

CRITICAL EXPONENTS OF DISCRETE GROUPS AND L^2 -SPECTRUM

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ABSTRACT. Let G be a noncompact semisimple Lie group and Γ an arbitrary discrete, torsion-free subgroup of G . Let $\lambda_0(M)$ be the bottom of the spectrum of the Laplace-Beltrami operator on the locally symmetric space $M = \Gamma \backslash X$, and let $\delta(\Gamma)$ be the exponent of growth of Γ . If G has rank 1, then these quantities are related by a well-known formula due to Elstrodt, Patterson, Sullivan and Corlette. In this note we generalize that relation to the higher rank case by estimating $\lambda_0(M)$ from above and below by quadratic polynomials in $\delta(\Gamma)$. As an application we prove a rigidity property of lattices.

1. INTRODUCTION

In the study of geometric aspects of discrete subgroups of semisimple Lie groups the main interest has been in *lattices* (a lattice is a discrete subgroup Γ of a Lie group G such that the coset space $\Gamma \backslash G$ has finite Haar measure). Indeed, lattices enjoy several remarkable properties. For instance, in higher rank Lie groups, lattices are strongly rigid in the sense of G. D. Mostow, super-rigid in the sense of G. A. Margulis and, in particular, arithmetic (see [15], [17] and [25]). On the other hand, there are various interesting classes of discrete subgroups of semisimple Lie groups that are *not* lattices. First of all there is the elaborated theory of Kleinian groups (i.e., discrete subgroups of $PSL(2, \mathbb{C})$). These groups are closely related to the geometry and topology of three-manifolds and often admit interesting deformations (see e.g. [24]). In the higher rank case there are, for instance, the generalized Schottky groups constructed by Y. Benoist, the groups considered by V. G. Kac and E. B. Vinberg and those studied by G. A. Margulis and G. A. Soifer (see [3], [12] and [16] respectively).

In contrast to the lattice case and the case of Kleinian groups, one knows only very little about discrete subgroups of *infinite* covolume in higher rank Lie groups. We mention the work of Y. Guivarc'h and J.-P. Conze on limit sets and that of P. Albuquerque and J.-F. Quint on Patterson-Sullivan measures (see e.g. [9], [6] and [1], [19], [20]).

In the following, G denotes a connected, semisimple Lie group without compact factors and with trivial center. Let Γ be an arbitrary discrete, torsion-free subgroup

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of G . For a maximal compact subgroup K of G , $X = G/K$ is a Riemannian symmetric space of noncompact type on which G and Γ act by isometries. Since Γ is torsion-free, the quotient manifold $M = \Gamma \backslash X$ is a locally symmetric space.

In this note we study two numerical quantities which measure the size of such groups and manifolds. The first one, the *critical exponent* (or *exponent of growth*) of a discrete group $\Gamma \subset G$, is defined as

$$\delta(\Gamma) := \limsup_{R \rightarrow \infty} \frac{\log n_R}{R} \geq 0$$

where n_R is the number of points in some orbit $\Gamma \cdot x \subset X$ which are contained in a ball of radius R about x (using the triangle inequality it is easy to see that $\delta(\Gamma)$ does not depend on the chosen point $x \in X$). The critical exponent of Γ thus measures the exponential growth rate of Γ -orbits in X . This rate is bounded from above by the volume growth rate of X (also called volume entropy). The latter is equal to $2\|\rho\|$, where 2ρ is the sum of positive restricted roots counted with multiplicity (see Section 2). Examples of subgroups Γ with minimal critical exponent, $\delta(\Gamma) = 0$, are, for instance, discrete abelian subgroups generated by semisimple elements of G . Lattices in G have maximal critical exponent $\delta(\Gamma) = 2\|\rho\|$ (see Section 2 where further examples are discussed).

The second quantity that we are going to consider is $\lambda_0(M)$, the *bottom of the spectrum of the Laplace-Beltrami operator* on the locally symmetric space $M = \Gamma \backslash X$.

If the (real) rank of the semisimple group G is 1, then the above two quantities are known to be equivalent. More precisely, the following assertion holds:

Theorem (Elstrodt-Patterson-Sullivan-Corlette). *If $\text{rank}_{\mathbb{R}} G = 1$, then*

$$\begin{aligned} \lambda_0(M) &= \|\rho\|^2 \quad \text{if } 0 \leq \delta(\Gamma) \leq \|\rho\| \quad \text{and} \\ \lambda_0(M) &= \delta(\Gamma)(2\|\rho\| - \delta(\Gamma)) \quad \text{if } \|\rho\| \leq \delta(\Gamma) \leq 2\|\rho\|. \end{aligned}$$

For Fuchsian groups these equations have been proved by J. Elstrodt [8] and S. Patterson [18], for discrete groups of isometries in higher-dimensional hyperbolic spaces the result is due to D. Sullivan [23], and for discrete subgroups of semisimple Lie groups of rank one to K. Corlette [7].

The goal of this note is to generalize the previous theorem to the higher rank case by estimating $\lambda_0(M)$ from above and below by quadratic polynomials in $\delta(\Gamma)$. Our main result (compare Theorem 1) is the following.

Theorem A. *Let Γ be a torsion-free, discrete subgroup of a semisimple Lie group G without compact factors and with trivial center. Then the following estimates hold:*

$$\begin{aligned} &\text{if } \delta(\Gamma) \in [0, \rho_m], \text{ then } \lambda_0(M) = \|\rho\|^2; \\ &\text{if } \delta(\Gamma) \in [\rho_m, \|\rho\|], \text{ then } \|\rho\|^2 - (\delta(\Gamma) - \rho_m)^2 \leq \lambda_0(M) \leq \|\rho\|^2 \text{ and} \\ &\text{if } \delta(\Gamma) \in [\|\rho\|, 2\|\rho\|], \text{ then} \\ &\quad \max\{0, \|\rho\|^2 - (\delta(\Gamma) - \rho_m)^2\} \leq \lambda_0(M) \leq \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2. \end{aligned}$$

As an application we obtain the following characterization of lattices among all discrete subgroups of a given semisimple group G that has Kazhdan's property (T) (see Theorem 2 for a more precise statement).

Theorem B. *Let Γ be a torsion-free, discrete subgroup of a semisimple Lie group G without compact factors and with trivial center. If G has Kazhdan's property (T), then Γ is a lattice iff $\delta(\Gamma) = 2\|\rho\|$.*

Notation. We write $f \approx g$ for two real-valued functions f and g if there is a constant $C \geq 1$ such that $\frac{1}{C}f(x) \leq g(x) \leq Cf(x)$ for all x .

2. CRITICAL EXPONENTS

2.1. Definitions. Let d denote the distance function associated to a G -invariant Riemannian metric on $X = G/K$. The *Poincaré series* of a discrete subgroup Γ of G is defined as

$$P_s(x, y) := \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}, \quad x, y \in X, s \in \mathbb{R}.$$

It is then well known that, for any $x, y \in X$, $P_s(x, y)$ converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$ (see e.g. [22]).

Let \mathfrak{g} be the Lie algebra of the semisimple group G , let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, and let \mathfrak{a} be a maximal abelian subalgebra in \mathfrak{p} . Let further Σ^+ denote the set of positive restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ with respect to some Weyl chamber \mathfrak{a}^+ (see e.g. [10], Chapter VI, for definitions). We also denote by $m_\alpha = \dim \mathfrak{g}_\alpha$ the multiplicity of $\alpha \in \Sigma^+$ and set $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*$. Since \mathfrak{g} is semisimple the Killing form $\langle \cdot, \cdot \rangle$ is nondegenerate and defines an isomorphism $\mathfrak{a}^* \simeq \mathfrak{a}$; $\alpha \mapsto \vec{\alpha}$ by $\langle \vec{\alpha}, H \rangle := \alpha(H)$ for all $H \in \mathfrak{a}$; finally, we set $\langle \alpha, \beta \rangle := \langle \vec{\alpha}, \vec{\beta} \rangle$.

2.2. The upper bound for critical exponents. The volume of a ball of radius R about a point x in a symmetric space X of noncompact type and rank r is given by

$$\text{vol } B_R(x) \approx R^{\frac{r-1}{2}} e^{2\|\rho\|R};$$

see, e.g., [11] for a proof. Thus

$$\limsup_{R \rightarrow \infty} \frac{1}{R} \log \text{vol } B_R(x) = 2\|\rho\|.$$

For any discrete subgroup Γ we have $n_R = |B_R(x) \cap \Gamma \cdot y| \leq \text{const. vol } B_R(x)$. We thus conclude from the definition and the above approximation that $\delta(\Gamma) \leq 2\|\rho\|$.

2.3. Some examples.

2.3.1. Abelian groups of hyperbolic isometries. Let $G = KAN$ be an Iwasawa decomposition of the semisimple Lie group G . Pick elements a_1, \dots, a_m of A with $m \leq r = \text{rank}(X)$ such that $\log a_1, \dots, \log a_m \in \mathfrak{a}$ are linearly independent. If Γ is the discrete subgroup of G generated by a_1, \dots, a_m , then $\delta(\Gamma) = 0$. To see this note that the orbit of $x \in X = G/K$ under A is a totally geodesic flat submanifold. Thus the growth of $\Gamma \subset A$ in X is the same as that in the flat Ax , i.e., Euclidean: $n_R \approx R^m$. Hence for these groups, $\delta(\Gamma)$ attains the minimal possible value.

2.3.2. *Discrete groups containing unipotent elements.* If $\Gamma \subset G = SL(n, \mathbb{R})$ contains unipotent elements, then $\delta(\Gamma) > c(G) > 0$ where $c(G)$ is some universal constant.

Proof. Let $e \neq u \in \Gamma$ be unipotent and let $\Gamma^* := \langle u \rangle \subset \Gamma$ be the infinite cyclic group generated by u . Since it is clear from that definition that $\delta(\Gamma^*) \leq \delta(\Gamma)$, it suffices to prove the assertion for Γ^* . For $G = SL(n, \mathbb{R})$ and $X = SL(n, \mathbb{R})/SO(n)$ we have the well-known facts

$$d(gx, x) \approx d_G(g, e) \approx \log(1 + \|g - e\|) \quad \text{and} \quad \|u^m\| \approx |m|;$$

see, e.g., [14]. We thus get

$$\sum_{m=-\infty}^{\infty} e^{-sd(u^m x, x)} \approx \sum_{k=0}^{\infty} 2e^{-sa \log(1+bk)} = \sum_{k=0}^{\infty} 2(1+bk)^{-as}$$

for some positive constants a and b . Hence $0 < \frac{1}{a} = \delta(\Gamma^*) \leq \delta(\Gamma)$. □

2.3.3. *Generalized Schottky groups.* These groups have been constructed by Y. Benoist in [3]. As an abstract group a generalized Schottky group Γ_g is isomorphic to a free group with g generators (we assume in the following that the set S of generators is symmetric, i.e., $s \in S$ implies $s^{-1} \in S$ and thus $|S| = 2g$). Let Γ_g be endowed with the word metric d_Γ , which is defined as follows: $d_\Gamma(e, \gamma) = l$ if γ can be written as a minimal word of the form $\gamma = s_1 \cdots s_l$ with $s_i \in S$.

We choose a base point x_0 in X , and we set $c(g) := \max_{s \in S} d_X(x_0, sx_0)$. For $R \in \mathbb{R}^+$, denote by n_R^Γ (resp. n_R^X) the number of $\gamma \in \Gamma_g$ with $d_\Gamma(e, \gamma) \leq R$ (resp. $d_X(x_0, \gamma x_0) \leq R$). Then we have, for $\gamma \in \Gamma_g$ and $k \in \mathbb{N}$, firstly that $n_k^\Gamma = 2g(2g - 1)^{k-1}$; and secondly that

$$\begin{aligned} d_X(x_0, \gamma x_0) &= d_X(x_0, s_1 \cdots s_l x_0) \leq d_X(x_0, s_1 x_0) + d_X(s_1 x_0, s_1 \cdots s_l x_0) \\ &= d_X(x_0, s_1 x_0) + d_X(x_0, s_2 \cdots s_l x_0) \leq c(g) + d_X(x_0, s_2 \cdots s_l x_0). \end{aligned}$$

Hence by induction $d_X(x_0, \gamma x_0) \leq c(g)l = c(g)d_\Gamma(e, \gamma)$ and therefore

$$n_k^\Gamma = 2g(2g - 1)^{k-1} \leq n_{kc(g)}^X.$$

Eventually we thus get the following lower bound for the critical exponent of a generalized Schottky group:

$$\delta(\Gamma_g) = \limsup_{R \rightarrow \infty} \frac{\log n_R^X}{R} \geq \frac{1}{c(g)} \log(2g - 1) > 0.$$

2.3.4. *Zariski-dense discrete subgroups.* A discrete Zariski-dense subgroup Γ of a noncompact semisimple Lie group satisfies $0 < \delta(\Gamma)$ (see, e.g., [1], Proposition 4.7).

2.3.5. *Lattices.* If Γ is a lattice in G , then $\delta(\Gamma) = 2\|\rho\|$. This is proved in [1], Thm. 7.4. For lattices the critical exponent thus attains the maximal possible value (compare 2.2).

3. THE BOTTOM OF THE L^2 -SPECTRUM

3.1. Definition. We define the *Laplace-Beltrami operator* (see [5] and [21]) first on C^∞ -functions with compact support on the locally symmetric space $M = \Gamma \backslash X$ by $\Delta_M := -\operatorname{div} \operatorname{grad}$. This operator has a unique selfadjoint extension to $L^2(M)$, which we also denote by Δ_M . Let $\operatorname{Spec} \Delta_M$ denote the L^2 -spectrum of Δ_M and define the *bottom of the spectrum* $\lambda_0(M) := \inf \operatorname{Spec} \Delta_M \geq 0$. Notice that the closed set $\operatorname{Spec} \Delta_M$ is contained in the interval $[\lambda_0(M), \infty)$.

3.2. A characterization of $\lambda_0(M)$. The following proposition has been proved by K. Corlette (see [7], Section 4).

Proposition 1. (Corlette) (a) *The Green function (or resolvent kernel) g^λ for the resolvent $(\Delta_M - \lambda I)^{-1}$ can be written in the form*

$$(*) \quad g^\lambda(\tilde{x}, \tilde{y}) = \sum_{\gamma \in \Gamma} G^\lambda(x, \gamma y), \quad \tilde{x}, \tilde{y} \in M = \Gamma \backslash X$$

where G^λ is the Green function for $(\Delta_X - \lambda I)^{-1}$ on $X = G/K$ and x (resp. y) in X is a lift of \tilde{x} (resp. \tilde{y}).

(b) *The bottom $\lambda_0(M)$ of the L^2 -spectrum of M is the supremum of all $\lambda \in [0, \|\rho\|^2]$ for which the right-hand side of (*) converges for all $x \neq y$.*

In order to state the next proposition, we need some additional notation. Let $G = K\overline{A^+}K$ be a Cartan decomposition, and let x_0 be the base point of the globally symmetric space X corresponding to the maximal compact subgroup K of G . Any element $g \in G$ can be written as $g = k \exp Hk'$ where $H \in \overline{\mathfrak{a}^+}$ is uniquely determined by g (see [10], Ch. IX.1).

For any x, y in the homogeneous space $X = G/K$, there are $g, h \in G$ such that $x = gx_0$ and $y = hx_0$. For $\gamma \in \Gamma \subset G$ we then have $d(x, \gamma y) = d(x_0, g^{-1}\gamma hx_0)$ and $g^{-1}\gamma h = k \exp Hk'$. The isometries g and h are not unique; if we also have $x = \bar{g}x_0$ and $y = \bar{h}x_0$, then $\bar{g} = gk_1, \bar{h} = hk_2$ for $k_1, k_2 \in K$. But since $\bar{g}^{-1}\gamma\bar{h} = k_1^{-1}k \exp Hk'k_2$ the element H is well defined and (for x and y fixed) will be denoted by $H(\gamma)$ in what follows. In conclusion we thus have

$$d(x, \gamma y) = d(x_0, k \exp H(\gamma)k'x_0) = d(x_0, \exp H(\gamma)x_0) = \|H(\gamma)\|.$$

Notice that the last equality holds since Ax_0 is totally geodesic in X .

Let $\Sigma^{++} \subset \Sigma^+$ be the set of indivisible positive roots (i.e., those positive roots α for which $\frac{1}{2}\alpha$ is not a root). The next proposition has been proved by J.-P. Anker and L. Ji (see [2], 4.2.).

Proposition 2. (Anker–Ji) *For any $\lambda < \|\rho\|^2$ and any $H \in \overline{\mathfrak{a}^+}$ with $\|H\|$ sufficiently large, the Green function G^λ on the symmetric space X of rank r satisfies*

$$G^\lambda(\exp Hx_0, x_0) \approx \prod_{\alpha \in \Sigma^{++}} \{1 + \alpha(H)\} \|H\|^{\frac{1-r}{2} - |\Sigma^{++}|} e^{-\rho(H) - \|H\|} \sqrt{\|\rho\|^2 - \lambda}.$$

The following proposition characterizes the bottom of the L^2 -spectrum.

Proposition 3. *Let Γ be a torsion-free, discrete subgroup of a noncompact, semi-simple Lie group G . Then the bottom $\lambda_0(M)$ of the L^2 -spectrum of the locally symmetric space $M = \Gamma \backslash G/K$ is the supremum of all $\lambda \in [0, \|\rho\|^2]$ such that the*

series

$$s^\lambda(\check{x}, \check{y}) := \sum_{\gamma \in \Gamma} \prod_{\alpha \in \Sigma^{++}} \{1 + \alpha(H(\gamma))\} \|H(\gamma)\|^{\frac{1-r}{2} - |\Sigma^{++}|} e^{-\rho(H(\gamma)) - \|H(\gamma)\| \sqrt{\|\rho\|^2 - \lambda}}$$

converges for some lifts x (resp. y) to X of \check{x} (resp. \check{y}) in M and where $H(\gamma)$ is the unique element in $\overline{\mathfrak{a}^+}$ such that $d(x_0, \exp H(\gamma)x_0) = d(x, \gamma y)$.

Proof. Since Δ_X commutes with isometries, one has for all $x, y \in X$ that $G^\lambda(gx, gy) = G^\lambda(x, y)$ for all $g \in G$. In particular, we have (using the same notation as above)

$$\begin{aligned} G^\lambda(x, \gamma y) &= G^\lambda(gx_0, \gamma hx_0) = G^\lambda(x_0, g^{-1}\gamma hx_0) \\ &= G^\lambda(x_0, k \exp H(\gamma)k'x_0) = G^\lambda(x_0, \exp H(\gamma)x_0). \end{aligned}$$

Since the convergence of the series (*) in Proposition 1(a) depends only on the asymptotic behaviour of G^λ for x and y far away from one another, the claim follows from Proposition 2. □

3.3. Relation to the critical exponent. Using the definitions in 2.1 we denote by $\overline{\mathfrak{a}^+}(1)$ the set of all unit vectors in the closed Weyl chamber $\overline{\mathfrak{a}^+}$. We define $\rho_m := \min_{H \in \overline{\mathfrak{a}^+}(1)} \rho(H)$. Since $\vec{\rho} \in \mathfrak{a}^+$ (see [4], Ch. VI, 1.10, Prop. 29) we have $\max_{H \in \overline{\mathfrak{a}^+}(1)} \rho(H) = \|\rho\|$. We then get $\rho_m \|H(\gamma)\| \leq \rho(H(\gamma)) \leq \|\rho\| \|H(\gamma)\|$ for all $\gamma \in \Gamma$. We can now state our main result which estimates $\lambda_0(M)$ in terms of $\delta(\Gamma)$.

Theorem 1. *Let Γ be a torsion-free, discrete subgroup of a semisimple Lie group G without compact factors and with trivial center. Then the following estimates hold:*

- (1) if $\delta(\Gamma) \in [0, \rho_m]$, then $\lambda_0(M) = \|\rho\|^2$;
- (2) if $\delta(\Gamma) \in [\rho_m, \|\rho\|]$, then $\|\rho\|^2 - (\delta(\Gamma) - \rho_m)^2 \leq \lambda_0(M) \leq \|\rho\|^2$ and
- (3) if $\delta(\Gamma) \in [\|\rho\|, 2\|\rho\|]$, then

$$\max\{0; \|\rho\|^2 - (\delta(\Gamma) - \rho_m)^2\} \leq \lambda_0(M) \leq \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2.$$

Proof. Since $\alpha(H(\gamma)) \geq 0$ for all $\alpha \in \Sigma^{++}$ and all $H(\gamma)$, we have

$$1 \leq \prod_{\alpha \in \Sigma^{++}} \{1 + \alpha(H(\gamma))\}.$$

Moreover, if we fix some (large) radius R_0 , we have for $\|H(\gamma)\| \geq R_0$ also the estimate

$$\prod_{\alpha \in \Sigma^{++}} \left\{ \frac{1 + \alpha(H(\gamma))}{\|H(\gamma)\|} \right\} \leq \prod_{\alpha \in \Sigma^{++}} \left\{ \frac{1}{R_0} + c_\alpha \right\} =: C,$$

where $c_\alpha := \max_{H \in \overline{\mathfrak{a}^+}(1)} \alpha(H) > 0$.

We next set $s_{\min}(\lambda) := \rho_m + \sqrt{\|\rho\|^2 - \lambda}$ and $s_{\max}(\lambda) := \|\rho\| + \sqrt{\|\rho\|^2 - \lambda}$. The series in Proposition 3 and the above two estimates then yield the inequalities

$$\sum_{\gamma \in \Gamma} \|H(\gamma)\|^a e^{-s_{\max}(\lambda)\|H(\gamma)\|} \leq s^\lambda(\check{x}, \check{y}) \leq C \sum_{\gamma \in \Gamma} \|H(\gamma)\|^b e^{-s_{\min}(\lambda)\|H(\gamma)\|}$$

with $a = \frac{1-r}{2} - |\Sigma^{++}|$ and $b = \frac{1-r}{2}$. For $k \in \mathbb{N}$, let r_k denote the number of Γ -orbit points of x_0 lying in a half-open shell of radii in $(k - \frac{1}{2}, k + \frac{1}{2}]$ centered at x_0 . Using

the notation from Section 2 and observing that $\|H(\gamma)\| = d(x, \gamma y)$, we can write the above inequalities also in the following form:

$$(**) \quad \sum_{k=0}^{\infty} r_k k^a e^{-s_{\max}(\lambda)k} \leq s^\lambda(\tilde{x}, \tilde{y}) \leq C \sum_{k=0}^{\infty} r_k k^b e^{-s_{\min}(\lambda)k}$$

with $\lambda \leq \|\rho\|^2$. For the following note that the series $\sum_{k=0}^{\infty} r_k k^\beta e^{-sk}$, $\beta \in \mathbb{R}$ converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$.

Step (a): $\delta(\Gamma) \leq \rho_m$. We have $s_{\min}(\lambda) > \rho_m \geq \delta(\Gamma)$ for all $\lambda \in (-\infty, \|\rho\|^2)$. Since the Poincaré series $P_s(x, y)$ converges for any $s > \delta(\Gamma)$, the right-hand side of $(**)$ and hence $s^\lambda(\tilde{x}, \tilde{y})$ converges for all $\lambda < \|\rho\|^2$ and therefore, by Proposition 3, $\lambda_0(M) = \|\rho\|^2$.

Step (b): $\delta(\Gamma) \geq \rho_m$. Since the series $s^\lambda(\tilde{x}, \tilde{y})$ in Proposition 3 diverges for any $\lambda > \lambda_0(M)$, the right-hand side of $(**)$ diverges for any $\lambda > \lambda_0(M)$. From the definition of the critical exponent we then get

$$\begin{aligned} \delta(\Gamma) \geq s_{\min}(\lambda_0(M)) &\iff 0 \leq \delta(\Gamma) - \rho_m \leq \sqrt{\|\rho\|^2 - \lambda_0(M)} \\ &\iff (\delta(\Gamma) - \rho_m)^2 \leq \|\rho\|^2 - \lambda_0(M) \iff \lambda_0(M) \geq \|\rho\|^2 - (\delta(\Gamma) - \rho_m)^2. \end{aligned}$$

Step (c): $\delta(\Gamma) \geq \|\rho\|$. Since the series $s^\lambda(\tilde{x}, \tilde{y})$ converges for any $\lambda < \lambda_0(M)$, the left-hand side of $(**)$ converges for any $\lambda < \lambda_0(M)$. The definition of the critical exponent then yields that

$$\begin{aligned} \delta(\Gamma) \leq s_{\max}(\lambda_0(M)) &\iff (\delta(\Gamma) - \|\rho\|)^2 \leq \|\rho\|^2 - \lambda_0(M) \\ &\iff \lambda_0(M) \leq \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2. \end{aligned}$$

This completes the proof of Theorem 1. □

In the special case where $\text{rank}_{\mathbb{R}}G = 1$ ($= \text{rank } X$) we have $\rho_m = \|\rho\|$ and Theorem 1 thus reduces to the following.

Corollary 1. (Elstrodt–Patterson–Sullivan–Corlette) *If $\text{rank}_{\mathbb{R}}G = 1$ and Γ and M are as in Theorem 2, then*

$$\begin{aligned} \lambda_0(M) &= \|\rho\|^2 \quad \text{if } 0 \leq \delta(\Gamma) \leq \|\rho\| \quad \text{and} \\ \lambda_0(M) &= \delta(\Gamma)(2\|\rho\| - \delta(\Gamma)) \quad \text{if } \|\rho\| \leq \delta(\Gamma) \leq 2\|\rho\|. \end{aligned}$$

4. A RIGIDITY PHENOMENON FOR LATTICES

In the rank one case, K. Corlette proved the following remarkable gap or rigidity theorem (see [7], Thm. 4.4):

Theorem (Corlette). *Let Γ be a discrete subgroup of the isometry group of a quaternionic hyperbolic space (resp. the Cayley hyperbolic plane) that is not a lattice. Then the critical exponent of Γ is $\leq 4n$ (resp. ≤ 16).*

Notice that for the above hyperbolic spaces $2\|\rho\|$ (which by 2.3.5 is equal to the critical exponent of lattices) equals $4n + 2$ (resp. 22). That result of Corlette is extended to the higher rank case by the following theorem, which characterizes lattices among discrete subgroups for all those symmetric spaces whose isometry groups have Kazhdan’s property (T). It is well known that a connected, semisimple Lie group has property (T) if and only if it has no simple factors locally isomorphic to the rank one groups $SO(n, 1)$ or $SU(n, 1)$ (see [15], [25]).

Theorem 2. *Let G be a semisimple Lie group G without compact factors and with trivial center. Let Γ be a discrete, torsion-free subgroup of G that is not a lattice. If G has Kazhdan's property (T), then there is a constant $c^*(G) > 0$ depending on G but not on Γ such that $\delta(\Gamma) \leq 2\|\rho\| - c^*(G)$.*

Proof. The assumption that G has property (T) implies that there is a constant $c(G) > 0$ depending on G but not on Γ such that $0 < c(G) \leq \lambda_0(\Gamma \backslash X)$ (see [13], Thm. 2). We may assume that $\delta(\Gamma) \geq \|\rho\|$. Then by Theorem 1 we have $0 < c(G) \leq \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2$ and thus $\delta(\Gamma) \leq \sqrt{\|\rho\|^2 - c(G)} + \|\rho\| =: 2\|\rho\| - c^*(G)$ with $c^*(G) > 0$. \square

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