

## ROBUST TRANSITIVITY AND TOPOLOGICAL MIXING FOR $C^1$ -FLOWS

FLAVIO ABDENUR, ARTUR AVILA, AND JAIRO BOCHI

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ABSTRACT. We prove that nontrivial homoclinic classes of  $C^r$ -generic flows are topologically mixing. This implies that given  $\Lambda$ , a nontrivial  $C^1$ -robustly transitive set of a vector field  $X$ , there is a  $C^1$ -perturbation  $Y$  of  $X$  such that the continuation  $\Lambda_Y$  of  $\Lambda$  is a topologically mixing set for  $Y$ . In particular, robustly transitive flows become topologically mixing after  $C^1$ -perturbations. These results generalize a theorem by Bowen on the basic sets of generic Axiom A flows. We also show that the set of flows whose nontrivial homoclinic classes are topologically mixing is *not* open and dense, in general.

### 1. STATEMENT OF THE RESULTS

Throughout this paper  $M$  denotes a compact  $d$ -dimensional boundaryless manifold,  $d \geq 3$ , and  $\mathfrak{X}^r(M)$  is the space of  $C^r$  vector fields on  $M$  endowed with the usual  $C^r$  topology, where  $r \geq 1$ . We shall prove that, generically (residually) in  $\mathfrak{X}^r(M)$ , nontrivial homoclinic classes are topologically mixing. As a consequence, nontrivial  $C^1$ -robustly transitive sets (and  $C^1$ -robustly transitive flows in particular) become topologically mixing after arbitrarily small  $C^1$ -perturbations of the flow.

These results generalize the following theorem by Bowen [B]: nontrivial basic sets of  $C^r$ -generic Axiom A flows are topologically mixing. Note that  $C^1$ -robustly transitive sets are a natural generalization of hyperbolic basic sets; they are the subject of several recent papers, such as [BD1] and [BDP].

In order to announce precisely our results, let us introduce some notation and definitions.

Given  $t \in \mathbb{R}$  and  $X \in \mathfrak{X}^r(M)$ , we shall denote by  $X^t$  the induced time  $t$  map. A subset  $\mathcal{R}$  of  $\mathfrak{X}^r(M)$  is *residual* if it contains the intersection of a countable number of open dense subsets of  $\mathfrak{X}^r(M)$ . Residual subsets of  $\mathfrak{X}^r(M)$  are dense. Given an open subset  $U$  of  $\mathfrak{X}^r(M)$ , then property (P) is *generic in  $U$*  if it holds for all flows in a residual subset  $\mathcal{R}$  of  $U$ ; (P) is *generic* if it is generic in all of  $\mathfrak{X}^r(M)$ .

A compact invariant set for  $X$  is *nontrivial* if it is neither a periodic orbit nor a single point. A compact invariant set  $\Lambda$  of  $X$  is *transitive* if there is some point  $x \in \Lambda$  such that the future orbit  $\{X^t(x) : t > 0\}$  of  $x$  is dense in  $\Lambda$ ;  $\Lambda$  is *topologically mixing* for  $X$  if given any nonempty open subsets  $U$  and  $V$  of  $\Lambda$ , then there is some

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$t_0 > 0$  such that  $X^t(U) \cap V \neq \emptyset$  for all  $t \geq t_0$ . A nontrivial  $X$ -invariant transitive set  $\Lambda$  is  $\Omega$ -isolated if there is some open neighborhood  $U$  of  $\Lambda$  such that  $U \cap \Omega(X) = \Lambda$ . Furthermore,  $\Lambda$  is isolated if there is a neighborhood  $U$  of  $\Lambda$  (called an isolating block) such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} X^t(U).$$

Given a hyperbolic closed orbit  $\gamma$  of  $X$ , the homoclinic class of  $\gamma$  relative to  $X$  is given by

$$H_X(\gamma) = \overline{W^s(\gamma) \pitchfork W^u(\gamma)},$$

where  $\pitchfork$  denotes points of transverse intersection of the invariant manifolds.  $H_X(\gamma)$  is a transitive compact  $X$ -invariant subset of the non-wandering set  $\Omega(X)$ . Moreover, if  $\gamma$  is a closed orbit of index  $i$ , then the set  $P_i(H_X(\gamma)) \equiv \{p \in H_X(\gamma) \cap \text{Per}(X) : p \text{ is hyperbolic with index } i\}$  is dense in  $H_X(\gamma)$ . (See [BDP]).  $H_X(\gamma)$  is not necessarily hyperbolic, but if  $X$  is Axiom A, then its basic sets are hyperbolic homoclinic classes. In the absence of ambiguity, we may write  $H(\gamma)$  for  $H_X(\gamma)$ .

An attractor is a transitive set  $\Lambda$  of  $X$  that admits a neighborhood  $U$  such that

$$X^t(U) \subset U \text{ for all } t > 0, \text{ and } \bigcap_{t \in \mathbb{R}} X^t(U) = \Lambda.$$

A repeller is an attractor for  $-X$ . Clearly any attractor or repeller is  $\Omega$ -isolated.

An isolated  $X$ -invariant compact set  $\Lambda$  is  $C^1$ -robustly transitive if there is some open neighborhood  $\mathcal{V}$  of  $X$  in  $\mathfrak{X}^1(M)$  and some isolating block  $U$  of  $\Lambda$  such that given any  $Y \in \mathcal{V}$ , then

$$\Lambda_Y \equiv \bigcap_{t \in \mathbb{R}} Y_t(U)$$

is a compact transitive nontrivial set of  $Y$ .

Finally, a flow  $X$  is  $C^1$ -robustly transitive if there is some open neighborhood  $\mathcal{W}$  of  $X$  in  $\mathfrak{X}^1(M)$  such that given any  $Y \in \mathcal{W}$ , then  $Y$  is transitive.

Our main result is the following:

**Theorem A.** *There is a residual subset  $\mathcal{R}$  of  $\mathfrak{X}^1(M)$  such that if  $\Lambda$  is an isolated nontrivial transitive set of  $X \in \mathcal{R}$ , then  $\Lambda$  is topologically mixing for  $X$ .*

Theorem A has the following immediate consequence for robustly transitive sets or flows:

**Corollary A'.** *Let  $\Lambda$  be a nontrivial robustly transitive set, with  $\mathcal{V}$  and  $U$  as in the definition above. Then there is some residual subset  $\mathcal{R}$  of  $\mathcal{V}$  such that if  $Y \in \mathcal{R}$ , then  $\Lambda_Y$  is topologically mixing for  $Y$ . In particular, given an open set  $\mathcal{W} \subset \mathfrak{X}^1(M)$  of transitive flows, then there is some residual subset  $\mathcal{R}$  of  $\mathcal{W}$  such that any  $Y \in \mathcal{R}$  is topologically mixing.*

Theorem A is very much a nonhyperbolic,  $C^1$  version of Bowen's aforementioned result. It is actually a consequence of the proof of the following result:

**Theorem B.** *Given any  $r \in \mathbb{N}$ , there is a residual subset  $\mathcal{R}$  of  $\mathfrak{X}^r(M)$  such that if  $Y \in \mathcal{R}$  and  $H(\gamma)$  is a nontrivial homoclinic class of  $Y$ , then  $H(\gamma)$  is topologically mixing for  $Y$ .*

Theorem B follows from general properties of homoclinic classes combined with simple topological arguments. All of the arguments in the proof of Theorem B hold

in any  $C^r$  topology with  $r \geq 1$ , whereas Theorem A requires the use of  $C^1$ -generic properties, which are not known in finer topologies.

Pugh’s General Density Theorem [Pu] and Theorem B of [BD2] (which is stated for diffeomorphisms, but holds for flows via the same arguments) imply that  $C^1$ -generically any  $\Omega$ -isolated transitive set coincides with some homoclinic class. Therefore Theorem B implies the following corollary:

**Corollary B’.** *There is a residual subset  $\mathcal{R}$  of  $\mathfrak{X}^1(M)$  such that if  $Y \in \mathcal{R}$  and  $\Lambda$  is a nontrivial transitive  $\Omega$ -isolated set of  $Y$ , then  $\Lambda$  is topologically mixing for  $Y$ . In particular,  $C^1$ -generically any nontrivial attractor/repeller is topologically mixing.*

Note that Corollary B’ generalizes the “mixing” aspect of [MP]. We remark that the dependence of our proofs on the ( $C^1$ ) Closing and Connecting Lemmas means that extending our results (with the exception of Theorem B) to finer topologies is probably very difficult.

Of course, in general not every nontrivial homoclinic class is topologically mixing: the basic sets of suspensions of Axiom A diffeomorphisms, for example, are not mixing. One may therefore ask how large is the set of the flows that have a nontrivial homoclinic class that is *not* mixing. A partial answer to this question is given by:

**Theorem C.** *There exists a 4-manifold  $M$  and an open set  $\mathcal{U} \subset \mathfrak{X}^1(M)$  such that each flow  $X$  in a dense subset  $\mathcal{D} \subset \mathcal{U}$  has a nontrivial homoclinic class that is not topologically mixing for  $X$ .*

Theorem C shows that the residual set  $\mathcal{R}$  of Theorem B is, in general, not open. The construction in Theorem C relies on the wild diffeomorphisms from [BD2] and [BD3].

On the other hand, robustly transitive flows have relatively tame dynamics. We pose the following:

**Question.** Is the set of ( $C^1$ -)robustly topologically mixing flows dense in the set of robustly transitive flows?

In [AA] the first two authors prove analogues of Theorems A, B, and C for diffeomorphisms. In addition, a robustly transitive but non-mixing diffeomorphism is constructed.

The next section first lists some definitions and properties needed for the proofs and then sets out the proofs themselves.

## 2. THE PROOFS

Given a hyperbolic periodic point  $p$ , let  $\gamma = \gamma(p)$  be its orbit and  $\Pi_X(p) = \Pi_X(\gamma)$  be its period. Set also

$$W^s(p) = \{x \in M : d(X^t(x), \gamma(p)) \rightarrow 0 \text{ as } t \rightarrow +\infty\},$$

$$W^{ss}(p) = \{x \in M : d(X^t(x), X^t(p)) \rightarrow 0 \text{ as } t \rightarrow +\infty\}.$$

We define  $W^u(p)$  and  $W^{uu}(p)$  as the corresponding sets for  $-X$ . Note that the set  $W^s(p)$  is  $X^t$ -invariant for all  $t \in \mathbb{R}$ , whereas  $W^{ss}(p)$  is  $X^t$ -invariant only for  $t \in \Pi_X(p) \cdot \mathbb{Z}$ . The *index* of  $\gamma$  is the dimension of the stable manifold  $W^s(\gamma) = W^s(p)$ .

**Lemma 1.** *Given any  $r \in \mathbb{N}$ , there exists a residual subset  $\mathcal{R}_1$  of  $\mathfrak{X}^r(M)$  such that if  $X \in \mathcal{R}_1$ , then given any distinct closed orbits  $\gamma, \gamma'$ , we have that*

$$\frac{\Pi_X(\gamma)}{\Pi_X(\gamma')} \in \mathbb{R} \setminus \mathbb{Q}.$$

*Proof.* For  $N \in \mathbb{N}$ , let  $A_N \subset \mathfrak{X}^r(M)$  be the set of vector fields  $X$  such that all singularities of  $X$  are hyperbolic and all closed orbits with periods less than  $N$  are hyperbolic. It follows from the standard proof of the Kupka–Smale theorem that the set  $A_N$  is open and dense in  $\mathfrak{X}^r(M)$ .

Now let  $a_1, a_2, \dots$  be an enumeration of the positive rational numbers and let  $B_N \subset \mathfrak{X}^r(M)$  be the set of vector fields  $X \in A_N$  such that if  $\gamma, \gamma'$  are distinct closed orbits with periods less than  $N$ , then  $\Pi_X(\gamma)/\Pi_X(\gamma')$  does not belong to  $\{a_1, \dots, a_N\}$ .

If  $X \in A_N$ , then the number of orbits with periods less than  $N$  is finite. Moreover, each of these orbits has a continuation and the period varies continuously. It follows that the set  $B_N$  is open.

Let us show that  $B_N$  is also dense, so that we can define  $\mathcal{R}_1 = \bigcap_N B_N$ . Given any  $X_0 \in \mathfrak{X}^1(M)$ , first approximate it by  $X_1 \in A_N$ . Let  $\gamma_1, \dots, \gamma_k$  be the  $X_1$ -orbits with periods less than  $N$ . Let  $\delta > 0$  be small enough such that the neighborhoods  $B(\gamma_i, \delta)$  are disjoint. Take  $C^r$  functions  $\psi_i : M \rightarrow [0, 1]$  such that  $\psi_i$  equals 1 in  $\gamma_i$  and equals 0 outside  $B(\gamma_i, \delta)$ . For  $s \in \mathbb{R}_+^k$  close to 0, let

$$Y_s = \left( \prod_{i=1}^k (1 + s_i \psi_i)^{-1} \right) X_1.$$

Then  $Y_s$  has the same orbits as  $X_1$ , and  $Y_s$  converges to  $X_1$  in the  $C^r$  topology as  $s \rightarrow 0$ . Moreover,

$$\Pi_{Y_s}(\gamma_i) = (1 + s_i)\Pi_{X_1}(\gamma_i).$$

Therefore, we can find  $s \in \mathbb{R}_+^k$  close to 0 such that  $Y_s \in A_N$  and the quotients  $\Pi_{Y_s}(\gamma_i)/\Pi_{Y_s}(\gamma_j)$ ,  $i \neq j$ , do not intersect  $\{a_1, \dots, a_N\}$ . If  $\gamma$  is another closed orbit of  $Y_s$ , then  $\Pi_{Y_s}(\gamma) \geq \Pi_{X_1}(\gamma) \geq N$ . This proves that  $Y_s \in B_N$ .  $\square$

We shall use the following simple fact, whose proof is omitted (it follows easily from the transitivity of the future orbits of irrational rotations of the circle):

**Lemma 2.** *Given numbers  $a > 0, b > 0$  and  $\varepsilon > 0$ , with  $a/b$  irrational, the set*

$$\{ma + nb + s : m, n \in \mathbb{N}, |s| < \varepsilon\}$$

*contains an interval of the form  $[T, +\infty)$ .*

We may now prove Theorem B:

*Proof of Theorem B.* Let  $\mathcal{R}_1 \subset \mathfrak{X}^r(M)$  be the residual set given by Lemma 1, and let  $H = H_X(\gamma_0)$  be a nontrivial homoclinic class of some  $f \in \mathcal{R}_1$ . Take two nonempty open sets  $U, V$  intersecting  $H$ . We shall prove that there exists  $t_0 > 0$  such that  $X^t(U) \cap V \neq \emptyset$  for every  $t \geq t_0$ .

Let  $\gamma$  and  $\gamma'$  be distinct periodic orbits in  $H$  with the same index, such that  $\gamma \cap U \neq \emptyset$  and  $\gamma' \cap V \neq \emptyset$ . In order to simplify the notation, let  $a = \Pi_X(\gamma)$  and  $b = \Pi_X(\gamma')$ . Recall that  $a/b \in \mathbb{R} \setminus \mathbb{Q}$ .

Take  $p \in \gamma \cap U$  and  $q \in \gamma' \cap V$ . Notice that  $W^u(p) \cap W^s(q)$  is nonempty. Fix a point  $y$  in this intersection. There exists  $\tau_1 > 0$  such that  $X^{-\tau_1}(y) \in W^{uu}(p)$  and, consequently, the sequence  $\{X^{(-\tau_1 - ma)}(y)\}_{m \in \mathbb{N}}$  is contained in  $W^{uu}(p)$  and converges to  $p$ . Therefore we can find  $t_1 > 0$  such that

$$X^{(-t_1 - ma)}(y) \in U \text{ for every } m \in \mathbb{N}.$$

Analogously, there exist  $t_2 > 0$  and  $\varepsilon > 0$  such that

$$X^{(t_2+nb+s)}(y) \in V \text{ for every } m \in \mathbb{N} \text{ and } |s| < \varepsilon.$$

Let  $T > 0$ , depending on  $a, b$  and  $\varepsilon$ , be given by Lemma 2. Set  $t_0 = t_1 + t_2 + T$ . Then, for any  $t \geq t_0$ , there exist numbers  $m, n \in \mathbb{N}$  and  $|s| < \varepsilon$  such that  $t = t_1 + t_2 + ma + nb + s$ . So  $X^t(U) \cap V$  contains the point  $X^{(t_2+nb+s)}(y)$ . This concludes the proof.  $\square$

Now we explain how Theorem A follows from Theorem B. We first need a couple of definitions:

**Definition 1.** Let  $\Lambda$  be a compact invariant set of  $X \in \mathfrak{X}^1(M)$ . Then we set  $P_i(\Lambda) \equiv \{p \in \Lambda : p \text{ is a hyperbolic periodic point of } X \text{ with index } i\}$ .

The next definition comes from [BDP]:

**Definition 2.** Let  $p$  be a periodic point of a flow  $X \in \mathfrak{X}^1(M)$ , and let  $U$  be a neighborhood of  $p$  in  $M$ . Then the *homoclinic class of  $p$  relative to  $U$*  is given by

$$HR_X(p, U) \equiv \text{cl} \{q \in H_X(p) \cap \text{Per}(X) : \text{the orbit } \gamma(q) \text{ is contained in } U\}.$$

It is easily seen that  $HR_X(p, U)$  is a compact transitive invariant set. Moreover, if  $\text{ind}(p) = i$ , then  $P_i(HR_X(p, U))$  is dense in  $HR_X(p, U)$ .

We need the following lemma, which is a consequence of a theorem by Arnaud [Ar] together with an argument from [BDP]:

**Lemma 3.** *There is a residual subset  $\mathcal{R}$  of  $\mathfrak{X}^1(M)$  such that if  $\Lambda$  is an isolated transitive set of  $X \in \mathcal{R}$ , then  $\Lambda = HR_X(p, U)$  for some periodic point  $p \in \Lambda$ .*

*Proof.* Let  $\mathcal{R}_2$  be as in Theorem 1 of [Ar] and let  $\mathcal{R}_3$  be as in Theorem B of [BD2], and set  $\mathcal{R} \equiv \mathcal{R}_2 \cap \mathcal{R}_3$ . Let  $\Lambda$  be an isolated transitive set of  $X \in \mathcal{R}$ , with  $U$  an isolating block of  $\Lambda$ .

By Theorem 1 of [Ar], there is a sequence of periodic orbits  $\gamma_k$  that converge to  $\Lambda$  in the Hausdorff topology. The orbit  $\gamma_k$  is contained in  $U$  for  $k$  sufficiently large. Since  $\Lambda$  is the maximal invariant set of  $U$ , it follows that for large  $k$  the orbit  $\gamma_k$  is contained in  $\Lambda$ . Since the sequence  $\{\gamma_k\}$  converges to  $\Lambda$  in the Hausdorff topology, the set of periodic points contained in  $\Lambda$  must be a dense subset of  $\Lambda$ .

Now, since  $\Lambda$  is transitive and has a dense subset of periodic points, we apply an argument from [BDP], which uses Theorem B of [BD2], to conclude that given any periodic point  $p \in \Lambda$ , then

$$\Lambda = HR_X(p, U).$$

$\square$

We are now ready to prove Theorem A:

*Proof of Theorem A.* It is easy to see that the proof of Theorem B actually implies the following result:

**Theorem D.** *There is a residual subset  $\mathcal{R}$  of  $\mathfrak{X}^1(M)$  such that if  $Y \in \mathcal{R}$  and  $\Lambda$  is a nontrivial transitive set of  $Y$  such that for some  $i \in \{1, \dots, d - 1\}$  the set  $P_i(\Lambda)$  is dense in  $\Lambda$ , then  $\Lambda$  is topologically mixing for  $Y$ .*

Now by Lemma 3 above we have that  $\Lambda$  coincides with some relative homoclinic class  $HR_X(p, U)$ . Let  $i$  be the index of the periodic point  $p$ . Since  $P_i(HR_X(p, U))$  is dense in  $HR_X(p, U)$ , we conclude that  $\Lambda$  satisfies the hypotheses of Theorem D above, and therefore that  $\Lambda$  is a mixing set for  $X$ .  $\square$

At last, we give the:

*Proof of Theorem C.* Let  $S$  be a compact 3-manifold, and let  $\text{Diff}^1(S)$  be the set of  $C^1$  diffeomorphisms of  $S$  endowed with the  $C^1$ -topology. The key of the construction is the following result of Bonatti and Díaz ([BD3, Theorem 3.2]): *There exist an open set  $\mathcal{U}_0 \subset \text{Diff}^1(S)$  and a dense subset  $\mathcal{D}_0 \subset \mathcal{U}_0$  such that for every  $f \in \mathcal{D}_0$  there are an open set  $B \subset S$  and an integer  $n \in \mathbb{N}$  such that every  $x \in B$  is a periodic point of  $f$  of (prime) period  $n$ .*

Let  $f_0 : S \rightarrow S$  be a diffeomorphism from the set  $\mathcal{D}_0$  above. Let  $X_0^t : M \rightarrow M$  be the suspension flow. As usual,  $M$  is the 4-manifold obtained from  $S \times [0, 1]$  by gluing points  $(x, 1)$  and  $(f_0(x), 0)$ . We will identify  $S$  with the submanifold  $\{(x, 0) \in M : x \in S\}$  of  $M$ .

Let  $\mathcal{U} \subset \mathfrak{X}^1(M)$  be a small neighborhood of  $X_0$  such that every vector field  $X \in \mathcal{U}$  is transverse to  $S$  and, moreover, the first-return map  $f_X \in \text{Diff}^1(S)$  belongs to  $\mathcal{U}_0$ . For  $X \in \mathcal{U}$ , we let  $\tau_X : S \rightarrow \mathbb{R}_+$  be the return-time map, which is a  $C^1$ -smooth function depending continuously (in the  $C^1$  topology) on  $X \in \mathcal{U}$ .

We will omit the proof of the following:

**Lemma 4.** *For every  $X \in \mathcal{U}$  and every neighborhood  $\mathcal{V} \ni X$ , if  $\tilde{f}$  is a small perturbation of  $f_X$  and  $\tilde{\tau}$  is a small perturbation of  $\tau_X$ , then there is  $\tilde{X} \in \mathcal{V}$  such that  $f_{\tilde{X}} = \tilde{f}$  and  $\tau_{\tilde{X}} = \tilde{\tau}$ .*

Now let  $X_1 \in \mathcal{U}$ . We shall prove that there exists  $X_4$  arbitrarily close to  $X_1$  and that has a nontrivial homoclinic class that is not topologically mixing.

Let  $f_1 = f_{X_1}$  and  $\tau_1 = \tau_{X_1}$ . Take  $f_2 \in \mathcal{D}_0$  close to  $f_1$ . Since  $f_2 \in \mathcal{D}_0$ , there is a ball  $B \subset S$  of points that are  $f_2$ -periodic, of period  $n$ . Let  $\tau_1^n : S \rightarrow \mathbb{R}_+$  be defined by  $\tau_1^n = \sum_{j=0}^{n-1} \tau_1 \circ f_1^j$ .

Using a chart, we identify  $B$  with a ball  $B(0, r) \subset \mathbb{R}^3$ , in such a way that the kernel of the differential  $D\tau_1^n(0)$  contains the plane  $xy$ .

Let  $f_3 \in \text{Diff}^1(S)$  be a perturbation of  $f_2$  such that:

- $f_3$  equals  $f_2$  outside  $\bigcup_{j=0}^{n-1} f_2^j(B)$ ;
- there exists a ball  $B_1 = B(0, r_1)$ , with  $0 < r_1 < r$ , such that  $f_3^n(B_1) = B_1$ ;
- $f_3^n$  restricted to  $B_1$  is an orthogonal rotation (indicated by  $R$ ) of angle  $2\pi/m$ , where  $m \in \mathbb{N}$ , along the axis  $y$ .

It is easy to construct a map  $\tau_3 : S \rightarrow \mathbb{R}$   $C^1$  close to  $\tau_1$ , such that  $\tau_3^n = \sum_{j=0}^{n-1} \tau_3 \circ f_3^j$  is an affine map in a smaller ball  $B_2$  around 0 and such that  $D\tau_3^n(0) = D\tau_1^n(0)$ . That is, if  $x \in B_2$ , then  $\tau_3^n(x) = \tau_3^n(0) + D\tau_3^n(0) \cdot x$ .

Using Lemma 4, we find a flow  $X_3$  close to  $X_1$  and such that  $f_{X_3} = f_3$  and  $\tau_{X_3} = \tau_3$ .

Let  $x \in B_2 \setminus \{0\}$ . Its successive returns to  $B_1$  under the flow  $X_3$  are  $R(x), \dots, R^{m-1}(x), R^m(x) = x$ . In particular,  $x$  is a periodic point. Summing the respective return times we get that the period of  $x$  is  $\sum_{j=0}^{m-1} \tau_3^n(R^j(x)) = m\tau_3^n(0)$ , since  $\sum_{j=0}^{m-1} R^j(x)$  belongs to the  $y$ -axis. That is, all points in  $B_2 \setminus \{0\}$  are periodic under  $X_3$  of (prime) period  $m\tau_3^n(0)$ .

Let  $B_3 \subset B_2$  be a ball (not centered at 0) such that  $\text{cl} B_3, \text{cl} R(B_3), \dots, \text{cl} R^{m-1}(B_3)$  are pairwise disjoint.

Now choose some  $f_4 : S \rightarrow S$  that is  $C^1$  close to  $f_3$  and such that:

- $f_4$  equals  $f_3$  outside  $B_3$ ;
- $f_4^{nm}$  restricted to  $B_3$  has a nontrivial homoclinic class (say, a solenoid attractor).

Also let  $\tau_4 : S \rightarrow \mathbb{R}_+$  be given by  $\tau_4 = \tau_3 \circ f_3^{-1} \circ f_4$ . Then  $\tau_4$  is  $C^1$  close to  $\tau_3$ . Using Lemma 4 again, we obtain  $X_4$   $C^1$  close to  $X_3$  such that  $f_{X_4} = f_4$  and  $\tau_{X_4} = \tau_4$ . The return time of a point  $x \in B_3$  to  $B_3$  under  $X_4$  is  $\tau_4^{nm}(x) = \tau_3^{nm}(f_3^{-1} \circ f_4(x))$ , which is independent of  $x$  (where, as usual, we let  $\tau_4^{nm} = \sum_{j=0}^{nm-1} \tau_4 \circ f_4^j$  and  $\tau_3^{nm} = \sum_{j=0}^{nm-1} \tau_3 \circ f_3^j$ ). Therefore,  $X_4^t$  has a nontrivial homoclinic class that is not topologically mixing.  $\square$

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REFERENCES

[AA] F. Abdenur and A. Avila, *Robust transitivity and topological mixing for diffeomorphisms*, preprint IMPA (www.preprint.impa.br), (2002).

[Ar] M.-C. Arnaud, *Approximation des ensembles  $\omega$ -limites des difféomorphismes par des orbites périodiques*, Ann. Scient. École Norm. Sup. Paris, **36**, 173-190, (2003).

[B] R. Bowen, *Mixing Anosov flows*, Topology, **15**, 77-79, (1976). MR **52**:11999

[BD1] Ch. Bonatti and L. J. Díaz, *Persistence of transitive diffeomorphisms*, Ann. Math., **143**, 367-396, (1995).

[BD2] Ch. Bonatti and L. J. Díaz, *Connexions hétéroclines et généricité d'une infinité de puits ou de sources*, Ann. Scient. École Norm. Sup. Paris, **32**, 135-150, (1999). MR **2000e**:37015

[BD3] Ch. Bonatti and L. J. Díaz, *On maximal transitive sets of generic diffeomorphisms*, preprint, (2001).

[BDP] Ch. Bonatti, L. J. Díaz, and E. Pujals, *A  $C^1$ -generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources*, to appear in Ann. Math.

[H] S. Hayashi, *Connecting invariant manifolds and the solution of the  $C^1$  stability and  $\Omega$ -stability conjectures for flows*, Ann. Math. **145**, 81-137, (1997). MR **98b**:58096

[MP] C. A. Morales and M. J. Pacifico, *Mixing attractors for 3-flows*, Nonlinearity **14**, 359-378, (2001). MR **2002a**:37036

[Pu] C. Pugh, *An improved closing lemma and a general density theorem*, Amer. J. Math., **89**, 1010-1021, (1967). MR **37**:2257

IMPA, ESTR. D. CASTORINA 110, 22460-320 RIO DE JANEIRO, BRAZIL  
*E-mail address*: flavio@impa.br

COLLÈGE DE FRANCE, 3 RUE D'ULM, 75005 PARIS, FRANCE  
*E-mail address*: avila@impa.br

IMPA, ESTR. D. CASTORINA 110, 22460-320 RIO DE JANEIRO, BRAZIL  
*E-mail address*: bochi@impa.br