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THE SHARP LOWER BOUND FOR THE FIRST POSITIVE EIGENVALUE OF A SUB-LAPLACIAN ON A PSEUDO-HERMITIAN MANIFOLD

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ABSTRACT. This paper studies, using the Bochner technique, a sharp lower bound of the first eigenvalue of a subelliptic Laplace operator on a strongly pseudoconvex CR manifold in terms of its pseudo-Hermitian geometry. For dimensions greater than or equal to 7, the lower bound under a condition on the Ricci curvature and the torsion was obtained by Greenleaf. We give a proof for all dimensions greater than or equal to 5. For dimension 3, the sharp lower bound is proved under a condition which also involves a distinguished covariant derivative of the torsion.

1. INTRODUCTION AND MAIN RESULTS

Let M be a $(2n+1)$-dimensional strongly pseudoconvex CR manifold and $H(M)$ the structure bundle, where $H(M)$ is a subbundle of the complexified tangent bundle $T_{\mathbb{C}}(M)$ of which each fiber is an n-dimensional complex vector space. Let θ be a real nonvanishing one-form on M that annihilates $H(M) \oplus \overline{H(M)}$. Then, (M, θ) is a strongly pseudoconvex pseudo-Hermitian manifold in the sense of Webster [9]. Locally, one can choose n complex one-forms θ^α, so that $(\theta, \theta^\alpha, \theta^{\overline{\alpha}})$ form a basis of complex covectors and

$$d\theta = i\,\theta^\alpha \wedge \theta^{\overline{\alpha}}, \quad \theta^{\overline{\alpha}} = \overline{\theta^\alpha}. \tag{1.1}$$

The local coframe $(\theta, \theta^\alpha, \theta^{\overline{\alpha}})$ is uniquely determined up to

$$\theta = \theta', \quad \theta^\alpha = \theta'^{\beta} U^\alpha_\beta, \quad \theta^{\overline{\alpha}} = \theta'^{\overline{\beta}} U^{\overline{\alpha}}_{\overline{\beta}} \tag{1.2}$$

where

$$U^\alpha_\beta U^{\overline{\alpha}}_{\overline{\gamma}} = \delta_{\beta\overline{\gamma}}, \quad U^{\overline{\alpha}}_{\overline{\beta}} = \overline{U^\alpha_\beta}. \tag{1.3}$$

If we compare the dual frame

$$X_0 = \overline{X_0}, \quad X_\alpha, \quad X_{\overline{\alpha}} = \overline{X}_\alpha \tag{1.4}$$

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to $(\theta, \theta^\alpha, \theta^{\overline{\alpha}})$, then the transformation (1.2) gives

$$X'_0 = X_0, \quad X'_\alpha = U^\beta_\alpha X_\beta, \quad X'_{\overline{\alpha}} = U^{\overline{\beta}}_{\overline{\alpha}} X_{\overline{\beta}}, \tag{1.5}$$

which singles out a unique transversal X_0 to $H(M) \oplus \overline{H(M)}$. Furthermore,

$$d\theta^\alpha = \theta^\beta \wedge w^\alpha_\beta + \theta \wedge \tau^\alpha \tag{1.6}$$

where w^α_β are connection 1-forms that are skew-Hermitian:

$$w^\alpha_\beta = -w^{\overline{\beta}}_{\overline{\alpha}} \tag{1.7}$$

and the τ^α are torsion 1-forms of type $(0,1)$:

$$\tau^\alpha = A_{\overline{\alpha}\overline{\gamma}} \theta^{\overline{\gamma}}, \quad A_{\overline{\alpha}\overline{\gamma}} = A_{\overline{\gamma}\overline{\alpha}}. \tag{1.8}$$

Moreover, if we define curvature 2-forms Ω^α_β by

$$\Omega^\alpha_\beta = dw^\alpha_\beta - w^\gamma_\beta \wedge w^\alpha_\gamma - i\theta^{\overline{\beta}} \wedge \tau^\alpha + i\tau^{\overline{\beta}} \wedge \theta^\alpha, \tag{1.9}$$

then

$$\Omega^\alpha_\beta = R_{\beta\overline{\alpha}\rho\overline{\sigma}} \theta^\rho \wedge \theta^{\overline{\sigma}} + \lambda_{\beta\overline{\alpha}} \wedge \theta, \tag{1.10}$$

where $\lambda_{\beta\overline{\alpha}}$ are 1-forms and the curvature tensor components $R_{\beta\overline{\alpha}\rho\overline{\sigma}}$ satisfy

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = R_{\overline{\alpha}\beta\overline{\sigma}\rho} = R_{\rho\overline{\alpha}\beta\overline{\sigma}}. \tag{1.11}$$

Let

$$\Gamma^\alpha_{\beta j} = \omega^\alpha_\beta(X_j), \quad \Gamma^{\overline{\alpha}}_{\overline{\beta} j} = \omega^{\overline{\alpha}}_{\overline{\beta}}(X_j) \tag{1.12}$$

where $\alpha, \beta \in I$ with $I = \{1, \cdots, n\}$ and $j \in \{0\} \cup I \cup \overline{I}$. Then $R_{\alpha\overline{\beta}\rho\overline{\sigma}}$ can also be written as

$$R_{\alpha\overline{\beta}\rho\overline{\sigma}} = X_\rho(\Gamma^\beta_{\alpha\overline{\sigma}}) - X_{\overline{\sigma}}(\Gamma^\beta_{\alpha\rho}) + \Gamma^\beta_{\alpha\gamma}\Gamma^\gamma_{\rho\overline{\sigma}} - \Gamma^\beta_{\alpha\overline{\gamma}}\Gamma^{\overline{\gamma}}_{\overline{\sigma}\rho} + \Gamma^\gamma_{\alpha\overline{\sigma}}\Gamma^\beta_{\gamma\rho} - \Gamma^\gamma_{\alpha\rho}\Gamma^\beta_{\gamma\overline{\sigma}} + i\delta_{\rho\overline{\sigma}}\Gamma^\beta_{\alpha 0}.$$

For $X = \sum_{\alpha=1}^n x^\alpha X_\alpha$ where x^α are local functions, we let

$$\mathrm{Ric}(X,X) = R_{\alpha\overline{\beta}} x^\alpha \overline{x^\beta}, \quad R_{\alpha\overline{\beta}} = g^{\rho\overline{\sigma}} R_{\alpha\overline{\beta}\rho\overline{\sigma}} = R_{\alpha\overline{\beta}\rho\overline{\rho}}, \tag{1.13}$$

and

$$\mathrm{Tor}(X,X) = i(A_{\overline{\alpha}\overline{\beta}} \overline{x^\alpha}\,\overline{x^\beta} - A_{\alpha\beta} x^\alpha x^\beta). \tag{1.14}$$

Then the covariant derivatives and the sub-Laplacian of a function f on M are given by

$$f_j = X_j f, \quad f_{\alpha j} = X_j f_\alpha - \Gamma^\gamma_{\alpha j} f_\gamma, \quad f_{\overline{\alpha} j} = X_j f_{\overline{\alpha}} - \Gamma^{\overline{\gamma}}_{\overline{\alpha} j} f_{\overline{\gamma}} \tag{1.15}$$

and

$$\tilde{\Delta} f = 2\mathrm{Re}(\mathrm{tr}(\pi_+ D^2 f)) = \sum_\alpha f_{\alpha\overline{\alpha}} + f_{\overline{\alpha}\alpha}. \tag{1.16}$$

Let λ_1 be the first positive eigenvalue of $\tilde{\Delta}$. We shall prove the following theorems.

Theorem 1.1. *Let $n \geq 2$ and let M be a $(2n+1)$-dimensional strongly pseudoconvex pseudo-Hermitian manifold in the sense of Webster. If*

$$\mathrm{Ric}_m(X,X) + (n/2)\,\mathrm{Tor}_m(X,X) \geq k_0\, g_m(X,X)$$

for all $m \in M$ and $X \in H_m(M)$, for some positive constant k_0, then $\lambda_1 \geq \frac{nk_0}{n+1}$.

When $n \geq 3$, Theorem 1.1 was established and proved by A. Greenleaf in [5], where he gave a fundamental Bochner formula for the sub-Laplacian. Our contribution here is proving the theorem when $n = 2$, but the argument in Section 2 works for $n \geq 2$.

For the case $n = 1$, we need one more condition on the torsion term A_{11}. We shall prove the following theorem.

Theorem 1.2. *Let M be a 3-dimensional strongly pseudoconvex pseudo-Hermitian manifold in the sense of Webster. Let*

$$Ric_m(X,X) + \frac{1}{2}Tor_m(X,X) - \frac{3}{k_0}B_m^2(X,X) \geq k_0\, g_m(X,X), \tag{1.17}$$

for all $m \in M$, $X \in H_m(M)$, and for some positive constant k_0, where

$$B_m^2(x_1X_1, x_1X_1) = 2|A_{11}|^2|x_1|^2 - \mathrm{Re}\, X_0(A_{\overline{1}\overline{1}})x_1^2 - 2\mathrm{Re}\, A_{\overline{1}\overline{1}}\Gamma_{10}^1 x_1^2. \tag{1.18}$$

Then $\lambda_1 \geq \frac{k_0}{2}$.

The above two theorems are sharp when M is the unit sphere in $\mathbb{C}^{n+1}$, in which case the torsion vanishes and $\lambda_1 = \frac{n}{n+1}k_0 = n$.

Observing that $(A_{\overline{1}\overline{1}})_0 = X_0A_{\overline{1}\overline{1}} - 2A_{\overline{1}\overline{1}}\Gamma_{\overline{1}0}^{\overline{1}} = X_0A_{\overline{1}\overline{1}} + 2A_{\overline{1}\overline{1}}\Gamma_{10}^1$, we remark that while the pseudo-Hermitian case differs from the Riemannian case in that torsion enters into the picture in addition to the Ricci curvature, the 3-dimensional pseudo-Hermitian case differs from the higher-dimensional cases in that the first covariant derivative of the torsion along the distinguished transversal X_0 also plays a role in Theorem 1.2.

2. Proof of Theorem 1.1

We shall start to prove Theorem 1.1.

Proof. Let $(X_0, X_\alpha, X_{\overline{\alpha}})$ be a local frame given by (1.4). Let X_α^* be the adjoint of X_α with respect to dv. Then

$$X_\alpha^* = -X_\alpha + (\sum_\beta \Gamma_{\beta\overline{\beta}}^\alpha) \quad \text{and} \quad \tilde{\Delta} = -\sum_\alpha (X_\alpha^* X_\alpha + X_{\overline{\alpha}}^* X_{\overline{\alpha}}). \tag{2.1}$$

Let $\tilde{\nabla} f = \sum_\alpha f_{\overline{\alpha}} X_\alpha \in \Gamma(H(M))$ and $\tilde{d}f = f_\alpha\theta^\alpha + f_{\overline{\alpha}}\theta^{\overline{\alpha}}$. We recall the following formulae proved in [5].

Bochner formula:

$$\begin{aligned}\frac{1}{2}\tilde{\Delta}|\tilde{\nabla} f|^2 &= \|\pi_+ D^2 f\|^2 + \|\pi_- D^2 f\|^2 + \mathrm{Re}\,(\tilde{\nabla} f, \tilde{\nabla}(\tilde{\Delta} f)) \\ &\quad + (\mathrm{Ric} + (n-2)/2\mathrm{Tor})(\tilde{\nabla} f, \tilde{\nabla} f) + i(D^2 f)(X_0, (\tilde{d}f)^*),\end{aligned} \tag{2.2}$$

$$\int_M i(D^2 f)(X_0, (\tilde{d}f)^*)dv = \frac{2}{n}\int_M \|\pi_+ D^2 f\|^2 - \|\pi_- D^2 f\|^2 - \mathrm{Ric}(\tilde{\nabla} f, \tilde{\nabla} f)dv \tag{2.3}$$

and

$$\int_M i(D^2 f)(X_0, (\tilde{d}f)^*)dv = \int_M -\frac{4}{n}|\mathrm{tr}(\pi_+ D^2 f)|^2 + \frac{1}{n}(\tilde{\Delta} f)^2 + \mathrm{Tor}(\tilde{\nabla} f, \tilde{\nabla} f)dv, \tag{2.4}$$

where $\|\pi_+ D^2 f\|^2$, $\|\pi_- D^2 f\|^2$, $\mathrm{tr}(\pi_+ D^2 f)$ and $D^2 f(X_0, (\tilde{d}f)^*)$ are locally given by $\sum f_{\beta\overline{\alpha}} f_{\beta\overline{\alpha}}$, $\sum f_{\beta\alpha} f_{\overline{\beta}\overline{\alpha}}$, $\sum f_{\alpha\overline{\alpha}}$ and $\sum f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\overline{\alpha}0}$, respectively.

If f is a non-constant real-valued function so that $\tilde{\Delta} f = -\lambda_1 f$, then $\int_M f dv = 0$ and

$$\lambda_1 \int_M |f|^2 dv = -\int_M (f, \tilde{\Delta} f) dv = 2 \int_M |\tilde{\nabla} f|^2 dv. \tag{2.5}$$

Consequently, for any $c \in [0,1]$, $(1-c) \times (2.3) + c \times (2.4)$ gives

$$\begin{aligned} &\int_M i(D^2 f)(X_0, (\tilde{d} f)^*) dv \\ &\quad = \int_M \Big[\frac{2(1-c)}{n} \|\pi_+ D^2 f\|^2 - \frac{4c}{n} |\mathrm{tr}(\pi_+ D^2 f)|^2 - \frac{2(1-c)}{n} \|\pi_- D^2 f\|^2 \\ &\qquad + \frac{2\lambda_1 c}{n} |\tilde{\nabla} f|^2 - \frac{2(1-c)}{n} \mathrm{Ric}(\tilde{\nabla} f, \tilde{\nabla} f) + c \mathrm{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \Big] dv. \end{aligned} \tag{2.6}$$

By the Cauchy-Schwarz inequality,

$$\|\pi_+ D^2 f\|^2 \geq \frac{1}{n} |\mathrm{tr}(\pi_+ D^2 f)|^2. \tag{2.7}$$

Then

$$\begin{aligned} 0 &= \int_M \frac{1}{2} \tilde{\Delta} |\tilde{\nabla} f|^2 dv \\ &= \int_M \Big[\|\pi_+ D^2 f\|^2 + \|\pi_- D^2 f\|^2 - \lambda_1 \mathrm{Re}\,(\tilde{\nabla} f, \tilde{\nabla} f)) \\ &\quad + (\mathrm{Ric} + \frac{(n-2)}{2} \mathrm{Tor})(\tilde{\nabla} f, \tilde{\nabla} f) \\ &\quad + \frac{2(1-c)}{n} \|\pi_+ D^2 f\|^2 - \frac{4c}{n} |\mathrm{tr}(\pi_+ D^2 f)|^2 - \frac{2(1-c)}{n} \|\pi_- D^2 f\|^2 \\ &\quad + \frac{2\lambda_1 c}{n} |\tilde{\nabla} f|^2 - \frac{2(1-c)}{n} \mathrm{Ric}(\tilde{\nabla} f, \tilde{\nabla} f) + c \mathrm{Tor}(\tilde{\nabla} f, \tilde{\nabla} f) \Big] dv \\ &\geq \int_M \Big[(\frac{1}{n} + \frac{2(1-c)}{n^2} - \frac{4c}{n}) |\mathrm{tr}(\pi_+ D^2 f)|^2 + (1 - \frac{2(1-c)}{n}) \|\pi_- D^2 f\|^2 \\ &\quad - \lambda_1 (1 - \frac{2c}{n}) |\tilde{\nabla} f|^2 + ((1 - \frac{2(1-c)}{n}) \mathrm{Ric} + \frac{(n-2+2c)}{2} \mathrm{Tor})(\tilde{\nabla} f, \tilde{\nabla} f) \Big] dv. \end{aligned}$$

To get rid of the $|\mathrm{tr}(\pi_+ D^2 f)|^2$ term, we solve $\frac{1}{n} + \frac{2(1-c)}{n^2} - \frac{4c}{n} = 0$ for c and let $c = \frac{n+2}{2(1+2n)}$. For this choice of c, the above inequality becomes

$$0 \geq \int_M \frac{2(n-1)}{1+2n} \Big[\|\pi_- D^2 f\|^2 - \frac{(n+1)\lambda_1}{n} |\tilde{\nabla} f|^2 + (\mathrm{Ric} + \frac{n}{2} \mathrm{Tor})(\tilde{\nabla} f, \tilde{\nabla} f) \Big] dv.$$

By the hypothesis, $(\mathrm{Ric} + \frac{n}{2} \mathrm{Tor})(\tilde{\nabla} f, \tilde{\nabla} f) \geq k_0 |\tilde{\nabla} f|^2$. Hence

$$0 \geq \int_M \frac{2(n-1)}{2n+1} \Big[\|\pi_- D^2 f\|^2 + \Big(k_0 - \frac{n+1}{n} \lambda_1 \Big) |\tilde{\nabla} f|^2 \Big] dv,$$

which implies that for $n \geq 2$, $\frac{n+1}{n} \lambda_1 \geq k_0$. Therefore, we have proved Theorem 1.1. □

3. Proof of Theorem 1.2

For $n = 1$, we need more on the pseudo-Hermitian geometry of M. First we compute the curvature 2-form Ω_1^1 in (1.9) to get $\lambda_{1\bar{1}}$ in (1.10) explicitly. The result is

$$\Omega_1^1 = d\omega_1^1 = R_{1\bar{1}1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + W_{1\bar{1}1}\theta^1 \wedge \theta - W_{\bar{1}1\bar{1}}\theta^{\bar{1}} \wedge \theta \tag{3.1}$$

where

$$\begin{gathered} R_{1\bar{1}1\bar{1}} = X_1\Gamma^1_{1\bar{1}} - X_{\bar{1}}\Gamma^1_{11} + \Gamma^1_{11}\Gamma^1_{1\bar{1}} - \Gamma^1_{1\bar{1}}\Gamma^{\bar{1}}_{\bar{1}1} + i\Gamma^1_{10}, \\ W_{1\bar{1}1} = X_1\Gamma^1_{10} - X_0\Gamma^1_{11} + \Gamma^1_{11}\Gamma^1_{10} - A_{11}\Gamma^1_{1\bar{1}} \end{gathered}$$

and $W_{\bar{1}1\bar{1}} = \overline{W_{1\bar{1}1}}$. There is another curvature 2-form Ω^1 defined by

$$\Omega^1 = d\tau^1 - \tau^1 \wedge \omega_1^1. \tag{3.2}$$

Explicit computation gives

$$\Omega^1 = (X_1A_{\bar{1}\bar{1}} + 2A_{\bar{1}\bar{1}}\Gamma^1_{11})\theta^1 \wedge \theta^{\bar{1}} - |A_{\bar{1}\bar{1}}|^2\theta^1 \wedge \theta + (-X_0A_{\bar{1}\bar{1}} - 2A_{\bar{1}\bar{1}}\Gamma^1_{10})\theta^{\bar{1}} \wedge \theta. \tag{3.3}$$

It was shown in [9] that the coefficient of $\theta^1 \wedge \theta^{\bar{1}}$ in Ω^1 is equal to $W_{\bar{1}1\bar{1}}$ in Ω_1^1. Equating the two expressions of $W_{\bar{1}1\bar{1}}$, we get

$$X_1A_{\bar{1}\bar{1}} = X_0\Gamma^1_{1\bar{1}} - X_{\bar{1}}\Gamma^1_{10} + \Gamma^1_{1\bar{1}}\Gamma^{\bar{1}}_{\bar{1}0} - A_{\bar{1}\bar{1}}\Gamma^1_{11}. \tag{3.4}$$

With this extra information, we will be able to prove the following lemma. For convenience, we shall henceforth write $A = A_{\bar{1}\bar{1}}$.

Lemma 3.1. *Let $\tilde{\Delta} f = -\lambda_1 f$ and $f_0 = X_0 f$. Then*

$$\frac{1}{2}\int_M \tilde{\Delta} f_0^2\, dv = -\lambda_1 \int_M f_0^2\, dv + 2\int_M |X_1 f_0|^2 dv - 4\mathrm{Re}\int_M A f_1 X_1 f_0\, dv \tag{3.5}$$

(both sides being zero).

Proof.

$$\begin{aligned} \frac{1}{2}\tilde{\Delta}(f_0^2) &= \frac{1}{2}\Big((f_0^2)_{1\bar{1}} + (f_0^2)_{\bar{1}1}\Big) \\ &= \frac{1}{2}\Big[X_{\bar{1}}(f_0^2)_1 - \Gamma^1_{1\bar{1}}(f_0^2)_1 + X_1(f_0^2)_{\bar{1}} - \Gamma^{\bar{1}}_{\bar{1}1}(f_0^2)_{\bar{1}}\Big] \\ &= X_{\bar{1}}(f_0X_1f_0) - \Gamma^1_{1\bar{1}}(f_0X_1f_0) + X_1(f_0X_{\bar{1}}f_0) - \Gamma^{\bar{1}}_{\bar{1}1}(f_0X_{\bar{1}}f_0) \\ &= 2|X_1f_0|^2 + 2f_0\mathrm{Re}\,(X_{\bar{1}}X_1f_0) - 2f_0\mathrm{Re}\Big(\Gamma^1_{1\bar{1}}X_1f_0\Big). \end{aligned}$$

Using the Lie bracket $[X_0, X_1] = \Gamma_{10}^1 X_1 - \overline{A} X_{\overline{1}}$, (1.15) and (1.7), we get

$$\begin{aligned}
X_{\overline{1}} X_1 f_0 &= X_{\overline{1}} X_0 X_1 f + X_{\overline{1}}[X_1, X_0] f \\
&= X_0 X_{\overline{1}} X_1 f + [X_{\overline{1}}, X_0] X_1 f + X_{\overline{1}}[X_1, X_0] f \\
&= X_0(f_{1\overline{1}} + \Gamma_{1\overline{1}}^1 f_1) + (A X_1 - \Gamma_{\overline{1}0}^{\overline{1}} X_{\overline{1}}) X_1 f + X_{\overline{1}}(\overline{A} X_{\overline{1}} - \Gamma_{10}^1 X_1) f \\
&= X_0 f_{1\overline{1}} + X_0(\Gamma_{1\overline{1}}^1) f_1 + \Gamma_{1\overline{1}}^1 X_0 f_1 + (A X_1 X_1 + \overline{A} X_{\overline{1}} X_{\overline{1}}) f \\
&\quad -\Gamma_{\overline{1}0}^{\overline{1}} X_{\overline{1}} X_1 f + X_{\overline{1}}(\overline{A}) X_{\overline{1}} f - X_{\overline{1}}(\Gamma_{10}^1) X_1 f - \Gamma_{10}^1 X_{\overline{1}} X_1 f \\
&= X_0 f_{1\overline{1}} + X_0(\Gamma_{1\overline{1}}^1) f_1 + \Gamma_{1\overline{1}}^1 X_1 f_0 + \Gamma_{1\overline{1}}^1(-\overline{A} X_{\overline{1}} + \Gamma_{10}^1 X_1) f \\
&\quad +(A X_1 X_1 + \overline{A} X_{\overline{1}} X_{\overline{1}}) f + X_{\overline{1}}(\overline{A}) X_{\overline{1}} f - X_{\overline{1}}(\Gamma_{10}^1) X_1 f \\
&= X_0 f_{1\overline{1}} + [X_0(\Gamma_{1\overline{1}}^1) - X_{\overline{1}}(\Gamma_{10}^1) + \Gamma_{1\overline{1}}^1 \Gamma_{10}^1] f_1 + (X_{\overline{1}}(\overline{A}) - \Gamma_{1\overline{1}}^1 \overline{A}) f_{\overline{1}} \\
&\quad +(A X_1 X_1 + \overline{A} X_{\overline{1}} X_{\overline{1}}) f + \Gamma_{1\overline{1}}^1 X_1 f_0.
\end{aligned}$$

Thus,

$$\begin{aligned}
2\mathrm{Re}\, X_{\overline{1}} X_1 f_0 &= X_0(f_{1\overline{1}} + f_{\overline{1}1}) + 2\mathrm{Re}\,[X_0(\Gamma_{1\overline{1}}^1) - X_{\overline{1}}(\Gamma_{10}^1) + \Gamma_{1\overline{1}}^1 \Gamma_{10}^1 + X_1 A - A\Gamma_{\overline{1}1}^{\overline{1}}] f_1 \\
&\quad +2(A X_1 X_1 + \overline{A} X_{\overline{1}} X_{\overline{1}}) f + 2\mathrm{Re}\, \Gamma_{1\overline{1}}^1 X_1 f_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\tilde{\Delta} f_0^2 &= 2|X_1 f_0|^2 - 2 f_0 \mathrm{Re}\left(\Gamma_{1\overline{1}}^1 X_1 f_0\right) \\
&\quad + f_0 \Big[X_0(f_{1\overline{1}} + f_{\overline{1}1}) + 2\mathrm{Re}\,[X_0(\Gamma_{1\overline{1}}^1) - X_{\overline{1}}(\Gamma_{10}^1) \\
&\qquad\qquad + \Gamma_{1\overline{1}}^1 \Gamma_{10}^1 + X_1 A - A\Gamma_{\overline{1}1}^{\overline{1}}] f_1 \\
&\qquad + 2(A X_1 X_1 + \overline{A} X_{\overline{1}} X_{\overline{1}}) f + 2\mathrm{Re}\, \Gamma_{1\overline{1}}^1 X_1 f_0 \Big] \\
&= 2|X_1 f_0|^2 + 2 f_0 (A X_1 X_1 + \overline{A} X_{\overline{1}} X_{\overline{1}}) f \\
&\quad + f_0 X_0 \tilde{\Delta} f + 2 f_0 \mathrm{Re}\,[X_0(\Gamma_{1\overline{1}}^1) - X_{\overline{1}}(\Gamma_{10}^1) + \Gamma_{1\overline{1}}^1 \Gamma_{10}^1 + X_1 A - A\Gamma_{\overline{1}1}^{\overline{1}}] f_1.
\end{aligned}$$

Using $X_{\overline{1}}^* = -X_1 + \Gamma_{\overline{1}1}^{\overline{1}}$, we get

$$\begin{aligned}
&2\int_M f_0\, A X_1 X_1 f\, dv \\
&= 2\int_M X_{\overline{1}}^*(f_0 A) f_1 dv \\
&= -2\int_M X_1(f_0 A) f_1 dv + 2\int_M \Gamma_{\overline{1}1}^{\overline{1}} f_0 A dv \\
&= -2\int_M X_1(A) f_1 f_0 dv - 2\int_M A f_1 X_1 f_0 dv + 2\int_M \Gamma_{\overline{1}1}^{\overline{1}} A f_0 f_1\, dv.
\end{aligned}$$

Then, by (3.4) with $A = A_{\overline{1}\overline{1}}$,

$$\begin{aligned}
2\int_M f_0\, A X_1 X_1 f\, dv &+ \int_M \Big(\Gamma^1_{1\overline{1}}\Gamma^1_{10} - X_{\overline{1}}(\Gamma^1_{10}) + X_0(\Gamma^1_{1\overline{1}}) + X_1 A - A\Gamma^{\overline{1}}_{\overline{1}1}\Big) f_1 f_0 dv \\
&= -2\int_M A f_1 X_1 f_0\, dv + \int_M \Big(\Gamma^1_{1\overline{1}}\Gamma^1_{10} - X_{\overline{1}}(\Gamma^1_{10}) \\
&\qquad + X_0(\Gamma^1_{1\overline{1}}) - X_1 A + A\Gamma^{\overline{1}}_{\overline{1}1}\Big) f_1 f_0 dv \\
&= -2\int_M A f_1 X_1 f_0\, dv.
\end{aligned}$$

Therefore,

$$\int_M \frac{1}{2}\tilde{\Delta} f_0^2\, dv = 2\int_M |X_1 f_0|^2 dv - \lambda_1 \int_M (f_0)^2\, dv - 4\mathrm{Re}\int_M A f_1 X_1 f_0\, dv$$

and the proof of the lemma is complete. □

Notice that

$$\begin{aligned}
&i(f_{\overline{1}} f_{10} - f_1 f_{\overline{1}0}) \\
&\quad = i f_{\overline{1}}(X_0 f_1 - \Gamma^1_{10} f_1) - i f_1 (X_0 f_{\overline{1}} - \Gamma^{\overline{1}}_{\overline{1}0} f_{\overline{1}}) \\
&\quad = i f_{\overline{1}}\Big(X_1 f_0 + (\Gamma^1_{10} f_1 - \overline{A} f_{\overline{1}}) - \Gamma^1_{10} f_1\Big) \\
&\qquad\qquad - i f_1 \Big(X_{\overline{1}} f_0 + (\Gamma^{\overline{1}}_{\overline{1}0} f_{\overline{1}} - A f_1) - \Gamma^{\overline{1}}_{\overline{1}0} f_{\overline{1}}\Big) \\
&\quad = i(f_{\overline{1}} X_1 f_0 - f_1 X_{\overline{1}} f_0) + i(A f_1^2 - \overline{A f_1^2}) \\
&\quad = i(f_{\overline{1}} X_1 f_0 - f_1 X_{\overline{1}} f_0) + \mathrm{Tor}(\tilde{\nabla} f, \tilde{\nabla} f),
\end{aligned} \tag{3.6}$$

and

$$X_1^* = -X_{\overline{1}} + \Gamma^1_{1\overline{1}} \quad \text{and} \quad [X_1, X_{\overline{1}}] = -iX_0 - \Gamma^1_{1\overline{1}} X_1 + \Gamma^{\overline{1}}_{\overline{1}1} X_{\overline{1}}. \tag{3.7}$$

By (3.7)

$$\begin{aligned}
\mathrm{Im}\int_M f_{\overline{1}} X_1 f_0\, dv &= \mathrm{Im}\int_M (-X_1 X_{\overline{1}} f + \Gamma^{\overline{1}}_{\overline{1}1} f_{\overline{1}}) f_0\, dv \\
&= \mathrm{Im}\int_M \frac{i}{2} X_0 f\, f_0\, dv \\
&= \frac{1}{2}\int_M f_0^2\, dv.
\end{aligned} \tag{3.8}$$

Hence, for $n = 1$, by (2.2), (3.6), (3.8) and the equation

$$\begin{aligned}
\int_M |f_{1\overline{1}}|^2 dv &= \int_M (\mathrm{Re}\, f_{1\overline{1}})^2 + (\mathrm{Im}\, f_{1\overline{1}})^2\, dv \\
&= \int_M \frac{1}{4}|\tilde{\Delta} f|^2 + \frac{1}{4}|f_0|^2 dv \\
&= \int_M \frac{\lambda_1}{2}|\tilde{\nabla} f|^2 + \frac{1}{4}|f_0|^2 dv,
\end{aligned}$$

we have

$$\begin{aligned}
0 &= \frac{1}{2}\int_M \tilde\Delta|\tilde\nabla f|^2 dv\\
&= \int_M \Big(\frac{\lambda_1}{2}|\tilde\nabla f|^2+\frac{1}{4}|f_0|^2\Big)+|f_{11}|^2-\lambda_1|\tilde\nabla f|^2\\
&\quad +\mathrm{Ric}(\tilde\nabla f,\tilde\nabla f)-\frac{1}{2}\mathrm{Tor}(\tilde\nabla f,\tilde\nabla f)+i(f_{\overline{1}}f_{10}-f_1f_{\overline{1}0})\,dv\\
&= \int_M \Big(\frac{\lambda_1}{2}|\tilde\nabla f|^2+\frac{1}{4}|f_0|^2\Big)+|f_{11}|^2-\lambda_1|\tilde\nabla f|^2\\
&\quad +\Big(\mathrm{Ric}+\frac{1}{2}\mathrm{Tor}\Big)(\tilde\nabla f,\tilde\nabla f)+i(f_{\overline{1}}X_1f_0-f_1X_{\overline{1}}f_0)\,dv\\
&= \int_M (-\frac{1}{2}\lambda_1|\tilde\nabla f|^2+\frac{1}{4}|f_0|^2+|f_{11}|^2\\
&\quad +\Big(\mathrm{Ric}+\frac{1}{2}\mathrm{Tor}\Big)(\tilde\nabla f,\tilde\nabla f)-2\mathrm{Im}\,(f_{\overline{1}}X_1f_0)\,dv\\
&= \int_M (-\frac{1}{2}\lambda_1|\tilde\nabla f|^2+|f_{11}|^2+\Big(\mathrm{Ric}+\frac{1}{2}\mathrm{Tor}\Big)(\tilde\nabla f,\tilde\nabla f)-\frac{3}{2}\mathrm{Im}\,(f_{\overline{1}}X_1f_0)\,dv.
\end{aligned}$$

By Lemma 3.1, where $\int_M \tilde\Delta f_0^2 dv=0$, and (3.8), we have

$$2\int_M |X_1f_0|^2-4\mathrm{Re}\int_M Af_1X_1f_0dv=\lambda_1\int_M f_0^2dv=2\lambda_1\mathrm{Im}\int_M f_{\overline{1}}X_1f_0dv. \tag{3.9}$$

Thus,

$$\begin{aligned}
\int_M |X_1f_0|^2 &= 2\mathrm{Re}\int_M Af_1X_1f_0dv+\lambda_1\mathrm{Im}\int_M f_{\overline{1}}X_1f_0dv\\
&\le 2\mathrm{Re}\int_M Af_1X_1f_0dv+\lambda_1\Big(\int_M |f_{\overline{1}}|^2dv\Big)^{1/2}\Big(\int_M |X_1f_0|^2dv\Big)^{1/2}\\
&\le 2\mathrm{Re}\int_M Af_1X_1f_0dv+\frac{\lambda_1^2}{2}\int_M |f_{\overline{1}}|^2dv+\frac{1}{2}\int_M |X_1f_0|^2dv.
\end{aligned} \tag{3.10}$$

Since

$$\int_M X_0(Af_1^2)\,dv=0, \tag{3.11}$$

we have

$$\int_M Af_1X_0f_1dv=-\frac{1}{2}\int_M X_0(A)f_1^2dv. \tag{3.12}$$

Thus,

$$\begin{aligned}
&\mathrm{Re}\int_M Af_1X_1f_0\,dv\\
&\quad =\mathrm{Re}\int_M Af_1(X_0f_1+\overline{A}X_{\overline{1}}f-\Gamma_{10}^1f_1)\,dv\\
&\quad =\mathrm{Re}\int_M (|A|^2|f_1|^2-\frac{1}{2}X_0(A)f_1^2-A\Gamma_{10}^1f_1^2)dv\\
&\quad =\int_M \frac{1}{2}B^2(\tilde\nabla f,\tilde\nabla f)\,dv.
\end{aligned} \tag{3.13}$$

If there is no confusion, we shall simply write

$$B^2(\tilde\nabla f, \tilde\nabla f) = B^2|f_1|^2. \tag{3.14}$$

(3.10)–(3.14) imply that

$$\begin{aligned}\int_M |X_1 f_0|^2 dv &\le 4\mathrm{Re}\int_M A f_1 X_1 f_0 dv + \lambda_1^2 \int_M |f_{\overline{1}}|^2 dv \\ &\le \int_M (2B^2+\lambda_1^2)|f_1|^2 dv.\end{aligned} \tag{3.15}$$

Therefore,

$$\begin{aligned}&-\frac{3}{2}\mathrm{Im}\int_M f_{\overline{1}} X_1 f_0 dv \\ &\quad\ge -\frac{3}{2}\Big(\int_M |f_1|^2 dv\Big)^{1/2}\Big(\int_M |X_1 f_0|^2 dv\Big)^{1/2} \\ &\quad\ge -\frac{3b}{4}\int_M |f_1|^2 dv - \frac{3}{4b}\int_M |X_1 f_0|^2 dv \\ &\quad\ge -\frac{3b}{4}\int_M |f_1|^2 dv - \frac{3}{4b}\int_M (2B^2+\lambda_1^2)|f_1|^2 dv.\end{aligned} \tag{3.16}$$

For simplicity, we will use the notation

$$\mathrm{Ric}(\tilde\nabla f, \tilde\nabla f) + \frac{1}{2}\mathrm{Tor}(\tilde\nabla f, \tilde\nabla f) = k|f_1|^2. \tag{3.17}$$

Therefore,

$$\begin{aligned}0 \;\ge\; & -\frac{1}{2}\lambda_1 \int_M |f_1|^2 dv + \int_M k|f_1|^2 dv \\ & -\frac{3b}{4}\int_M |f_1|^2 dv - \frac{3}{4b}\int_M (2B^2+\lambda_1^2)|f_1|^2 dv \\ =\; & -\lambda_1 \int_M \Big(\frac{1}{2}+\frac{3\lambda_1}{4b}\Big)|f_1|^2 dv + \int_M \Big(k - \frac{3}{4}b - \frac{3B^2}{2b}\Big)|f_1|^2 dv.\end{aligned}$$

Let $b = k_0/2$. Then by (1.17) and (1.18),

$$k - \frac{3B^2}{2b} = k - \frac{3B^2}{k_0} \ge k_0.$$

Thus,

$$\lambda_1 \ge \frac{(k_0 - \frac{3b}{4})}{\frac{1}{2}+\frac{3\lambda_1}{4b}} = \frac{(4k_0-3b)b}{2b+3\lambda_1} = \frac{5k_0^2}{4(k_0+3\lambda_1)}.$$

This holds if and only if $12\lambda_1^2 + 4k_0\lambda_1 \ge 5k_0^2$, i.e., $(2\lambda_1 - k_0)(6\lambda_1+5k_0) \ge 0$. Since $\lambda_1 > 0$, we have $6\lambda_1 + 5k_0 > 0$. Hence $\lambda_1 \ge \frac{k_0}{2}$. Therefore, the proof of Theorem 1.2 is complete. □

Finally, we remark that for $n = 1$, (1.2) and (1.5) reduce to $\theta = \theta'$, $\theta^1 = e^{i\alpha}\theta_1'$ and $X_0' = X_0$, $X_1' = e^{i\alpha}X_1$ where $\alpha \in \mathbb{R}$. Under these transformations, it can be checked that the quantities considered in Theorem 1.2 also have intrinsic meaning even though they are expressed locally.

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