CONTINUOUS-TRACE GROUPOID CROSSED PRODUCTS

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Abstract. Let $G$ be a second countable, locally compact groupoid with Haar system, and let $\mathcal{A}$ be a bundle of $C^*$-algebras defined over the unit space of $G$ on which $G$ acts continuously. We determine conditions under which the associated crossed product $C^*(G; \mathcal{A})$ is a continuous trace $C^*$-algebra.

1. Introduction

Throughout this note, $G$ will denote a second countable, locally compact groupoid with Haar system $\{\lambda^u\}_{u \in G(0)}$. Also, we shall fix a bundle $\mathcal{A}$ of (separable) $C^*$-algebras over the unit space $G(0)$ of $G$. We shall write $p$ for the projection of $\mathcal{A}$ onto $G(0)$. We shall assume that there is a continuous action, denoted $\sigma$, of $G$ on $\mathcal{A}$.

This means the following: First of all, $\sigma$ must be a homomorphism from $G$ into the isomorphism groupoid of $\mathcal{A}$, $\text{Iso}(\mathcal{A})$, so that, in particular, $\sigma_\gamma : A(s(\gamma)) \to A(r(\gamma))$ is a $C^*$-isomorphism for each $\gamma \in G$. Second, let $s^*(\mathcal{A})$ and $r^*(\mathcal{A})$ be the bundles on $G$ obtained by pulling back $\mathcal{A}$ via $s$ and $r$, so that $s^*(\mathcal{A}) = \{(\gamma, a) \mid a \in A(s(\gamma))\}$ and similarly for $r^*(\mathcal{A})$. Then $\sigma$ determines a bundle map $\sigma^* : s^*(\mathcal{A}) \to r^*(\mathcal{A})$ by the formula $\sigma^*(\gamma, a) = (\gamma, \sigma_\gamma(a))$. The continuity assumption that we make is that for each continuous section $f$ of $s^*(\mathcal{A})$, $\sigma^* \circ f$ is a continuous section of $r^*(\mathcal{A})$.

Let $C_c(G, r^*(\mathcal{A}))$ denote the space of continuous sections of $r^*(\mathcal{A})$ with compact support and, for $f, g \in C_c(G, r^*(\mathcal{A}))$, set

$$f \ast g(\gamma) := \int f(\eta)\sigma_\eta(g(\eta^{-1}\gamma)) d\lambda^{\gamma}(\eta)$$

and

$$f^*(\gamma) = \sigma_\gamma(f(\gamma^{-1})^*).$$

Then, with respect to these operations and pointwise addition and scalar multiplication, $C_c(G, r^*(\mathcal{A}))$ becomes a topological $*$-algebra in the inductive limit topology to which Renault’s disintegration theorem [16] applies. The enveloping $C^*$-algebra of $C_c(G, r^*(\mathcal{A}))$ is called the crossed product of $G$ acting on $\mathcal{A}$ and is denoted $C^*(G; \mathcal{A})$ [16]. The basic problem we study in this note is
**Question 1.** Under what circumstances on $G$ and $\mathcal{A}$ is $C^*(G^0, \mathcal{A})$ a continuous trace $C^*$-algebra?

The first systematic investigation into questions of this type that we know of is Green’s pioneering study [5]. There, Green deals with the case where $G$ is the transformation group groupoid obtained by letting a locally compact group $H$ act on a locally compact Hausdorff space $X$ and where $\mathcal{A}$ is the trivial line bundle over $G^0 = X$. His principal result asserts that if $H$ acts freely on $X$, then $C^*(G^0, \mathcal{A})$ is continuous trace if and only if the action of $H$ is proper. In this event, in fact, $C^*(G; \mathcal{A})$ is strongly Morita equivalent to $C_0(X/H)$.

Another precedent to this investigation is the work of the second two authors in [10]. Here, the hypothesis is that $G$ is a principal groupoid and the bundle $\mathcal{A}$ is again the trivial line bundle. It was shown that $C^*(G; \mathcal{A}) = C^*(G)$ has continuous trace if and only if the action of $G$ on $G^0$ is proper. In this event, again, $C^*(G)$ is strongly Morita equivalent to $C_0(G^0/G)$.

The first example where the bundle $\mathcal{A}$ is non-trivial was considered by Raeburn and Rosenberg in [14]. They considered a locally compact group acting on a continuous trace $C^*$-algebra $A$ and showed that if the natural action of $G$ on the spectrum of $A$, $\hat{A}$, is free and proper, then the cross product $C^*$-algebra $A \rtimes G$ has continuous trace. In [13], Olesen and Raeburn proved a conditioned converse: if the group $G$ is abelian and acts freely on $\hat{A}$, then $A \rtimes G$ is continuous trace, the action of $G$ on $\hat{A}$ must be proper. Quite recently, Deicke [2] used non-abelian duality theory to remove the hypothesis that $G$ is abelian. Thus, the best result in this direction is:

If $G$ acts on $A$ yielding a free action on $\hat{A}$, then $A \rtimes G$ is continuous trace if and only if the action of $G$ on $\hat{A}$ is proper.

Our objective in this note, Theorem 1, is to prove a result that contains all of these examples as special cases — and much more, as well. It is based on two hypotheses and some ancillary considerations that we will elaborate. The first hypothesis is

**Hypothesis 1.** The $C^*$-algebra $C_0(G^0, \mathcal{A})$ is continuous trace.

Here, $C_0(G^0, \mathcal{A})$ denotes the $C^*$-algebra of continuous sections of the bundle $\mathcal{A}$ that vanish at infinity on $G^0$. An hypothesis on the bundle $\mathcal{A}$ of this nature is natural and reasonable, in view of the fact that in the trivial case $G = G^0$, we have $C^*(G; \mathcal{A}) \cong C_0(G^0, \mathcal{A})$. Furthermore, we note that a compact group can act on an antiliminal $C^*$-algebra in such a way that the crossed product is continuous trace [17].

To state our second hypothesis, we need a couple of remarks about the spectrum of $C_0(G^0, \mathcal{A})$. We shall denote it by $X$ throughout this note. Observe that $X$ may be expressed as the disjoint union of spectra $\coprod_{u \in G^0} A(u)^\wedge$ and the natural projection $\hat{p}$ from $X$ to $G^0$ is continuous and open [8], [12]. The groupoid $G$ acts on $X$ (using the map $\hat{p}$) as follows. If $x \in X$, we shall write $x = [\pi_x]$ in order to specify a particular irreducible representation in the equivalence class represented by $x$. Thus, if $X \ast G$ denotes the space $\{ (x, \gamma) \in X \times G \mid \hat{p}(x) = r(x) \}$ and if $(x, \gamma) \in X \ast G$, then $x \cdot \gamma$ is defined to be $[\pi_x \circ \sigma_{\gamma}]$. The fact that the action of $G$ on $X$ is well defined and is continuous is easily checked (as in [15] Lemma 7.1 for example). Our second hypothesis is

**Hypothesis 2.** The action of $G$ on $X$ is free.
For $x \in X$, we shall write $\mathcal{K}(x)$ for the quotient $\mathcal{A}(\tilde{p}(x))/\ker(\pi_x)$. Then $\mathcal{K}(x)$ is well defined (i.e., it is independent of the choice of $\pi_x$) and is an elementary $C^*$-algebra. Hypothesis 4 guarantees that the collection $\{\mathcal{K}(x)\}_{x \in X}$ may be given the structure of an elementary $C^*$-algebra bundle over $X$ satisfying Fell’s condition [3 Proposition 10.5.8]. As a result, we find that $C_0(G^0, \mathcal{A})$ is naturally isomorphic to $C_0(X, \mathcal{K})$. The action of $G$ on $X$ induces one on $\mathcal{K}$ that we shall use. We find it preferable to express this in terms of the action groupoid $X \ast G$ defined as follows.

As a set, $X \ast G := \{(x, \gamma) \mid \tilde{p}(x) = r(\gamma)\}$ and the groupoid operations are defined by the formulae

$$(x, \alpha)(x\alpha, \beta) := (x, \alpha\beta)$$

$$(x, \alpha)^{-1} := (x\alpha, \alpha^{-1}),$$

$(x, \alpha), (x, \beta) \in X \ast G$. Note, in particular, that the unit space of $X \ast G$ may be identified with $X$ via: $(x, \tilde{p}(x)) \mapsto x$. Note, too, that the range and source maps on $X \ast G$, denoted $\tilde{r}$ and $\tilde{s}$, are given by the equations $\tilde{r}(x, \alpha) = (x, r(\alpha))$ and $\tilde{s}(x, \alpha) = (x\alpha, s(\alpha))$. The groupoid $X \ast G$ in the product topology is clearly locally compact, Hausdorff, and separable. It has a Haar system $\{\lambda^x\}_{x \in X}$ given by the formula $\lambda^x = \delta_x \times \lambda^{\tilde{r}(x)}$. Observe that the action of $G$ on $X$ is free (resp. proper) if $X \ast G$ is principal (resp. proper).

The groupoid $X \ast G$ acts on $\mathcal{K}$ via the formula

$$\tilde{s}_{(x, \gamma)}(k) := \sigma_\gamma(a) + \ker(\pi_x),$$

where $k = a + \ker(\pi_x)$, $G \in \mathcal{K}(x, \gamma)$. Note that this action is well defined since $\pi_x \circ \sigma_\gamma = \pi_{x, \gamma}$. We promote this action of $X \ast G$ on $\mathcal{K}$ to one on $X \ast \mathcal{K} := \{(x, k) \mid k \in \mathcal{K}(x)\}$: $(x, k) : (x, \gamma) := (x \cdot \gamma, \sigma_{(x, \gamma)}^{-1}(k))$. If the action of $G$ on $X$ is free and proper, then the action of $X \ast G$ on $X \ast \mathcal{K}$ is also free and proper. In this case, we write $\mathcal{K}^X$ for the quotient space $X \ast \mathcal{K}/X \ast G$. Then $\mathcal{K}^X$ is naturally a bundle of elementary $C^*$-algebras over $X/G$. In fact, using the methods of Theorem 1.1 of [14], it is easy to see that $\mathcal{K}^X$ satisfies Fell’s condition. Thus, in particular, $C_0(X/G, \mathcal{K}^X)$ has continuous trace, if $G$ acts on $X$ freely and properly.

With these preliminaries at our disposal, we are able to state the main result of this paper as

**Theorem 1.** Under Hypotheses 4 and 2, $C^*(G; \mathcal{A})$ has continuous trace if and only if the action of $G$ on $X$ is proper. In this event, $C^*(G; \mathcal{A})$ is strongly Morita equivalent to $C_0(X/G, \mathcal{K}^X)$, where $\mathcal{K}^X$ is the elementary $C^*$-bundle over $X/G$, satisfying Fell’s condition, that was just defined.

**2. Sufficiency in Theorem 4**

First we reduce the proof of Theorem 4 to the case when $G$ is a principal groupoid. This reduction is accomplished with the aid of

**Theorem 2.** In the notation established above, $C^*(G; \mathcal{A})$ is isomorphic to $C^*(X \ast G; \mathcal{K})$.

*Proof.* For $u \in G^0$, identify $A(u)$ with $C_0(\tilde{p}^{-1}(u), \mathcal{K})$. Also, set $C_{cc}(G, r^*(A)) := \{f \in C_c(G, r^*(A)) \mid \langle x, \gamma \rangle \mapsto \|f(\gamma)(x)\|\}$ has compact support in $X \ast G$. Recall that $r^*(A) = \{\gamma, a \mid a \in A(r(\gamma))\}$ and, likewise, $\tilde{r}^*(\mathcal{K}) = \{((x, \gamma), a) \mid a \in \mathcal{K}(x)\}$. So, in the definition of $C_{cc}(G, r^*(A)), f(\gamma)$, which nominally is in $A(r(\gamma))$, is to be viewed in $C_0(\tilde{p}^{-1}(u), \mathcal{K})$. In particular, $f(\gamma)(x)$ lies in $\mathcal{K}(x)$, when $\tilde{p}(x) = x \ast \gamma$. In this case, $\tilde{s}(x \ast \gamma) = x \ast \gamma$ and $\tilde{s}(x \ast \gamma) : x \ast \gamma := (x \ast \gamma, (x \ast \gamma, \gamma, \gamma) \circ ((x, \gamma) \circ \sigma_{(x, \gamma)}^{-1}(\lambda^x(k))))$. Note that $\tilde{s}(x \ast \gamma) : (x \ast \gamma) = (x \ast \gamma, \sigma_{(x, \gamma)}^{-1}(\lambda^x(k)))$. If the action of $G$ on $X$ is free and proper, then the action of $X \ast G$ on $X \ast \mathcal{K}$ is also free and proper.
serves as a bundle of Morita equivalences between \( A \) to \((G;A)\). This second algebra is a continuous-trace \( C^* \)-algebra since \( \mathcal{A}^{G(0)} \) is a bundle of elementary \( C^* \)-algebras satisfying Fell’s condition, as we noted earlier.

On the basis of Theorem 2, we may and shall assume from now on that \( X = G^{(0)} \) and that \( A(u) \) is an elementary \( C^* \)-algebra for every \( u \in G^{(0)} \).

Proof of the sufficiency. If \( G \) acts freely and properly on \( G^{(0)} \), then \( G^{(0)} \) is an equivalence between \( G \) and the quotient space \( G^{(0)}/G \) (viewed as a cotrivial groupoid) in the sense of [9]. Further, if \( G^{(0)} \ast \mathcal{A} := \{ (u, a) \mid a \in A(u) \} \), then \( G^{(0)} \ast \mathcal{A} \) serves as a bundle of Morita equivalences between \( \mathcal{A} \) and \( \mathcal{A}^{G^{(0)}} \) in the sense of [7]. (See [16], too.) Indeed, recall that \( \mathcal{A}^{G^{(0)}} \) is the quotient \( (G^{(0)} \ast \mathcal{A})/G \), where \((r(\gamma), a) \cdot \gamma = (s(\gamma), \sigma^{-1}_\gamma(a))\). Then the \( \mathcal{A} \)-valued inner product on \( G^{(0)} \ast \mathcal{A} \) is given by the formula

\[
\langle (u, a), (u, b) \rangle_{\mathcal{A}} = a^*b,
\]

while the \( \mathcal{A}^{G^{(0)}} \)-valued inner product on \( G^{(0)} \ast \mathcal{A} \) is given by the formula

\[
\mathcal{A}^{G^{(0)}} \langle (u, a), (u, b) \rangle = [u, ab^*],
\]

where \([u, ab^*]\) denotes the image of \((u, ab^*)\) in \( \mathcal{A}^{G^{(0)}} \). By Corollaire 5.4 of [16],

\[C^*(G;\mathcal{A}) \] is Morita equivalent to \( C_0(G^{(0)}/G, \mathcal{A}^{G^{(0)}})\).

This second algebra is a continuous-trace \( C^* \)-algebra since \( \mathcal{A}^{G^{(0)}} \) is a bundle of elementary \( C^* \)-algebras satisfying Fell’s condition, as we noted earlier.
3. Necessity in Theorem 1

The proof of the necessity in Theorem 1 is modeled on the proofs in [10] and [11], which in turn are inspired by ideas in [5]. There are, however, a number of new difficulties that must be overcome.

For each \( u \in G^{(0)} \), we fix an irreducible representation \( \pi_u \) of \( A(u) \) on a Hilbert space \( H_u \). Since each \( A(u) \) is elementary, the \( \pi_u \)'s are unique up to unitary equivalence. We define \( L^u \) to be the representation of \( C^*(G; A) \) on the Hilbert space \( L^2(\lambda_u) \otimes H_u \) according to the formula

\[
L^u(f)\xi(\gamma) = \int \pi_u \circ \sigma_\gamma^{-1}(f(\gamma\alpha))\xi(\alpha^{-1})\,d\lambda_u(\alpha),
\]

where \( f \in C_c(G, r^*(A)) \) and \( \xi \in L^2(\lambda_u) \otimes H_u \). (Recall that \( \lambda_u \) is the image of \( \lambda_u \) under inversion.) Thus, \( L^u \) is the representation of \( C^*(G; A) \) induced by the irreducible representation \( \pi_u \) viewed as a representation of \( C_0(G^{(0)}, A) \). This implies, in particular, that replacing \( \pi_u \) by a unitarily equivalent representation does not affect the unitary equivalence class of \( L^u \). The following lemma and corollary capture the salient features of the \( L^u \) that we shall use.

**Lemma 3.** Under the hypothesis that the action of \( G \) on \( G^{(0)} \) is free (and Hypothesis 7 on \( A \)), the following assertions hold:

1. Each representation \( L^u \) is irreducible.
2. \( L^u \) is unitarily equivalent to \( L^v \) if and only if \( u \) and \( v \) lie in the same orbit.
3. The map \( u \to L^u \) is continuous.

**Proof.** The proof follows the lines of the arguments in [11] Lemma 2.4 and Proposition 2.5. Only minor changes need to be made to accommodate the presence of \( A \). The key point is that \( L^u \) is unitarily equivalent to the representation \( R^u \) of \( C^*(G; A) \) defined by the formula

\[
R^u(f)\xi(\gamma \cdot u) = \int \pi_u \circ \sigma_\gamma^{-1}(f(\gamma\alpha))\xi(\alpha^{-1} \cdot u)\,d\lambda_u(\alpha),
\]

where \( f \in C_c(G, r^*(A)) \), \( \xi \in L^2([u], \mu_u) \otimes H_u \), where \([u]\) denotes the orbit of \( u \) and \( \mu_u \) is the image of \( \lambda_u \) under the map \( r|s^{-1}(u) \). The fact that the action of \( G \) on \( G^{(0)} \) is free (i.e., \( G \) is a principal groupoid) implies that \( r|s^{-1}(u) \) is a bijection between \( s^{-1}(u) \) and \([u]\). It is a Borel isomorphism, of course, because of our separability hypotheses and the fact that \( r|s^{-1}(u) \) is continuous.

The value of \( R^u \) for us lies in the fact that it is evident how to express \( R^u \) as the integrated form of a representation of \( (G, A) \) in the sense of [11] Definition 3.4. The measure class on \( G^{(0)} \) is, of course, that determined by \( \mu_u \) and the Hilbert bundle \( H \) is the constant bundle determined by \( H_u \) over the orbit of \( u \), i.e.,

\[
H(v) = \begin{cases} 
\{v\} \times H_u, & v \in [u], \\
0, & \text{otherwise}.
\end{cases}
\]

Thus, \( \int H(v) \, d\mu_u \) is identified with \( L^2([u], \mu_u) \otimes H_u \) in the standard fashion. The groupoid \( G \) is represented on \( H \) according to the formula

\[
U_\gamma((s(\gamma), \xi)) = (r(\gamma), \xi),
\]

\( \xi \in H_u, s(\gamma) \in [u], \) i.e., \( \{U_\gamma\}_{\gamma \in G} \) is just the translation representation, and \( A \) is represented on \( H \) according to the formula

\[
\sigma(a) \cdot (v, \xi) = (v, \pi_v \circ \sigma_\gamma(a)\xi),
\]
Observe that the $C_0(G(0), A)$ acts as multipliers on $C^*(G; A)$ according to the formula $\phi \cdot f(\gamma) = \phi(r(\gamma))f(\gamma)$ for $\phi \in C_0(G(0), A)$ and $f \in C_c(G, r^*(A))$. The extension $\tilde{R}^u$ of $R^u$ to the multiplier algebra $C^*(G; A)$ represents $C_0(G(0), A)$ on $L^2([u], \mu_u) \otimes H_u$ via the equation

$$\tilde{R}^u(\phi)(v) = \pi_u \circ \sigma_\gamma(\phi(v))\xi(v),$$

again, where $\gamma$ is the unique element in $G$ with source $v$ and range $u$. It is clear from this that the weak closure of the algebra $\tilde{R}^u(C_0(G(0), A))$ is the full algebra of decomposable operators on $L^2([u], \mu_u) \otimes H_u$. It follows that any projection that commutes with $R^u(C^*(G; A))$ must be diagonal. On the other hand, it follows from the definition of the representation of $G$, $\{U_\gamma\}_{\gamma \in G}$, that a diagonal operator commuting with $R^u(C^*(G; A))$ must commute with $\{U_\gamma\}_{\gamma \in G}$, and therefore must be constant a.e. $\mu_u$. This proves that $R^u$, and hence $L^u$, is irreducible.

If $u$ and $v$ lie in the same orbit, it is clear that translation by the (unique) $\gamma$ with source $v$ and range $u$ implements an equivalence between $L^u$ and $L^v$. On the other hand, if $u$ and $v$ lie in different orbits, then $L^u$ and $L^v$ are disjoint. Indeed, the representations $N_u$ and $N_v$ of $C_0(G(0))$ obtained by restricting $\tilde{R}^u$ and $\tilde{R}^v$ to $C_0(G(0))$, viewed as a subalgebra of $M(C^*(G; A))$, are supported on the disjoint sets $[u]$ and $[v]$. Arguing just as we did in the proof of [10 Proposition 2.5], using [18 Lemma 4.15], we conclude $L^u$ and $L^v$ are disjoint.

Finally, to see that the map $u \mapsto L^u$ is continuous, observe that Hypothesis [11] guarantees that for each point $u \in G(0)$ we can find a neighborhood $V_u$ of $u$ on which the $H_v$’s can be chosen to be the fibres of a (topological) Hilbert bundle $\mathcal{H}$ and on which we can choose the $\pi_v$’s so that for any section $\phi \in C_0(G(0), A)$ that is supported on $V_u$ and any two $C_0$-sections of $\mathcal{H}$ over $V_u$, $\xi$ and $\eta$, the function $v \mapsto (\pi_v(\phi(v))\xi(v)|\eta(v))_{H_v}$ is continuous. It follows from the continuity of the Haar system that given such sections $\xi$ and $\eta$ of $\mathcal{H}$ and any two functions $g$ and $h$ in $C_c(G)$, the function $v \mapsto (L^u(f)(g \otimes \xi)|(h \otimes \eta))$ (where the inner products are taken in $L^2(\lambda_u) \otimes H_v$) is continuous for all $f \in C_c(G; r^*(A))$. This shows that the map $u \mapsto L^u$ is continuous.

**Corollary 4.** Assume that $G$ is principal and that $A$ is an elementary $C^*$-bundle over $G(0)$, satisfying Fell’s condition, on which $G$ acts. If $C^*(G; A)$ has continuous trace, then the map that sends $u \in G(0)$ to the unitary equivalence class of $L^u$ defines a continuous open surjection of $G(0)$ onto $C^*(G; A)\vee$ that is constant on $G$-orbits. In particular, orbits are closed and $G(0)/G$ is homeomorphic to $C^*(G; A)\vee$.

**Proof.** The proof is also essentially the same as the proof in [10 Proposition 2.5]. Here is an outline. Write $\Psi$ for the map $u \mapsto [L^u]$. Then by Lemma [8], $\Psi$ is continuous and constant on $G$-orbits. Thus $\Psi$ passes to a continuous map on $G(0)/G$ with the quotient topology (no matter how bad that might be). Since, however, $C^*(G; A)\vee$ is Hausdorff by hypothesis, we conclude that $G(0)/G$ is Hausdorff.

Suppose that $L$ is an irreducible representation of $C^*(G; A)$ and let $M$ be the representation of $C_0(G(0))$ obtained by extending $L$ to the multiplier algebra of $C^*(G; A)$ and then restricting to $C_0(G(0))$. The kernel $J$ of $M$ is the set of functions in $C_0(G(0))$ that vanish on a closed set $F$ in $G(0)$. Then $F$ is easily seen to be invariant. Indeed, one may do this directly or use the fact that it supports the
quasi-invariant measure associated to the disintegrated form of $L$ guaranteed by [10]. Further, since $L$ is irreducible, $F$ cannot be expressed as the union of two disjoint, closed, $G$-invariant sets. Since the quotient map from $G^{(0)}$ to $G^{(0)}/G$ is continuous and open, we may apply the lemma on page 222 of [13] to conclude that $F$ is an orbit closure. Since we now know that orbits are closed, $F$ is, in fact, an orbit. Thus $L$ factors through $C^*(G[[u]]; A)$. However, $G[[u]]$ is a transitive principal groupoid. A little reflection, using Theorem 3.1 of [9], reveals that every irreducible representation of $C^*(G[[u]]; A)$ is unitarily equivalent to $L^u$. □

We now assume the action of $G$ on $G^{(0)}$ is not proper and use Lemma 2.6 of [10] to choose a sequence $\{\gamma_n\} \subseteq G$ such that $\gamma_n \to \infty$ in the sense that $\{\gamma_n\}$ eventually escapes each compact subset of $G$, and such that $r(\gamma_n), s(\gamma_n) \to z$, for some $z \in G^{(0)}$. We shall fix this sequence for the remainder of the proof. We also choose a relatively compact neighborhood $U$ of $z \in G^{(0)}$ and a section $g$ in the Pedersen ideal of $C_0(G^{(0)}, A)$ such that $g$ is non-negative, compactly supported, and satisfies $\text{tr}(\pi_u(g(u))) \equiv 1$ on $U$. The fact that $A$ satisfies Fell’s condition guarantees that such choices are possible.

With these ingredients fixed, we want to build a special neighborhood $E$ of $z$ in $G$, following the analysis on pages 236–238 of [10]. First observe, as we have above, that since $G$ is principal, $r$ maps $G_z$ bijectively onto $[z]$ while $s$ maps $G^z$ bijectively onto $[z]$, where, recall, $[z]$ denotes the orbit of $z$. Since $[z]$ is closed by Corollary 4 while $r$ is continuous and open on $G$, we see that $r$ maps $G_z$ homeomorphically onto $[z]$. Likewise, $s$ maps $G^z$ homeomorphically onto $[z]$. Also, since $G$ is principal, multiplication induces a homeomorphism between $G_z \times G^z$ and $G[[z]]$. Let $N$ be the closed support of $g$, a compact subset of $G^{(0)}$, and set $F_z := G_z \cap r^{-1}([z] \cap N)$ and $F^z := G^z \cap s^{-1}([z] \cap N)$, obtaining compact subsets of $G_z$ and $G^z$, respectively. Then we see that if $\gamma \in G[[z]]$ and if $g(s(\gamma)) \neq 0$ and $g(r(\gamma)) \neq 0$, then $\gamma \in F_z F^z$.

According to Lemma 2.7 of [10], we may select symmetric, conditionally compact open neighborhoods $W_0$ and $W_1$ of $G^{(0)}$ such that $W_0 \subseteq W_1$. (Recall that a neighborhood $W$ of $G^{(0)}$ is conditionally compact in case $VW$ and $WV$ are relatively compact subsets of $G$ for each relatively compact subset $V$ in $G$.) We select such a pair, as we may, with the additional property that $F_z F^z \subseteq W_0 z W_0$. Then from the preceding paragraph, we see that if $\gamma \notin W_0 z W_0$, then either $g(s(\gamma)) = 0$ or $g(r(\gamma)) = 0$.

By construction,

$$W_0^{-1} z \setminus W_0 z \subseteq r^{-1}(G^{(0)} \setminus N).$$

So we may find relatively compact open neighborhoods $V_0$ and $V_1$ of $z$ in $G$ so that $V_0 \subseteq W_0, V_0 \subseteq V_1$, and so that

$$W_1^{-1} V_1 \setminus W_0 V_0 \subseteq r^{-1}(G^{(0)} \setminus N).$$

With these $V_0$ and $V_1$ so chosen, the special open neighborhood $E$ of $z$ in $G$ that we want is defined to be $E := W_0 V_0 W_0$.

Observe that we have

$$W_1^{-1} V_1 W_1^{-1} \setminus E = W_1^{-1} V_1 W_1^{-1} \setminus W_0 V_0 W_0 \subseteq r^{-1}(G^{(0)} \setminus N).$$

Set

$$g^1(\gamma) := \begin{cases} g(r(\gamma)), & \gamma \in W_1^{-1} V_1 W_1^{-1}, \\ 0, & \gamma \notin E. \end{cases}$$
Then, since \( W_1^{-1} V_1 W_1^{-1} \setminus E \subset r^{-1}(G(0) \setminus N) \), \( g^1 \) is a continuous section of \( r^*(A) \) on \( G \) that vanishes outside \( E \).

Observe the following containment relations among relatively compact sets: \( E^2 = W_0 V_0 W_0^2 V_0 W_0 \subset W_0^2 V_0 W_0^2 V_0 W_0 \subset W_0^2 V_0 W_0^2 V_0 W_0 \subset W_0^4 V_0 W_0^2 \). Hence, we may find a compactly supported function \( b \) on \( G \) such that \( 0 \leq b(\gamma) \leq 1 \) for all \( \gamma \), \( b \equiv 1 \) on \( E^2 \) and \( b \equiv 0 \) off \( W_0^4 V_0 W_0^4 \). Replacing \( b \) by \( \frac{b + b^2}{2} \), if necessary, we may assume that \( b \) is a selfadjoint element of the convolution algebra of scalar-valued functions \( C_c(G) \).

Define \( F(\gamma) := g(r(\gamma)) \sigma_{r}(g(s(\gamma))) b(\gamma) \). By our choices of \( g \) and \( b \), \( F \) belongs to \( C_c(G, r^*(A)) \), \( F \) is selfadjoint and

\[
L^u(F)\xi(\gamma) = \pi_u(\sigma^{-1}_\gamma(g(r(\gamma)))) \pi_u(\sigma^{-1}_\alpha(g(r(\alpha)))) \beta(\gamma\alpha) \xi(\alpha) d\lambda_u(\alpha)
\]

for all \( u \) by the definition of \( L^u \) (cf. [1]). Let \( P_{u,1} \) be the projection onto \( E_u := L^2(G_u \cap E) \otimes H_u \) and let \( P_{u,2} \) be the projection onto the orthocomplement, \( E_u := L^2(G_u \setminus E) \otimes H_u \). Then, if \( P_{u,1} \xi = \xi \), we see that

\[
L^u(F)\xi(\gamma) = \pi_u(\sigma^{-1}_\gamma(g(r(\gamma)))) \int_{G_u \cap E} \pi_u(\sigma^{-1}_\alpha(g(r(\alpha)))) \xi(\alpha) d\lambda_u(\alpha)
\]

\[
= \int_{G_u \cap E} \pi_u \circ \sigma^{-1}_\gamma(g^1(\gamma)) \pi_u \circ \sigma^{-1}_\alpha(g^1(\alpha)) \xi(\alpha) d\lambda_u(\alpha)
\]

for all \( \gamma \in G_u \cap E \) because \( b \) is identically 1 on \( E^2 \). However, by definitions of \( E \) and \( g^1 \), the equation persists when \( \gamma \in G_u \setminus E \), yielding 0. Thus, \( P_{u,1} \) commutes with \( L^u(F) \). Moreover, when \( u = z \), these formulas show that \( L^z(F)P_{z,1} = L^z(F) \).

We now want to show that \( L^u(F)P_{u,1} \geq 0 \) and we want to analyze the trace, \( \text{tr}(L^u(F)P_{u,1}) \). However, when \( \xi \) is in the range of \( P_{u,1} \), the formula for \( L^u(F)\xi \) shows that

\[
(L^u(F)\xi)(\gamma) = \int G_u \pi_u \circ \sigma^{-1}_\gamma(g^1(\gamma)) \pi_u \circ \sigma^{-1}_\alpha(g^1(\alpha)) \xi(\alpha) \xi(\gamma) d\lambda_u(\alpha) d\lambda_u(\gamma)
\]

\[
= \int \pi_u \circ \sigma^{-1}_\alpha(g^1(\alpha)) \xi(\alpha) \pi_u \circ \sigma^{-1}_\gamma(g^1(\gamma)) \xi(\gamma) d\lambda_u(\alpha) d\lambda_u(\gamma)
\]

\[
\geq 0.
\]

As for the trace, observe that if \( K_u \) is defined by the formula

\[
K_u(\gamma, \eta) = \pi_u(\sigma^{-1}_\gamma(g^1(\gamma))) \sigma^{-1}_\eta(g^1(\eta))
\]

on \( G_u \times G_u \), then our calculations show that \( K_u \) is continuous, positive semidefinite and supported on \( (G_u \cap E) \times (G_u \cap E) \), and that

\[
(L^u(F)P_{u,1})\xi(\gamma) = \int (K_u(\gamma, \eta) \xi(\eta), \xi(\gamma)) d\lambda_u(\eta) d\lambda_u(\gamma)
\]

Consequently, we may use Dufo’s generalization of Mercer’s theorem, [1] Proposition 3.1.1], and the fact that \( K_u(\gamma, \gamma) = \pi_u \circ \sigma^{-1}_\gamma(g(r(\gamma)))^2 \) to conclude that

\[
\text{tr}(L^u(F)P_{u,1}) = \int_{G_u \cap E} \text{tr}(\pi_u \circ \sigma^{-1}_\gamma(g(r(\gamma)))^2) d\lambda_u(\gamma)
\]

By our choice of \( g \), this expression is continuous in \( u \) and when \( u = z \) yields the value \( \text{tr}(L^z(F)) \).

We will show that there is a positive number \( a \) such that

\[
\| (L^u(\gamma_n)) P_{s(\gamma_n),2} \| \geq 2a
\]
eventually, where \((L^{s(\gamma_n)}(F)P_{s(\gamma_n),2})^+\) denotes the positive part of the selfadjoint operator \(L^{s(\gamma_n)}(F)P_{s(\gamma_n),2}\). Therefore, eventually
\[
\text{(the largest eigenvalue of } L^{s(\gamma_n)}(F)P_{s(\gamma_n),2} \text{)} \geq 2a.
\]
Assume that we have shown this, and set
\[
q(t) = \begin{cases} 
0, & t \leq a, \\
2(t-a), & a \leq t \leq 2a, \\
t, & 2a < t.
\end{cases}
\]
Then \(q(F)\) is a positive element in the Pedersen ideal of \(C^*(G,A)\), and so the function
\[
u \rightarrow \text{tr}(L^u(q(F))) = \text{tr}(L^u(q(F))P_{u,1}) + \text{tr}(L^u(q(F))P_{u,2})
\]
is finite and continuous in \(u\), with value \(\text{tr}(L^z(q(F)))\) at \(u = z\). (Recall that \(L^z(F) = L^z(F)P_{z,1}\) and so \(L^z(q(F)) = L^z(q(F))P_{z,1}\).) On the other hand, we showed that \(L^u(F)P_{u,1}\) is positive. Since \(P_{u,1}\) commutes with \(L^u(F)\), \(L^u(F)P_{u,1} = L^u(F^+)P_{u,1}\). But also we showed that \(u \rightarrow \text{tr}(L^u(F)P_{u,1})\) is continuous at \(z\). Consequently, so is \(u \rightarrow \text{tr}(L^u(F^+)P_{u,1})\). Since \(q(F) \leq F^+\), the function \(u \rightarrow \text{tr}(L^u(q(F))P_{u,1})\) is continuous by Lemma 4.4.2(i) in [3], with value \(\text{tr}(L^z(q(F))) = \text{tr}(L^z(q(F))P_{z,1})\) at \(u = z\), also. Therefore
\[
\lim_{u \to z} \text{tr}(L^u(q(F))P_{u,2}) = 0.
\]
Since the largest eigenvalue of \(L^{s(\gamma_n)}(F)P_{s(\gamma_n),2} \geq 2a\), the largest eigenvalue of \(L^{s(\gamma_n)}(q(F))P_{s(\gamma_n),2} \geq 2a\) also. Consequently,
\[
\liminf_n \text{tr}(L^{s(\gamma_n)}(q(F))P_{s(\gamma_n),2}) \geq 2a.
\]
This contradiction will complete the proof.

We will finish by verifying the asserted inequality [2]. To this end, choose an open neighborhood of \(z\) in \(G\), \(V_2\), that is contained in \(V_0\) and choose a conditionally compact neighborhood \(Y\) of \(G^{(0)}\) such that if \(v \in V_2\), then \(r(v)\) maps \(Yv\) into \(U\). Without loss of generality, we may assume that \(Y \subseteq W_0\). Observe that if \(\gamma_n \notin \overline{W_1^2 V_1 W_1^2}\), then for \(\gamma \notin Y\gamma_n, \gamma \notin E\). Indeed, if \(\gamma = \gamma'\gamma_n \in E \cap Y\gamma_n\), then \(\gamma_n \in (\gamma')^{-1}E \subseteq W_0^2 V_0 W_0 \subseteq \overline{W_1^2 V_1 W_1^2}\) contrary to assumption. So, since \(r(\gamma_n)\) and \(s(\gamma_n)\) are tending to \(z\), while \(\gamma_n\) eventually escapes \(\overline{W_1^2 V_1 W_1^2}\), we can conclude that for \(n\) sufficiently large, whenever \(\gamma\) lies in \(Y\gamma_n\), then \(\gamma \notin E\) while \(r(\gamma)\) and \(s(\gamma)\) lie in \(U\). From now on, we will assume that \(n\) is sufficiently large so that these conditions are satisfied.

Next observe that since for each \(n\), the map \(\gamma \rightarrow \pi_{s(\gamma_n)} \circ \sigma_{\gamma_n}^{-1}(g(r(\gamma)))\) defines a continuous family of rank 1 projections on the Hilbert space \(H_{s(\gamma_n)}\), we can find a Borel family of unit vectors \(\gamma \rightarrow v_\gamma^n\) such that \(\pi_{s(\gamma_n)} \circ \sigma_{\gamma_n}^{-1}(g(r(\gamma)))\) is the rank 1 projection determined by \(v_\gamma^n\).

Let \(h_n(\gamma) = 1_{Y\gamma_n}(\gamma) \times v_\gamma^n\) where \(1_{Y\gamma_n}\) denotes the characteristic function of \(Y\gamma_n\). Observe that if \(\gamma, \alpha \in Y\gamma_n\), then \(\gamma \alpha^{-1} \in Y\gamma_n\gamma_n^{-1}Y \subseteq YV_0 Y \subseteq W_0 V_0 W_0 = E\).
Consequently, \( b(\gamma \alpha^{-1}) = 1 \) and we may calculate to find that if \( \gamma \in Y_{\gamma_n} \), then

\[
L^{s(\gamma_n)}(F)h_n(\gamma) = \pi_{s(\gamma_n)}(\sigma_\alpha^{-1}(g(\tau(\alpha)))) \int \pi_{s(\gamma_n)}(\sigma_\alpha^{-1}(g(\tau(\alpha)))) b(\gamma \alpha^{-1}) h_n(\alpha) d\lambda_{s(\gamma_n)}(\alpha)
\]

\[
= \nu^s L_r(\gamma_n)(Y).
\]

Hence \( (L^{s(\gamma_n)}(F)h_n|_{h_n}) = \lambda_r(\gamma_n)(Y)^2 \) but by our assumption on \( \gamma_n, h_n \) lies in \( \mathcal{E}(\gamma_n) \), so

\[
((L^{s(\gamma_n)}(F)P_{s(\gamma_n),2})^+ h_n|_{h_n}) \geq (L^{s(\gamma_n)}(F)P_{s(\gamma_n),2} h_n|_{h_n}) = (L^{s(\gamma_n)}(F)h_n|_{h_n}) = \lambda_r(\gamma_n)(Y)^2.
\]

This shows that \( \| (L^{s(\gamma_n)}(F)P_{s(\gamma_n),2})^+ \| = \lambda_r(\gamma_n)(Y) \) provided \( n \) is sufficiently large. But the continuity of the Haar system implies that \( \liminf \lambda_r(\gamma_n)(Y) > 0 \), as \( n \to \infty \). This verifies equation (2) and completes the proof of Theorem 1.

**References**


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