2)\,d\lambda_u(\gamma).$$

By our choice of $g$, this expression is continuous in $u$ and when $u = z$ yields the value $\mathrm{tr}(L^z(F))$.

We will show that there is a positive number $a$ such that

$$\big\|(L^{s(\gamma_n)}(F)P_{s(\gamma_n),2})^+\big\| \ge 2a \tag{2}$$

eventually, where $(L^{s(\gamma_n)}(F)P_{s(\gamma_n),2})^+$ denotes the positive part of the selfadjoint operator $L^{s(\gamma_n)}(F)P_{s(\gamma_n),2}$. Therefore, eventually

$$\big(\text{the largest eigenvalue of } L^{s(\gamma_n)}(F)P_{s(\gamma_n),2}\big) \geq 2a.$$

Assume that we have shown this, and set

$$q(t) = \begin{cases} 0, & t \leq a, \\ 2(t-a), & a \leq t \leq 2a, \\ t, & 2a. \end{cases}$$

Then $q(F)$ is a positive element in the Pedersen ideal of $C^*(G, \mathcal{A})$, and so the function

$$u \to \operatorname{tr}(L^u(q(F))) = \operatorname{tr}(L^u(q(F))P_{u,1}) + \operatorname{tr}(L^u(q(F))P_{u,2})$$

is finite and continuous in $u$, with value $\operatorname{tr}(L^z(q(F)))$ at $u = z$. (Recall that $L^z(F) = L^z(F)P_{z,1}$ and so $L^z(q(F)) = L^z(q(F))P_{z,1}$.) On the other hand, we showed that $L^u(F)P_{u,1}$ is positive. Since $P_{u,1}$ commutes with $L^u(F)$, $L^u(F)P_{u,1} = L^u(F^+)P_{u,1}$. But also we showed that $u \to \operatorname{tr}(L^u(F)P_{u,1})$ is continuous at $z$. Consequently, so is $u \to \operatorname{tr}(L^u(F^+)P_{u,1})$. Since $q(F) \leq F^+$, the function $u \to \operatorname{tr}(L^u(q(F))P_{u,1})$ is continuous by Lemma 4.4.2(i) in [3], with value $\operatorname{tr}(L^z(q(F))) = \operatorname{tr}(L^z(q(F))P_{z,1})$ at $u = z$, also. Therefore

$$\lim_{u \to z} \operatorname{tr}(L^u(q(F))P_{u,2}) = 0.$$

Since the largest eigenvalue of $L^{s(\gamma_n)}(F)P_{s(\gamma_n),2} \geq 2a$, the largest eigenvalue of $L^{s(\gamma_n)}(q(F))P_{s(\gamma_n),2} \geq 2a$ also. Consequently,

$$\liminf_n \operatorname{tr}(L^{s(\gamma_n)}(q(F))P_{s(\gamma_n),2}) \geq 2a.$$

This contradiction will complete the proof.

We will finish by verifying the asserted inequality (2). To this end, choose an open neighborhood of $z$ in $G$, $V_2$, that is contained in $V_0$ and choose a conditionally compact neighborhood $Y$ of $G^{(0)}$ such that if $v \in V_2$, then $r$ maps $Yv$ into $U$. Without loss of generality, we may assume that $Y \subseteq W_0$. Observe that if $\gamma_n \notin \overline{W_1}^2 \overline{V_1}\,\overline{W_1}^2$, then for $\gamma \in Y\gamma_n$, $\gamma \notin E$. Indeed, if $\gamma = \gamma'\gamma_n \in E \cap Y\gamma_n$, then $\gamma_n \in (\gamma')^{-1}E \subseteq W_0^2 V_0 W_0 \subseteq \overline{W_1}^2 \overline{V_1}\,\overline{W_1}^2$ contrary to assumption. So, since $r(\gamma_n)$ and $s(\gamma_n)$ are tending to $z$, while $\gamma_n$ eventually escapes $\overline{W_1}^2 \overline{V_1}\,\overline{W_1}^2$, we can conclude that for $n$ sufficiently large, whenever $\gamma$ lies in $Y\gamma_n$, then $\gamma \notin E$ while $r(\gamma)$ and $s(\gamma)$ lie in $U$. From now on, we will assume that $n$ is sufficiently large so that these conditions are satisfied.

Next observe that since for each $n$, the map $\gamma \to \pi_{s(\gamma_n)} \circ \sigma_\gamma^{-1}(g(r(\gamma)))$ defines a continuous family of rank 1 projections on the Hilbert space $H_{s(\gamma_n)}$, we can find a Borel family of unit vectors $\gamma \to v_\gamma^n$ such that $\pi_{s(\gamma_n)} \circ \sigma_\gamma^{-1}(g(r(\gamma)))$ is the rank 1 projection determined by $v_\gamma^n$.

Let $h_n(\gamma) = 1_{Y\gamma_n}(\gamma) \times v_\gamma^n$ where $1_{Y\gamma_n}$ denotes the characteristic function of $Y\gamma_n$. Observe that if $\gamma, \alpha \in Y\gamma_n$, then $\gamma\alpha^{-1} \in Y\gamma_n\gamma_n^{-1}Y \subseteq YV_0Y \subseteq W_0V_0W_0 = E$.

Consequently, $b(\gamma\alpha^{-1}) = 1$ and we may calculate to find that if $\gamma \in Y\gamma_n$, then

$$\begin{aligned} &L^{s(\gamma_n)}(F)h_n(\gamma) \\ &\quad= \pi_{s(\gamma_n)}(\sigma_\gamma^{-1}(g(r(\gamma)))) \int \pi_{s(\gamma_n)}(\sigma_\alpha^{-1}(g(r(\alpha))))b(\gamma\alpha^{-1})h_n(\alpha)\,d\lambda_{s(\gamma_n)}(\alpha) \\ &\quad= v_\gamma^n \lambda_{r(\gamma_n)}(Y). \end{aligned}$$

Hence $(L^{s(\gamma_n)}(F)h_n | h_n) = \lambda_{r(\gamma_n)}(Y)^2$. However, by our assumption on $\gamma_n$, $h_n$ lies in $\mathcal{E}_{s(\gamma_n),2}$, so

$$\begin{aligned} ((L^{s(\gamma_n)}(F)P_{s(\gamma_n),2})^+h_n | h_n) &\geq (L^{s(\gamma_n)}(F)P_{s(\gamma_n),2}h_n | h_n) \\ &= (L^{s(\gamma_n)}(F)h_n | h_n) = \lambda_{r(\gamma_n)}(Y)^2. \end{aligned}$$

This shows that $\|(L^{s(\gamma_n)}(F)P_{s(\gamma_n),2})^+\| \geq \lambda_{r(\gamma_n)}(Y)$ provided $n$ is sufficiently large. But the continuity of the Haar system implies that $\liminf \lambda_{r(\gamma_n)}(Y) > 0$, as $n \to \infty$. This verifies equation (2) and completes the proof of Theorem 1.

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