THE SHARP LOWER BOUND FOR THE FIRST POSITIVE EIGENVALUE OF A SUB-LAPLACIAN ON A PSEUDO-HERMITIAN MANIFOLD

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Abstract. This paper studies, using the Bochner technique, a sharp lower bound of the first eigenvalue of a subelliptic Laplace operator on a strongly pseudoconvex CR manifold in terms of its pseudo-Hermitian geometry. For dimensions greater than or equal to 7, the lower bound under a condition on the Ricci curvature and the torsion was obtained by Greenleaf. We give a proof for all dimensions greater than or equal to 5. For dimension 3, the sharp lower bound is proved under a condition which also involves a distinguished covariant derivative of the torsion.

1. Introduction and main results

Let $M$ be a $(2n+1)$-dimensional strongly pseudoconvex CR manifold and $H(M)$ the structure bundle, where $H(M)$ is a subbundle of the complexified tangent bundle $T_C(M)$ of which each fiber is an $n$-dimensional complex vector space. Let $\theta$ be a real nonvanishing one-form on $M$ that annihilates $H(M) \oplus \overline{H(M)}$. Then, $(M, \theta)$ is a strongly pseudoconvex pseudo-Hermitian manifold in the sense of Webster [9]. Locally, one can choose $n$ complex one-forms $\theta^\alpha$, so that $(\theta, \theta^\alpha, \theta^\overline{\alpha})$ form a basis of complex covectors and

\begin{equation}
\begin{aligned}
d\theta &= i \theta^\alpha \wedge \theta^\overline{\alpha}, \\
\theta^\overline{\alpha} &= \overline{\theta^\alpha}.
\end{aligned}
\end{equation}

The local coframe $(\theta, \theta^\alpha, \theta^\overline{\alpha})$ is uniquely determined up to

\begin{equation}
\begin{aligned}
\theta &= \theta', \\
\theta^\alpha &= \theta'^\beta U^\alpha_\beta, \\
\theta^\overline{\alpha} &= \theta'^\beta U^\overline{\alpha}_\beta
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
U^\alpha_\beta U^\overline{\gamma}_\beta &= \delta^\alpha^\gamma, \\
U^\overline{\alpha}_\beta &= U^\overline{\gamma}_\beta.
\end{aligned}
\end{equation}

If we compare the dual frame

\begin{equation}
\begin{aligned}
X_0 &= \overline{X}_0, \\
X_\alpha &= \overline{X}_\alpha
\end{aligned}
\end{equation}

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to \((\theta, \theta^\alpha, \theta^\beta)\), then the transformation (1.2) gives
\[
X'_0 = X_0, \quad X'_\alpha = U^\alpha_\beta X_\beta, \quad X'_\beta = U^\beta_\alpha X_\alpha,
\]
which singles out a unique transversal \(X_0\) to \(H(M) \oplus \overline{H(M)}\). Furthermore,
\[
d\theta^\alpha = \theta^\beta \wedge w^\beta_\alpha + \theta \wedge \tau^\alpha
\]
where \(w^\beta_\alpha\) are connection 1-forms that are skew-Hermitian:
\[
w^\beta_\alpha = -w^\alpha_\beta
\]
and the \(\tau^\alpha\) are torsion 1-forms of type \((0,1)\):
\[
\tau^\alpha = A^\alpha_\beta \theta^\beta, \quad A^\alpha_\beta = A^\alpha_\beta.
\]
Moreover, if we define curvature 2-forms \(\Omega^\alpha_\beta\) by
\[
\Omega^\alpha_\beta = dw^\alpha_\beta - w^\gamma_\beta \wedge w^\alpha_\gamma - i\theta^\beta \wedge \tau^\alpha + i\tau^\beta \wedge \theta^\alpha,
\]
then
\[
\Omega^\alpha_\beta = R_{\beta \alpha \tau \sigma} \theta^\rho \wedge \theta^\sigma + \lambda_{\beta \alpha \tau \sigma} \wedge \theta,
\]
where \(\lambda_{\beta \alpha \tau \sigma}\) are 1-forms and the curvature tensor components \(R_{\beta \alpha \tau \sigma}\) satisfy
\[
R_{\beta \alpha \tau \sigma} = R_{\tau \sigma \beta \alpha} = R_{\alpha \beta \tau \sigma}.
\]
Let
\[
\Gamma^\alpha_\beta_j = \omega^\alpha_\beta(X_j), \quad \Gamma^\alpha_\beta_j = \omega^\alpha_\beta(X_j)
\]
where \(\alpha, \beta \in I\) with \(I = \{1, \cdots, n\}\) and \(j \in \{0\} \cup I \cup \overline{I}\). Then \(R_{\alpha \beta \rho \sigma}\) can also be written as
\[
R_{\alpha \beta \rho \sigma} = X_\rho (\Gamma^\beta_\alpha_\sigma) - X_\rho (\Gamma^\beta_\alpha_\sigma) + \Gamma^\beta_\alpha_\rho \Gamma^\alpha_\sigma - \Gamma^\beta_\alpha_\rho \Gamma^\alpha_\sigma + \Gamma^\gamma_\alpha_\rho \Gamma^\beta_\sigma - \Gamma^\gamma_\alpha_\rho \Gamma^\beta_\sigma + i\delta^\rho_\sigma \Gamma^\beta_\alpha.
\]
For \(X = \sum_{\alpha=1}^n x^\alpha X_\alpha\) where \(x^\alpha\) are local functions, we let
\[
\text{Ric}(X, X) = R_{\alpha \beta \rho \sigma} x^\alpha x^\beta, \quad R_{\alpha \beta} = g^{\rho \sigma} R_{\alpha \beta \rho \sigma} = R_{\alpha \beta \rho \sigma},
\]
and
\[
\text{Tor}(X, X) = i(A_{\alpha \beta \gamma \delta} x^\alpha x^\beta - A_{\alpha \beta} x^\alpha x^\beta).
\]
Then the covariant derivatives and the sub-Laplacian of a function \(f\) on \(M\) are given by
\[
f_j = X_j f, \quad f_{\alpha \beta} = X_j f_\alpha - \Gamma^\gamma_\alpha_j f_\gamma, \quad f_{\alpha \rho} = X_j f_\sigma - \Gamma^\gamma_\alpha_j f_\gamma
\]
and
\[
\tilde{\Delta} f = 2\text{Re}(\text{tr}(\pi + D^2 f)) = \sum_{\alpha} f_{\alpha \alpha} + f_{\alpha \rho}.
\]
Let \(\lambda_1\) be the first positive eigenvalue of \(\tilde{\Delta}\). We shall prove the following theorems.

**Theorem 1.1.** Let \(n \geq 2\) and let \(M\) be a \((2n + 1)\)-dimensional strongly pseudo-convex pseudo-Hermitian manifold in the sense of Webster. If
\[
\text{Ric}_m(X, X) + (n/2) \text{Tor}_m(X, X) \geq k_0 g_m(X, X)
\]
for all \(m \in M\) and \(X \in H_m(M)\), for some positive constant \(k_0\), then \(\lambda_1 \geq \frac{k_0}{n+1}\).
When \( n \geq 3 \), Theorem 1.1 was established and proved by A. Greenleaf in [3], where he gave a fundamental Bochner formula for the sub-Laplacian. Our contribution here is proving the theorem when \( n = 2 \), but the argument in Section 2 works for \( n \geq 2 \).

For the case \( n = 1 \), we need one more condition on the torsion term \( A_{11} \). We shall prove the following theorem.

**Theorem 1.2.** Let \( M \) be a 3-dimensional strongly pseudoconvex pseudo-Hermitian manifold in the sense of Webster. Let

\[
(1.17) \quad \text{Ric}_m(X, X) + \frac{1}{2} \text{Tor}_m(X, X) - \frac{3}{k_0} B_m^2(X, X) \geq k_0 g_m(X, X),
\]

for all \( m \in M \), \( X \in H_m(M) \), and for some positive constant \( k_0 \), where

\[
(1.18) \quad B_m^2(x_1 X_1, x_1 X_1) = 2|A_{11}|^2 |x_1|^2 - \text{Re} X_0 (A_{11}^2) x_1^2 = 2 \text{Re} A_{11}^2 x_1^2.
\]

Then \( \lambda_1 \geq \frac{2}{n} \).

The above two theorems are sharp when \( M \) is the unit sphere in \( \mathbb{C}^{n+1} \), in which case the torsion vanishes and \( \lambda_1 = \frac{n}{n+1} k_0 = n \).

Observing that \( (A_{11}^2)^0 = X_0 A_{11} A_{11} + 2 A_{11} A_{11} A_{11} = X_0 A_{11} + 2 A_{11} A_{11} A_{11} \), we remark that while the pseudo-Hermitian case differs from the Riemannian case in that torsion enters into the picture in addition to the Ricci curvature, the 3-dimensional pseudo-Hermitian case differs from the higher-dimensional cases in that the first covariant derivative of the torsion along the distinguished transversal \( X_0 \) also plays a role in Theorem 1.2.

2. **Proof of Theorem 1.1**

We shall start to prove Theorem 1.1.

**Proof.** Let \( (X_0, X_\alpha, X_\bar{\alpha}) \) be a local frame given by (1.4). Let \( X_\alpha^* \) be the adjoint of \( X_\alpha \) with respect to \( dv \). Then

\[
(2.1) \quad X_\alpha^* = -X_\alpha + \left( \sum_\beta \Gamma_{\alpha \beta \gamma} \right) \text{ and } \hat{\Delta} = - \sum_\alpha (X_\alpha^* X_\alpha + X_\bar{\alpha} X_\bar{\alpha}).
\]

Let \( \hat{\nabla} f = \sum_\alpha f_\alpha X_\alpha \in \Gamma(H(M)) \) and \( \hat{\Delta} f = f_\alpha \theta_\alpha + f_\bar{\alpha} \bar{\theta}_{\bar{\alpha}} \). We recall the following formulae proved in [3].

**Bochner formula:**

\[
(2.2) \quad \frac{1}{2} \hat{\Delta} |\hat{\nabla} f|^2 = \|\pi_+ D^2 f\|^2 + \|\pi_- D^2 f\|^2 + \text{Re} (\hat{\nabla} f, \hat{\nabla} (\hat{\Delta} f))
\]

\[\quad + (\text{Ric} + (n-2)/2 \text{Tor})(\hat{\nabla} f, \hat{\nabla} f) + i(D^2 f)(X_0, (\hat{\Delta} f)^*),\]

\[
(2.3) \quad \int_M i(D^2 f)(X_0, (\hat{\Delta} f)^*) dv = \frac{2}{n} \int_M \|\pi_+ D^2 f\|^2 - \|\pi_- D^2 f\|^2 - \text{Ric}(\hat{\nabla} f, \hat{\nabla} f) dv
\]

and

\[
(2.4) \quad \int_M i(D^2 f)(X_0, (\hat{\Delta} f)^*) dv = \int_M -
\frac{4}{n} \text{tr}(\pi_+ D^2 f)^2 + \frac{1}{n} (\hat{\Delta} f)^2 + \text{Tor}(\hat{\nabla} f, \hat{\nabla} f) dv,
\]

where \( \|\pi_+ D^2 f\|^2, \|\pi_- D^2 f\|^2, \text{tr}(\pi_+ D^2 f) \) and \( D^2 f(X_0, (\hat{\Delta} f)^*) \) are locally given by \( \sum f_\alpha \beta X_\alpha X_\beta, \sum f_\beta \alpha X_\beta X_\alpha, \sum f_\alpha \bar{\alpha} X_\alpha X_\bar{\alpha} \) and \( \sum f_\alpha f_\alpha^* - f_\alpha f_\alpha, \) respectively.
If \( f \) is a non-constant real-valued function so that \( \Delta f = -\lambda_1 f \), then \( \int_M f \, dv = 0 \) and

\[
\lambda_1 \int_M |f|^2 \, dv = -\int_M (f, \Delta f) \, dv = 2 \int_M |\nabla f|^2 \, dv.
\]

Consequently, for any \( c \in [0, 1] \), \((1 - c) \times (2.3) + c \times (2.4) \) gives

\[
\int_M i(D^2 f)(X_0, (\bar{d} f)^*) \, dv
\]

\[
= \int_M \left[ \frac{2(1-c)}{n} ||\pi^+ D^2 f||^2 - \frac{4c}{n} |\text{tr}(\pi^+ D^2 f)|^2 - \frac{2(1-c)}{n} ||\pi^- D^2 f||^2 + \frac{2\lambda_1 c}{n} |\nabla f|^2 - \frac{2(1-c)}{n} \text{Ric}(\nabla f, \nabla f) + c \text{Tor}(\nabla f, \nabla f) \right] \, dv.
\]

By the Cauchy-Schwarz inequality,

\[
\|\pi^+ D^2 f\|^2 \geq \frac{1}{n} |\text{tr}(\pi^+ D^2 f)|^2.
\]

Then

\[
0 = \int_M \frac{1}{2} \Delta |\nabla f|^2 \, dv
\]

\[
= \int_M \left[ \|\pi^+ D^2 f\|^2 + \|\pi^- D^2 f\|^2 - \lambda_1 \text{Re}(\nabla f, \nabla f) \right. \\
+ (\text{Ric} + \frac{(n-2)}{2} \text{Tor})(\nabla f, \nabla f) \\
+ \frac{2(1-c)}{n} \|\pi^+ D^2 f\|^2 - \frac{4c}{n} |\text{tr}(\pi^+ D^2 f)|^2 - \frac{2(1-c)}{n} \|\pi^- D^2 f\|^2 \\
+ \frac{2\lambda_1 c}{n} |\nabla f|^2 - \frac{2(1-c)}{n} \text{Ric}(\nabla f, \nabla f) + c \text{Tor}(\nabla f, \nabla f) \Big] \, dv
\]

\[
\geq \int_M \left[ \left( \frac{1}{n} + \frac{2(1-c)}{n^2} - \frac{4c}{n} \right) |\text{tr}(\pi^+ D^2 f)|^2 + (1 - \frac{2(1-c)}{n}) \|\pi^- D^2 f\|^2 \\
- \lambda_1 (1 - \frac{2c}{n}) |\nabla f|^2 + ((1 - \frac{2(1-c)}{n}) \text{Ric} + \frac{(n-2+2c)}{2} \text{Tor})(\nabla f, \nabla f) \right] \, dv.
\]

To get rid of the \(|\text{tr}(\pi^+ D^2 f)|^2\) term, we solve \( \frac{1}{n} + \frac{2(1-c)}{n^2} - \frac{4c}{n} = 0 \) for \( c \) and let \( c = \frac{n+1}{2(n+2)} \). For this choice of \( c \), the above inequality becomes

\[
0 \geq \int_M \frac{2(n-1)}{1+2n} \left[ \|\pi^- D^2 f\|^2 - \frac{(n+1)\lambda_1}{n} |\nabla f|^2 + (\text{Ric} + \frac{n}{2} \text{Tor})(\nabla f, \nabla f) \right] \, dv.
\]

By the hypothesis, \( (\text{Ric} + \frac{n}{2} \text{Tor})(\nabla f, \nabla f) \geq k_0 |\nabla f|^2 \). Hence

\[
0 \geq \int_M \frac{2(n-1)}{2n+1} \left[ \|\pi^- D^2 f\|^2 + \left( k_0 - \frac{n+1}{n} \lambda_1 \right) |\nabla f|^2 \right] \, dv,
\]

which implies that for \( n \geq 2 \), \( \frac{n+1}{n} \lambda_1 \geq k_0 \). Therefore, we have proved Theorem 1.1. \( \square \)
3. Proof of Theorem 1.2

For $n = 1$, we need more on the pseudo-Hermitian geometry of $M$. First we compute the curvature 2-form $\Omega^1_1$ in (1.9) to get $\lambda_1$ in (1.10) explicitly. The result is

\begin{equation}
\Omega^1_1 = d\omega^1 = R_{\Pi\Pi\Pi} \theta^1 \wedge \theta^\Pi + W_{\Pi\Pi} \theta^1 \wedge \theta - W_{\Pi\Pi} \theta^\Pi \wedge \theta
\end{equation}

where

\begin{align*}
R_{\Pi\Pi\Pi} &= X_1 \Gamma^1_{1\Pi} - X_{\Pi} \Gamma^1_{11} + \Gamma^1_{11} \Gamma^1_{1\Pi} - \Gamma^1_{1\Pi} \Gamma^1_{11} + i \Gamma^1_{10}, \\
W_{\Pi\Pi} &= X_1 \Gamma^1_{10} - X_{\Pi} \Gamma^1_{11} + \Gamma^1_{11} \Gamma^1_{10} - A_1 \Gamma^1_{1\Pi}
\end{align*}

and $W_{\Pi\Pi} = W_{\Pi\Pi}$. There is another curvature 2-form $\Omega^1$ defined by

\begin{equation}
\Omega^1 = d\tau^1 - \tau^1 \wedge \omega^1.
\end{equation}

Explicit computation gives

\begin{equation}
\Omega^1 = (X_1 A_{\Pi\Pi} + 2 A_{\Pi\Pi} \Gamma^1_{1\Pi}) \theta^1 \wedge \theta + (X_0 A_{\Pi\Pi} - 2 A_{\Pi\Pi} \Gamma^1_{10}) \theta^\Pi \wedge \theta.
\end{equation}

It was shown in [9] that the coefficient of $\theta^1 \wedge \theta^\Pi$ in $\Omega^1$ is equal to $W_{\Pi\Pi}$ in $\Omega^1_1$. Equating the two expressions of $W_{\Pi\Pi}$, we get

\begin{equation}
X_1 A_{\Pi\Pi} = X_0 \Gamma^1_{1\Pi} - X_{\Pi} \Gamma^1_{10} + \Gamma^1_{11} \Gamma^1_{1\Pi} - A_{\Pi\Pi} \Gamma^1_{11}.
\end{equation}

With this extra information, we will be able to prove the following lemma. For convenience, we shall henceforth write $A = A_{\Pi\Pi}$.

**Lemma 3.1.** Let $\tilde{\Delta} f = -\lambda_1 f$ and $f_0 = X_0 f$. Then

\begin{equation}
\frac{1}{2} \int_M \tilde{\Delta} f_0^2 \, dv = -\lambda_1 \int_M f_0^2 \, dv + 2 \int_M |X_1 f_0|^2 \, dv - 4 \text{Re} \int_M A f_1 X_1 f_0 \, dv
\end{equation}

(both sides being zero).

**Proof.**

\begin{align*}
\frac{1}{2} \tilde{\Delta} (f_0^2) &= \frac{1}{2} \left( (f_0^2, f_0^2)_{\Pi\Pi} + (f_0^2, f_0^2)_{\Pi1} \right) \\
&= \frac{1}{2} \left[ X_{\Pi} (f_0^2)_{1} - \Gamma^1_{1\Pi} (f_0^2)_{1} + X_1 (f_0^2)_{1} - \Gamma^1_{11} (f_0^2)_{1} \right] \\
&= X_{\Pi} (f_0 X_1 f_0) - \Gamma^1_{1\Pi} (f_0 X_1 f_0) + X_1 (f_0 X_1 f_0) - \Gamma^1_{11} (f_0 X_1 f_0) \\
&= 2 |X_1 f_0|^2 + 2 f_0 \text{Re} (X_{\Pi} X_1 f_0) - 2 f_0 \text{Re} \left( \Gamma^1_{11} X_1 f_0 \right).
\end{align*}
Using the Lie bracket \([X_0, X_1] = \Gamma_{10}^1 X_1 - \overline{A}X_{\overline{T}}\) (1.15) and (1.7), we get

\[
X_{\overline{T}}X_1 f_0 = X_{\overline{T}}X_0 X_1 f + X_{\overline{T}}[X_1, X_0] f
= X_0 X_{\overline{T}} X_1 f + [X_{\overline{T}}, X_0] X_1 f + X_{\overline{T}}[X_1, X_0] f
= X_0 f_{\overline{T}} + X_0 (\Gamma_{1\overline{T}}^1 f_1) + (AX_1 - \Gamma_{10}^0 \overline{A}X_{\overline{T}}) X_1 f + X_{\overline{T}}(\overline{A}X_{\overline{T}} - \Gamma_{10}^1 X_1) f
= X_0 f_{\overline{T}} + X_0 (\Gamma_{1\overline{T}}^1 f_1) + \Gamma_{1\overline{T}}^1 X_1 f_0 + (AX_1 X_1 + \overline{A}X_{\overline{T}} X_{\overline{T}}) f - \Gamma_{10}^0 \overline{A}X_{\overline{T}} X_1 f
= X_0 f_{\overline{T}} + X_0 (\Gamma_{1\overline{T}}^1 f_1) + \Gamma_{1\overline{T}}^1 X_1 f_0 + \Gamma_{1\overline{T}}^1 (\overline{A}X_{\overline{T}} + \Gamma_{10}^1 X_1) f + (AX_1 X_1 + \overline{A}X_{\overline{T}} X_{\overline{T}}) f - \Gamma_{10}^0 \overline{A}X_{\overline{T}} X_1 f
= X_0 f_{\overline{T}} + X_0 (\Gamma_{1\overline{T}}^1 f_1) + \Gamma_{1\overline{T}}^1 X_1 f_0 + \Gamma_{1\overline{T}}^1 (\overline{A}X_{\overline{T}} + \Gamma_{10}^1 X_1) f + (AX_1 X_1 + \overline{A}X_{\overline{T}} X_{\overline{T}}) f
+ (AX_1 X_1 + \overline{A}X_{\overline{T}} X_{\overline{T}}) f + \Gamma_{1\overline{T}}^1 X_1 f_0.
\]

Thus,

\[
2 Re X_{\overline{T}} X_1 f_0 = X_0 (f_{\overline{T}} + f_{\overline{T}}) + 2 Re [X_0 (\Gamma_{1\overline{T}}^1) - X_{\overline{T}} (\Gamma_{10}^1) + \Gamma_{1\overline{T}}^1 \Gamma_{10}^1 + X_1 A - \overline{A} \Gamma_{1\overline{T}}^1] f_1
+ 2 (AX_1 X_1 + \overline{A}X_{\overline{T}} X_{\overline{T}}) f + 2 Re \Gamma_{1\overline{T}}^1 X_1 f_0.
\]

Therefore,

\[
\frac{1}{2} \overline{\Delta} f_0^2 = 2 |X_1 f_0|^2 - 2 f_0 Re \left( \Gamma_{1\overline{T}}^1 X_1 f_0 \right)
+ f_0 \left[ X_0 (f_{\overline{T}} + f_{\overline{T}}) + 2 Re [X_0 (\Gamma_{1\overline{T}}^1) - X_{\overline{T}} (\Gamma_{10}^1) + \Gamma_{1\overline{T}}^1 \Gamma_{10}^1 + X_1 A - \overline{A} \Gamma_{1\overline{T}}^1] f_1
+ 2 (AX_1 X_1 + \overline{A}X_{\overline{T}} X_{\overline{T}}) f + 2 Re \Gamma_{1\overline{T}}^1 X_1 f_0 \right]
= 2 |X_1 f_0|^2 + 2 f_0 (AX_1 X_1 + \overline{A}X_{\overline{T}} X_{\overline{T}}) f
+ f_0 X_0 \overline{\Delta} f + 2 f_0 Re [X_0 (\Gamma_{1\overline{T}}^1) - X_{\overline{T}} (\Gamma_{10}^1) + \Gamma_{1\overline{T}}^1 \Gamma_{10}^1 + X_1 A - \overline{A} \Gamma_{1\overline{T}}^1] f_1.
\]

Using \(X^+_{\overline{T}} = -X_1 + \Gamma_{1\overline{T}}^1\), we get

\[
2 \int_M f_0 AX_1 X_1 f dv
= 2 \int_M X^+_{\overline{T}}(f_0 A) f_1 dv
= -2 \int_M X_1 (f_0 A) f_1 dv + 2 \int_M \Gamma_{1\overline{T}}^1 f_0 Adv
= -2 \int_M X_1 (A) f_1 f_0 dv - 2 \int_M A f_1 X_1 f_0 dv + 2 \int_M \Gamma_{1\overline{T}}^1 A f_0 f_1 dv.
\]
Then, by (3.4) with $A = A_{TT}$,

$$
2 \int_M f_0 \nabla X_1 X_1 f \, dv + \int_M \left( \Gamma_{1T}^1 \Gamma_{10}^1 - X_T(\Gamma_{1T}^1) + X_0(\Gamma_{1T}^1) + X_1 A - A \Gamma_{1T}^1 \right) f_1 f_0 \, dv \\
= -2 \int_M A f_1 X_1 f_0 \, dv + \int_M \left( \Gamma_{1T}^1 \Gamma_{10}^1 - X_T(\Gamma_{1T}^1) \\
+ X_0(\Gamma_{1T}^1) - X_1 A + A \Gamma_{1T}^1 \right) f_1 f_0 \, dv \\
= -2 \int_M A f_1 X_1 f_0 \, dv.
$$

Therefore,

$$
\int_M \frac{1}{2} \Delta f_0^2 \, dv = 2 \int_M |X_1 f_0|^2 \, dv - \lambda_1 \int_M (f_0)^2 \, dv - 4 \text{Re} \int_M A f_1 X_1 f_0 \, dv
$$

and the proof of the lemma is complete.

Notice that

$$
\begin{align*}
\text{(3.6)} & \quad i(f_T f_{10} - f_1 f_{10}) \\
& = i f_T \left( X_0 f_1 - \Gamma_{10}^1 f_1 \right) - i f_1 \left( X_0 f_T - \Gamma_{10}^1 f_T \right) \\
& = i f_T \left( X_1 f_0 + (\Gamma_{10}^1 f_1 - \overline{X_T f_T}) - \Gamma_{10}^1 f_1 \right) \\
& \quad - i f_1 \left( X_T f_0 + (\overline{\Gamma_{10}^1 f_T} - A f_1) - \Gamma_{10}^1 f_T \right) \\
& = i(f_T X_1 f_0 - f_1 X_T f_0) + i(A f_T^2 - \overline{A f_1^2}) \\
& = i(f_T X_1 f_0 - f_1 X_T f_0) + \text{Tor}(\nabla f, \nabla f),
\end{align*}
$$

and

$$
\text{(3.7)} & \quad X_1^* = -X_T + \Gamma_{1T}^1 \quad \text{and} \quad [X_1, X_T] = -iX_0 - \Gamma_{1T}^1 X_1 + \Gamma_{1T}^1 X_T.
$$

By (3.7)

$$
\begin{align*}
\text{(3.8)} & \quad \text{Im} \int_M f_T X_1 f_0 \, dv = \text{Im} \int_M (-X_1 X_T f + \Gamma_{1T}^1 f_T f_0) f_0 \, dv \\
& \quad = \text{Im} \int_M i \frac{1}{2} X_0 f f_0 \, dv \\
& \quad = \frac{1}{2} \int_M f_0^2 \, dv.
\end{align*}
$$

Hence, for $n = 1$, by (2.2), (3.6), (3.8) and the equation

$$
\int_M |f_T|^2 \, dv = \int_M (\text{Re} f_T)^2 + (\text{Im} f_T)^2 \, dv \\
= \int_M \frac{1}{4} |\Delta f|^2 + \frac{1}{4} |f_0|^2 \, dv \\
= \int_M \frac{\lambda_1}{4} |\nabla f|^2 + \frac{1}{4} |f_0|^2 \, dv,
$$
we have
\[
0 = \frac{1}{2} \int_M \hat{\Delta} |\hat{\nabla} f|^2 dv \\
= \int_M \left( \frac{\lambda_1}{2} |\hat{\nabla} f|^2 + \frac{1}{4} |f_0|^2 \right) + |f_{111}|^2 - \lambda_1 |\hat{\nabla} f|^2 \\
+ \text{Ric}(\hat{\nabla} f, \hat{\nabla} f) - \frac{1}{2} \text{Tor}(\hat{\nabla} f, \hat{\nabla} f) + i(f_{11} f_0 - f_1 X f_0) dv \\
= \int_M \left( \frac{\lambda_1}{2} |\hat{\nabla} f|^2 + \frac{1}{4} |f_0|^2 \right) + |f_{111}|^2 - \lambda_1 |\hat{\nabla} f|^2 \\
+ \left( \text{Ric} + \frac{1}{2} \text{Tor} \right)(\hat{\nabla} f, \hat{\nabla} f) + i(f_{11} f_0 - f_1 X f_0) dv \\
= \int_M (-\frac{1}{2} \lambda_1 |\hat{\nabla} f|^2 + \frac{1}{4} |f_0|^2 + |f_{111}|^2 \\
+ \left( \text{Ric} + \frac{1}{2} \text{Tor} \right)(\hat{\nabla} f, \hat{\nabla} f) - 2 \text{Im} (f_{11} f_0) dv \\
= \int_M (-\frac{1}{2} \lambda_1 |\hat{\nabla} f|^2 + |f_{111}|^2 + \left( \text{Ric} + \frac{1}{2} \text{Tor} \right)(\hat{\nabla} f, \hat{\nabla} f) - \frac{3}{2} \text{Im} (f_{11} f_0) dv.
\]

By Lemma 3.1, where \( \int_M \hat{\Delta} f_0^2 dv = 0 \), and (3.8), we have
\[
(3.9) \quad 2 \int_M |X f_0|^2 - 4 \text{Re} \int_M A f_1 X f_0 dv = \lambda_1 \int_M f_0^2 dv = 2 \lambda_1 \text{Im} \int_M \hat{\nabla} f_0 X f_0 dv.
\]
Thus,
\[
(3.10) \quad \int_M |X f_0|^2 = 2 \text{Re} \int_M A f_1 X f_0 dv + \lambda_1 \text{Im} \int_M \hat{\nabla} f_0 X f_0 dv \\
\leq 2 \text{Re} \int_M A f_1 X f_0 dv + \lambda_1 \left( \int_M |f_{111}|^2 dv \right)^{1/2} \left( \int_M |X f_0|^2 dv \right)^{1/2} \\
\leq 2 \text{Re} \int_M A f_1 X f_0 dv + \frac{\lambda_1^2}{2} \int_M |f_{11}|^2 dv + \frac{1}{2} \int_M |X f_0|^2 dv.
\]
Since
\[
(3.11) \quad \int_M X_0 (A f_1^2) dv = 0,
\]
we have
\[
(3.12) \quad \int_M A f_1 X_0 f_1 dv = -\frac{1}{2} \int_M X_0 (A f_1^2) dv.
\]
Thus,
\[
\text{Re} \int_M A f_1 X f_0 dv \\
= \text{Re} \int_M A f_1 (X_0 f_1 + \hat{\nabla} X f_0 - \Gamma_{10} f_1) dv \\
= \text{Re} \int_M (|A|^2 |f_1|^2 - \frac{1}{2} X_0 (A f_1^2) - A \Gamma_{10} f_1^2) dv \\
= \int_M \frac{1}{2} B^2 (\hat{\nabla} f, \hat{\nabla} f) dv.
\]
If there is no confusion, we shall simply write
\begin{equation}
B^2(\nabla f, \nabla f) = B^2|f_1|^2.
\end{equation}

(3.10)–(3.14) imply that
\begin{equation}
\int_M |X_1 f_0|^2 dv \leq 4\text{Re} \int_M A f_1 X_1 f_0 dv + \lambda_1^2 \int_M |f_1|^2 dv \\
\leq \int_M (2B^2 + \lambda_1^2)|f_1|^2 dv.
\end{equation}

Therefore,
\begin{equation}
-\frac{3}{2}\text{Im} \int_M f_1 X_1 f_0 dv \\
\geq \frac{3}{2}\left( \int_M |f_1|^2 dv \right)^{1/2} \left( \int_M |X_1 f_0|^2 dv \right)^{1/2} \\
\geq -\frac{3b}{4} \int_M |f_1|^2 dv - \frac{3}{4b} \int_M |X_1 f_0|^2 dv \\
\geq -\frac{3b}{4} \int_M |f_1|^2 dv - \frac{3}{4b} \int_M (2B^2 + \lambda_1^2)|f_1|^2 dv.
\end{equation}

For simplicity, we will use the notation
\begin{equation}
\text{Ric}(\nabla f, \nabla f) + \frac{1}{2} \text{Tor}(\nabla f, \nabla f) = k|f_1|^2.
\end{equation}

Therefore,
\begin{align*}
0 &\geq -\frac{1}{2} \lambda_1 \int_M |f_1|^2 dv + \int_M k|f_1|^2 dv \\
&\quad -\frac{3b}{4} \int_M |f_1|^2 dv - \frac{3}{4b} \int_M (2B^2 + \lambda_1^2)|f_1|^2 dv \\
&\quad = -\lambda_1 \int_M \frac{1}{2} + \frac{3\lambda_1}{4b})|f_1|^2 dv + \int_M (k - \frac{3b}{4} - \frac{3B^2}{2b})|f_1|^2 dv.
\end{align*}

Let $b = k_0/2$. Then by (1.17) and (1.18),
\begin{equation*}
k - \frac{3B^2}{2b} = k - \frac{3B^2}{k_0} \geq k_0.
\end{equation*}

Thus,
\begin{equation*}
\lambda_1 \geq \frac{(k_0 - \frac{3b}{2})}{\frac{1}{2} + \frac{3\lambda_1}{4b}} = \frac{(4k_0 - 3b)b}{2b + 3\lambda_1} = \frac{5k_0^2}{4(k_0 + 3\lambda_1)}.
\end{equation*}

This holds if and only if $12\lambda_1^2 + 4k_0 \lambda_1 \geq 5k_0^2$, i.e., $(2\lambda_1 - k_0)(6\lambda_1 + 5k_0) \geq 0$. Since $\lambda_1 > 0$, we have $6\lambda_1 + 5k_0 > 0$. Hence $\lambda_1 \geq \frac{k_0}{2}$. Therefore, the proof of Theorem 1.2 is complete.

Finally, we remark that for $n = 1$, (1.2) and (1.5) reduce to $\theta = \theta'$, $\theta^1 = e^{i\alpha}\theta_1'$ and $X_0' = X_0$, $X_1' = e^{i\alpha}X_1$ where $\alpha \in \mathbb{R}$. Under these transformations, it can be checked that the quantities considered in Theorem 1.2 also have intrinsic meaning even though they are expressed locally.
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