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COMMUTANTS OF BOL LOOPS OF ODD ORDER

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Abstract. In this note we show that the commutant of a Bol loop of odd order is a subloop.

1. Introduction

A *loop* $(L, \cdot)$ is a set L with a binary operation $\cdot : L \times L \to L$ such that (i) for given $a, b \in L$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in L$, and (ii) there exists a neutral element $1 \in L$ satisfying $1 \cdot x = x \cdot 1 = x$ for all $x \in L$. Basic references for loop theory are [1], [6]. We will use the usual juxtaposition conventions to avoid excessive parenthesization, e.g., $ab \cdot c = (a \cdot b) \cdot c$. The *commutant* of a loop L is the set

$$C(L) = \{a \in L : ax = xa \quad \forall x \in L\}.$$

The *center* of L is the set of all $a \in C(L)$ such that $a \cdot xy = ax \cdot y$, $x \cdot ay = xa \cdot y$, and $xy \cdot a = x \cdot ya$ for all $x, y \in L$. The center is a normal subloop. For some varieties of loops, such as groups, the commutant and center coincide. For other varieties, the commutant is larger than the center, but is still "well-behaved" in the sense that it is a normal subloop. However, the commutant of an arbitrary loop need not be a subloop at all, and even when it is, it need not be normal.

The commutant is also known in the literature by other names such as "Moufang center", "commutative center", or "centrum". Since this object is not, in general, central in the sense of universal algebra, we prefer a term that does not suggest otherwise. Thus we have borrowed the term "commutant", which is used for a similar concept in other fields.

A loop is said to be a (left) *Bol loop* if it satisfies the identity $x(y \cdot xz) = (x \cdot yx)z$ for all x, y, z. A right Bol loop is similarly defined, and a loop that is both a left and right Bol loop is a *Moufang loop*. (This is one of many equivalent definitions; see [1], [6], [8].) In this paper, all Bol loops will be left Bol loops. The commutant of a Moufang loop is a subloop, but it is an open problem to characterize precisely those Moufang loops for which the commutant is normal [2]. On the other hand, the commutant of an arbitrary Bol loop need not be a subloop; see the web page [5]. (Note that [5] lists right Bol loops.) Our main result is the following.

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Theorem 1.1. *If every element of the commutant of a Bol loop has finite odd order, then the commutant is a subloop.*

Glauberman observed that a finite Bol loop has odd order if and only if every element has odd order [4]. Thus we have the following.

Corollary 1.2. *The commutant of a finite Bol loop of odd order is a subloop.*

Our investigations were aided by the automated reasoning tool OTTER, developed by McCune [7]. We thank Ken Kunen for introducing us to OTTER.

2. Proofs

Let L be a Bol loop. We recall that L has both the *left inverse property* (LIP) and the *left alternative property* (LAP) ([6], [8]). These are given, respectively, by the equations $x^{-1} \cdot xy = x \cdot x^{-1}y = y$ and $x \cdot xy = x^2y$, where $x^{-1}x = xx^{-1} = 1$. We shall make use of these without comment in what follows.

Lemma 2.1. *Let L be a Bol loop. If a, b are in $C(L)$, then so are a^2, b^{-1} and a^2b.*

Proof. Fix $x \in L$. We have $a(a^2x) = (a^2x)a = (a \cdot ax)a = (a \cdot xa)a = a \cdot xa^2$. Cancelling, we have $a^2x = xa^2$, that is, $a^2 \in C(L)$. Next, we have $b^{-1}x = b^{-1}(x \cdot b^{-1}b) = (b^{-1} \cdot xb^{-1})b = b(b^{-1} \cdot xb^{-1}) = xb^{-1}$, so that $b^{-1} \in C(L)$. Finally, $a^2b \cdot x = (a \cdot ba)x = a(b \cdot ax) = a(b \cdot xa) = a(b \cdot x(b \cdot b^{-1}a)) = a((b \cdot xb) \cdot ab^{-1}) = (a \cdot (b \cdot xb)a)b^{-1} = (a^2 \cdot (b \cdot xb))b^{-1} = ((b \cdot xb) \cdot a^2)b^{-1} = (b(x \cdot ba^2))b^{-1} = b^{-1} \cdot b(x \cdot ba^2) = x \cdot a^2b$. Therefore $a^2b \in C(L)$. □

For an element a of a loop L, let $\langle a \rangle$ denote the subloop generated by a. Bol loops are *power-associative* ([6], [8]), that is, if $x^0 := 1$, $x^{n+1} := xx^n$, and $x^{-n-1} := x^{-1}x^{-n}$ for $n \geq 0$, then $x^m x^n = x^{m+n}$ for all integers m, n.

Lemma 2.2. *Let L be a Bol loop. For each $a \in C(L)$, $\langle a \rangle \subseteq C(L)$.*

Proof. Since L is power-associative, this follows from Lemma 2.1 and induction. □

We now can prove Theorem 1.1. Given $a, b \in C(L)$, let $c \in L$ be the unique element such that $c^2 = a$. Since a has odd order, Lemma 2.2 implies that $c \in C(L)$. By Lemma 2.1, $ab = c^2b \in C(L)$. Since $C(L)$ is closed under products and inverses, we may apply the *left inverse property* $a^{-1} \cdot ab = b$, which holds in Bol loops ([6], [8]), and the commutativity of $C(L)$ to conclude that $C(L)$ is a subloop.

We conclude with three remarks. First, the previously cited result of Glauberman actually applies to any subset of a Bol loop containing the neutral element and closed under taking inverses and under the operation $(x, y) \mapsto x \cdot yx$ [3]. So, in fact, we have the following extension of Corollary 1.2: *If the commutant of a Bol loop has finite odd order, then the commutant is a subloop.*

Second, an element a of an arbitrary loop L is called a *Bol element* if $a(x \cdot ay) = (a \cdot xa)y$ for all $x, y \in L$. The set $B(L)$ of all Bol elements of L need not be a subloop. The proofs herein carry over nearly verbatim to obtain the following extension of Theorem 1.1: *If every element of $B(L) \cap C(L)$ has finite odd order, then $B(L) \cap C(L)$ is a subloop.*

Finally, we have been unable to find an example of a (necessarily infinite) uniquely 2-divisible Bol loop (i.e., a loop in which the mapping $x \mapsto x^2$ is a bijection) with a commutant element whose unique square root is not a commutant element. However, we conjecture that such loops do indeed exist.

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