

## A SIMPLE PROOF THAT SUPER-REFLEXIVE SPACES ARE $K$ -SPACES

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ABSTRACT. We demonstrate the title.

A quasi-Banach space  $Z$  is called a  $K$ -space [3] if every extension of  $Z$  by the ground field splits; that is, whenever  $X$  is a quasi-Banach space having a line  $L$  such that  $X/L$  is isomorphic to  $Z$ ,  $L$  is complemented in  $X$  (and so,  $X = L \oplus Z$ ). These spaces play an important rôle in the theory of extensions of (quasi) Banach spaces [1], [2].

The property of being a  $K$ -space is closely related to the behaviour of quasi-linear functionals. Recall that a homogeneous functional  $f : Z \rightarrow \mathbb{K}$  is said to be quasi-linear if there is a constant  $Q$  such that

$$|f(x+y) - f(x) - f(y)| \leq Q(\|x\| + \|y\|) \quad (x, y \in Z).$$

The least possible constant in the preceding inequality shall be denoted by  $Q(f)$ .

It is well known [1] that  $Z$  is a  $K$ -space if and only if each quasi-linear functional on  $Z$  can be approximated by a true linear (but not necessarily continuous!) functional  $\ell : Z \rightarrow \mathbb{K}$  in the sense that the distance

$$\text{dist}(f, \ell) \stackrel{\text{def}}{=} \inf\{K \geq 0 : |f(x) - \ell(x)| \leq K\|x\| \text{ for all } x \in Z\}$$

is finite.

The main examples of  $K$ -spaces are supplied by Kalton and co-workers: for instance,  $\mathcal{L}_p$  spaces ( $0 < p \leq \infty$ ) are  $K$ -spaces if and only if  $p \neq 1$  ([1], [4], [5], [6]). Also,  $B$ -convex spaces (Banach spaces having nontrivial type  $p > 1$ ) are  $K$ -spaces and so are quotients of Banach  $K$ -spaces.

In this short note, we present a very simple proof that super-reflexive Banach spaces are  $K$ -spaces. Of course this is contained in Kalton's result for  $B$ -convexity. Nevertheless, I believe that a simpler proof for this particular case is interesting because, in the presence of some unconditional structure (e.g., for Banach lattices),  $B$ -convexity is equivalent to super-reflexivity.

**Mini-Theorem.** *Every super-reflexive space is a  $K$ -space.*

*Proof.* Suppose on the contrary that  $Z$  is super-reflexive and there exists a quasi-linear function  $f : Z \rightarrow \mathbb{K}$  such that  $\text{dist}(f, \ell) = \infty$  for all linear maps  $\ell : Z \rightarrow \mathbb{K}$ .

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Let  $\mathcal{F}$  denote the family of all finite-dimensional subspaces of  $Z$ . For each  $E \in \mathcal{F}$ , let  $f_E$  denote the restriction of  $f$  to  $E$ . It is clear that  $Q(f_E) \leq Q(f)$ . Put

$$\delta_E = \text{dist}(f_E, E^*) = \inf\{\text{dist}(f_E, \ell) : \ell \in E^*\}.$$

Obviously,  $\delta_E$  is finite for all  $E \in \mathcal{F}$ . The hypothesis means that  $\delta_E \rightarrow \infty$  with respect to the natural (inclusion) order in  $\mathcal{F}$ . In particular,  $\delta_E > 0$  for  $E$  large enough. Now, for each  $E \in \mathcal{F}$ , take  $\ell_E \in E^*$  such that  $\text{dist}(f_E, \ell_E) = \delta_E$  and let  $g_E = \delta_E^{-1}(f_E - \ell_E)$  (if  $\delta_E = 0$ , take  $g_E = 0$ ). Clearly,  $|g_E(x)| \leq \|x\|$  provided  $x \in E$ . Also, it is clear that  $Q(g_E) \rightarrow 0$  as  $E$  increases in  $\mathcal{F}$ .

Let  $\mathfrak{U}$  be any ultrafilter refining the Fréchet filter on  $\mathcal{F}$ , and let  $\mathcal{F}_{\mathfrak{U}}$  denote the ultraproduct of  $\mathcal{F}$  with respect to  $\mathfrak{U}$ . Define  $g : \mathcal{F}_{\mathfrak{U}} \rightarrow \mathbb{K}$  by

$$g[(x_E)]_{\mathfrak{U}} = \lim_{\mathfrak{U}(E)} g_E(x_E),$$

where  $[(x_E)]_{\mathfrak{U}}$  denotes the class of  $(x_E)$  in  $\mathcal{F}_{\mathfrak{U}}$ .

Obviously,  $g$  is a (well-defined) bounded linear functional on  $\mathcal{F}_{\mathfrak{U}}$  and, in fact,  $\|g\| \leq 1$ . Since  $Z$  is super-reflexive,  $\mathcal{F}_{\mathfrak{U}}$  is reflexive and we have  $(\mathcal{F}_{\mathfrak{U}})^* = (\mathcal{F}^*)_{\mathfrak{U}}$ , where  $\mathcal{F}^* = \{E^* : E \in \mathcal{F}\}$  (see [7]). It follows that  $g = [(\ell'_E)]_{\mathfrak{U}}$ , where  $\ell'_E \in E^*$  and  $\|\ell'_E\| \leq 1$  for all  $E$ . Hence,

$$\lim_{\mathfrak{U}(E)} g_E(x_E) = \lim_{\mathfrak{U}(E)} \ell'_E(x_E)$$

and so

$$\lim_{\mathfrak{U}(E)} \text{dist}(g_E, \ell'_E) = 0.$$

In particular, for every  $\varepsilon > 0$ , the set  $\mathcal{S} = \{E \in \mathcal{F} : 0 < \text{dist}(g_E, \ell'_E) < \varepsilon\}$  belongs to  $\mathfrak{U}$ . But, for  $E \in \mathcal{S}$ , one has

$$\text{dist}(f_E, \ell_E + \delta_E \ell'_E) \leq \varepsilon \delta_E < \delta_E,$$

a contradiction. □

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