

HARDY INEQUALITIES RELATED TO GRUSHIN TYPE OPERATORS

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ABSTRACT. We prove some Hardy type inequalities related to the Grushin type operator $\Delta_x + |x|^{2\gamma}\Delta_y$.

1. INTRODUCTION

The purpose of this paper is to prove Hardy type inequalities related to the degenerate Grushin type operator $\Delta_\gamma := \Delta_x + |x|^{2\gamma}\Delta_y$ ($x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$).

The well-known Hardy inequality for the first-order derivative of a function $u \in \mathcal{C}_0^1(\Omega)$, where Ω is an open set of \mathbb{R}^n , is given by

$$(1.1) \quad c \int_{\Omega} \frac{u^2}{|x|^2} dx \leq \int_{\Omega} |\nabla u|^2 dx,$$

(see, for instance, [8] and the references therein). Much effort has been made to give explicit values of the constant c , and even more, to find its best value c_b (see [7], [9]).

The pre-eminent rule of the Hardy inequality in the study of linear and nonlinear partial differential equations is well-known. For instance, let us consider the linear initial value problem

$$(1.2) \quad \begin{cases} u_t - \Delta u = \lambda \frac{u}{|x|^2}, & x \in \mathbb{R}^n, \quad n \geq 3, \quad t \in]0, T[, \quad \lambda \in \mathbb{R}, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n, \quad u_0 \in L^2(\mathbb{R}^n), \quad u_0 > 0. \end{cases}$$

Problem (1.2) has a solution if and only if $\lambda \leq (\frac{n-2}{2})^2 = c_b$ (see [2] for more details). In recent years this result has been extended in several directions; see, e.g., [3], [4], [11], [13], [18]. As an example, consider the following quasilinear problem:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda \frac{|u|^{p-2}u}{|x|^p}, & (x, t) \in \Omega \times]0, T[, \quad n > p > 1, \quad \lambda > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \quad u_0 > 0, \\ u(x, t) = 0, & x \in \partial\Omega, \quad t \in]0, T[, \end{cases}$$

where u_0 satisfies suitable regularity assumptions and Ω is a bounded domain in \mathbb{R}^n containing the origin. In [11], the authors proved that the above problem with $p \geq \frac{2n}{n+1}$ has a solution if and only if $\lambda \leq c_{n,p} := (\frac{n-p}{p})^p$. Again, the result depends

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on the relation between λ and the best constant $c_{n,p}$ in the inequality of type (1.1) in L^p :

$$(1.3) \quad c \int_{\Omega} \frac{|u|^p}{|x|^p} dx \leq \int_{\Omega} |\nabla u|^p dx.$$

In the Heisenberg group setting, Garofalo and Lanconelli in [12], Niu, Zhang and Wang in [17] and the author in [5] proved, among other results, the following Hardy type inequality related to the Kohn Laplacian operator:

$$(1.4) \quad c \int_{\Omega} \frac{u^2}{\rho^2} \psi^2 d\xi \leq \int_{\Omega} |\nabla_H u|^2 d\xi, \quad u \in \mathcal{C}_0^1(\Omega)$$

where ∇_H denotes the vector field associated to the Kohn Laplacian ($\Delta_H = \nabla_H \cdot \nabla_H$), ρ and ψ are, respectively, a suitable distance from the origin and a weight function such that $0 \leq \psi \leq 1$ (for a detailed description of these results we refer the interested reader to [5], [12], [17]).

Recently, in [13], it has been pointed out that the analogue problem of (1.2) involving the Kohn Laplacian Δ_H , namely

$$\begin{cases} u_t - \Delta_H u = \lambda \psi^2 \frac{u}{\rho^2} & \text{on } \mathbb{R}^{2n+1} \times]0, T[, \quad \lambda \in \mathbb{R}, \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R}^{2n+1}, \quad u_0 \in L^2(\mathbb{R}^{2n+1}), \quad u_0 > 0, \end{cases}$$

has a positive solution if and only if $\lambda \leq c_{b,H}$, where $c_{b,H}$ is the best constant in (1.4).

Furthermore, an important connection between the Gelfand problem and Hardy inequality has been pointed out in [3] and [4].

Having in mind some extensions of these kinds of results in the setting of second-order linear degenerate (or singular) partial differential operators, it appears that an important step towards this programme is to establish some fundamental inequalities of Hardy type.

In this paper we shall prove some inequalities of the type (1.1) associated to the Grushin type operator $\Delta_{\gamma} := \Delta_x + |x|^{2\gamma} \Delta_y$ with $\gamma > 0$ ($x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$); see [14], [10] and the references therein.

Let ∇_{γ} be the gradient operator defined by

$$\nabla_{\gamma} := (\nabla_x, |x|^{\gamma} \nabla_y).$$

We have $\Delta_{\gamma} = \nabla_{\gamma} \cdot \nabla_{\gamma}$.

Our aim is to prove inequalities of the type

$$(1.5) \quad c \int_{\Omega} |u|^p w^p d\xi \leq \int_{\Omega} |\nabla_{\gamma} u|^p d\xi,$$

where w is one of the following functions: $1/|x|$, $1/[\xi]$ or $|x|^{\gamma}/[\xi]^{1+\gamma}$ and $[\cdot]$ denotes a suitable distance from the origin (see below for the definition). Furthermore, we give an estimate on the optimal constant in (1.5) and in some cases we show its sharp value.

For this goal we shall mainly use a technique developed in [5] and [16]. An interesting outcome of this approach is that, in some cases, one can easily obtain the best constant even for a higher-order generalization of (1.1). We refer to Niu, Zhang and Wang [17] for a different and interesting approach based on the Picone identity [1] in the Heisenberg group setting.

This paper is organized as follows. In the next section, we introduce notation and basic facts about the degenerate operator Δ_{γ} . Section 3 is devoted to the proof

of some Hardy inequalities of type (1.5). In the same section, as a consequence of the technique used in the proof of our main result, we prove a Poincaré inequality related to the operator Δ_γ .

2. NOTATION AND PRELIMINARY RESULTS

In this section we shall introduce some notation. Let γ be a positive real number and let $\xi = (x_1, \dots, x_d, y_1, \dots, y_k) = (x, y) \in \mathbb{R}^d \times \mathbb{R}^k = \mathbb{R}^N$ with $d, k \geq 1$ and $N = d + k$. We denote by $|x|$ (resp. $|y|$) the Euclidean norm in \mathbb{R}^d (resp. \mathbb{R}^k): $|x| := \sqrt{x_1^2 + \dots + x_d^2}$ (resp. $|y| := \sqrt{y_1^2 + \dots + y_k^2}$). Unless otherwise stated, $\Omega \subset \mathbb{R}^N$ will denote an open set.

The symbols div , ∇_ξ and Δ_ξ stand respectively for the usual divergence operator on \mathbb{R}^N , the gradient and the Laplace operator for functions defined on \mathbb{R}^N with respect to the ξ -variable.

For $i = 1, \dots, d$, and $j = 1, \dots, k$ consider the vector fields

$$X_i := \frac{\partial}{\partial x_i}, \quad Y_j := |x|^\gamma \frac{\partial}{\partial y_j},$$

and the associated gradient as follows:

$$\nabla_\gamma := (X_1, \dots, X_d, Y_1, \dots, Y_k) = (\nabla_x, |x|^\gamma \nabla_y).$$

The Grushin operator Δ_γ is the operator defined by

$$\Delta_\gamma := \sum_{i=1}^d X_i^2 + \sum_{j=1}^k Y_j^2 = \Delta_x + |x|^{2\gamma} \Delta_y = \nabla_\gamma \cdot \nabla_\gamma.$$

Defining on \mathbb{R}^N the dilation δ_λ as

$$(2.1) \quad \delta_\lambda(x, y) := (\lambda x, \lambda^{1+\gamma} y),$$

it is not difficult to check that X_i and Y_i are homogeneous of degree one with respect to the dilation $X_i(\delta_\lambda) = \lambda \delta_\lambda(X_i)$, $Y_i(\delta_\lambda) = \lambda \delta_\lambda(Y_i)$, and hence $\nabla_\gamma(\delta_\lambda) = \lambda \delta_\lambda(\nabla_\gamma)$.

Let $[\xi] = [(x, y)]$ be the following distance from the origin on \mathbb{R}^N :

$$[\xi] = [(x, y)] := \left(\left(\sum_{i=1}^d x_i^2 \right)^{1+\gamma} + (1 + \gamma)^2 \sum_{i=1}^k y_i^2 \right)^{\frac{1}{2+2\gamma}}.$$

The function $[\cdot]$ is related to the fundamental solution at the origin of the Grushin operator Δ_γ (see [6]). Furthermore, it is easy to see that $[\cdot]$ is homogeneous of degree one with respect to δ_λ . Let $R > 0$. We shall denote by B_R the set

$$B_R := \{\xi \in \mathbb{R}^N : [\xi] < R\}.$$

A function $u : \Omega \rightarrow \mathbb{R}$, such that $u(\xi) = u([\xi])$ (u depends only on $[\xi]$) is said to be *radial*.

Let $u \in \mathcal{C}^1(\Omega)$. If u is radial, then it is easy to check that

$$(2.2) \quad |\nabla_\gamma u(\xi)| = \frac{|x|^\gamma}{[\xi]^\gamma} |u'([\xi])|.$$

Moreover, if $u \in \mathcal{C}^2(\Omega)$ and is radial, we find

$$\Delta_\gamma u = \frac{|x|^{2\gamma}}{[\xi]^{2\gamma}} \left(u'' + \frac{d + (1 + \gamma)k - 1}{[\xi]} u' \right).$$

Let $\Omega = B_{R_2} \setminus \overline{B_{R_1}}$, with $0 \leq R_1 < R_2 \leq +\infty$ and $u \in \mathcal{C}(\Omega)$. As we shall see below, in some intermediate inequalities appearing in the proof of our main results, we shall need to compute $\int_{\Omega} u$. For this task we can proceed by considering the transformation $\xi := \Phi(\rho, \theta, \theta_1, \dots, \theta_{d-1}, \omega_1, \dots, \omega_{k-1})$, introduced in [6] and defined by

$$(2.3) \quad \begin{aligned} x_1 &= \rho \sin \theta (\sin^2 \theta)^{-\frac{\gamma}{2(1+\gamma)}} \cos \omega_1, \\ x_2 &= \rho \sin \theta (\sin^2 \theta)^{-\frac{\gamma}{2(1+\gamma)}} \sin \omega_1 \cos \omega_2, \\ &\dots \\ x_{d-1} &= \rho \sin \theta (\sin^2 \theta)^{-\frac{\gamma}{2(1+\gamma)}} \sin \omega_1 \sin \omega_2 \dots \cos \omega_{d-1}, \\ x_d &= \rho \sin \theta (\sin^2 \theta)^{-\frac{\gamma}{2(1+\gamma)}} \sin \omega_1 \sin \omega_2 \dots \sin \omega_{d-1}, \\ y_1 &= \frac{1}{1+\gamma} \rho^{1+\gamma} \cos \theta \cos \theta_1, \\ y_2 &= \frac{1}{1+\gamma} \rho^{1+\gamma} \cos \theta \sin \theta_1 \cos \theta_2, \\ &\dots \\ y_{k-1} &= \frac{1}{1+\gamma} \rho^{1+\gamma} \cos \theta \sin \theta_1 \sin \theta_2 \dots \cos \theta_{k-1}, \\ y_k &= \frac{1}{1+\gamma} \rho^{1+\gamma} \cos \theta \sin \theta_1 \sin \theta_2 \dots \sin \theta_{k-1}, \end{aligned}$$

for $R_1 < \rho < R_2$, $\theta_i, \omega_j \in]0, \pi[$ for $i = 1, \dots, k-2$, $j = 1, \dots, d-2$, $\theta_{k-1}, \omega_{d-1} \in]0, 2\pi[$ and $\theta \in]a_{\theta}, b_{\theta}[$ where a_{θ} and b_{θ} depend on d and k ; that is,

- $\theta \in]0, \frac{\pi}{2}[$, if $d, k \geq 2$,
- $\theta \in]0, \pi[$, if $k = 1$ and $d \geq 2$,
- $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, if $d = 1$ and $k \geq 2$,
- $\theta \in]0, 2\pi[$, if $d = k = 1$.

Let $J(\Phi)$ be the Jacobian of Φ . By a simple computation we have $|\det J(\Phi)| = \rho^{Q-1} \Theta(\theta, \theta_1, \dots, \theta_{d-2}, \omega_1, \dots, \omega_{k-2})$, where $Q := d + (1 + \gamma)k$ and

$$\Theta := \left(\frac{1}{1+\gamma}\right)^k |\sin \theta|^{\frac{d}{1+\gamma}-1} \cos^{k-1} \theta \sin^{k-2} \theta_1 \dots \sin \theta_{k-2} \sin^{d-2} \omega_1 \dots \sin \omega_{d-2}.$$

Therefore, if $u : \Omega \rightarrow \mathbb{R}$ is radial, i.e., $u(\xi) = u(\|\xi\|)$, then

$$(2.4) \quad \int_{\Omega} u(\|\xi\|) d\xi = s_n \int_{R_1}^{R_2} \rho^{Q-1} u(\rho) d\rho,$$

where

$$s_n := \int_{a_{\theta}}^{b_{\theta}} d\theta \int_0^{\pi} d\theta_1 \dots \int_0^{\pi} d\theta_{k-2} \int_0^{2\pi} d\theta_{k-1} \int_0^{\pi} d\omega_1 \dots \int_0^{\pi} d\omega_{d-2} \int_0^{2\pi} d\omega_{d-1} \Theta.$$

Let $|\cdot|$ be the Lebesgue measure on \mathbb{R}^N . We note that from (2.4) one gets

$$|B_R| = |B_1| R^Q,$$

where $Q := d + (1 + \gamma)k$ is the so-called *homogeneous dimension*.

Remark 2.1. Using (2.3), we can deduce the following criteria for the integrability of the function $|x|^p \|\xi\|^q$:

- i) if $d > -p$ and $Q > -p - q$, then

$$\int_{B_1} |x|^p \|\xi\|^q d\xi < +\infty;$$

- ii) if $d > -p$ and $Q < -p - q$, then

$$\int_{\mathbb{R}^N \setminus B_1} |x|^p \|\xi\|^q d\xi < +\infty.$$

Let $\mathcal{C}_0^k(\Omega)$ be the set of functions with compact support belonging to $\mathcal{C}^k(\Omega)$. Let p be such that $1 \leq p < \infty$. We shall denote by $D_\gamma^{1,p}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in the norm $(\int_\Omega |\nabla_\gamma u|^p d\xi)^{1/p}$. If $w \in L^1_{loc}(\Omega)$ and $w > 0$ a.e. on Ω , then $D_\gamma^{1,p}(\Omega, w)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ in the norm $(\int_\Omega |\nabla_\gamma u|^p w d\xi)^{1/p}$.

3. HARDY INEQUALITIES

Throughout this section we shall denote by Ω an open subset contained in \mathbb{R}^N . The results of this paper are the following.

Theorem 3.1. *Let $p > 1$, $d, k \geq 1$ and $\alpha, \beta \in \mathbb{R}$ be such that $d + (1 + \gamma)k > \alpha - \beta - p$ and $d > \gamma p - \beta$. Then, for every $u \in D_\gamma^{1,p}(\Omega, |x|^{\beta - \gamma p} [\xi]^{(1 + \gamma)p - \alpha})$, we have*

$$(3.1) \quad c_{Q,p,\alpha,\beta}^p \int_\Omega |u|^p \frac{|x|^\beta}{[\xi]^\alpha} d\xi \leq \int_\Omega |\nabla_\gamma u|^p |x|^{\beta - \gamma p} [\xi]^{(1 + \gamma)p - \alpha} d\xi,$$

where $c_{Q,p,\alpha,\beta} := (Q + \beta - \alpha)/p = (d + (1 + \gamma)k + \beta - \alpha)/p$.

If $0 \in \Omega$, then the constant $c_{Q,p,\alpha,\beta}^p$ in (3.1) is sharp.

In particular, if $Q := d + (1 + \gamma)k > p > 1$, then

$$(3.2) \quad \left(\frac{Q - p}{p}\right)^p \int_\Omega \frac{|u|^p |x|^{\gamma p}}{[\xi]^p} d\xi \leq \int_\Omega |\nabla_\gamma u|^p d\xi, \quad u \in D_\gamma^{1,p}(\Omega),$$

$$(3.3) \quad \left(\frac{Q - p}{p}\right)^p \int_\Omega \frac{|u|^p}{[\xi]^p} d\xi \leq \int_\Omega |\nabla_\gamma u|^p \frac{[\xi]^{\gamma p}}{|x|^{\gamma p}} d\xi, \quad d > \gamma p, u \in D_\gamma^{1,p}\left(\Omega, \frac{[\xi]^{\gamma p}}{|x|^{\gamma p}}\right),$$

$$(3.4) \quad \left(\frac{Q - p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} d\xi \leq \int_\Omega |\nabla_\gamma u|^p \frac{[\xi]^{(1 + \gamma)p}}{|x|^{(1 + \gamma)p}} d\xi,$$

$$d > (1 + \gamma)p, u \in D_\gamma^{1,p}\left(\Omega, \frac{[\xi]^{(1 + \gamma)p}}{|x|^{(1 + \gamma)p}}\right).$$

Remark 3.2. If $\gamma = 0$, then the operator Δ_γ is the standard Laplace operator acting on functions defined on \mathbb{R}^N and (3.2) is the classical Hardy inequality (see (1.3)).

Theorem 3.3. *Let $d > p > 1$. Then, for every $u \in D_\gamma^{1,p}(\Omega)$ the inequalities*

$$(3.5) \quad b_{d,p}^p \int_\Omega \frac{|u|^p}{|x|^p} d\xi \leq \int_\Omega |\nabla_\gamma u|^p d\xi,$$

$$(3.6) \quad b_{d,p}^p \int_\Omega \frac{|u|^p}{[\xi]^p} d\xi \leq \int_\Omega |\nabla_\gamma u|^p d\xi$$

hold with $b_{d,p} := (d - p)/p$.

In particular, if $p = 2$ and $d \geq 3$, we have

$$\left(\frac{d - 2}{2}\right)^2 \int_\Omega \frac{u^2}{[\xi]^2} d\xi \leq \left(\frac{d - 2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} d\xi \leq \int_\Omega |\nabla_\gamma u|^2 d\xi.$$

Remark 3.4. From the above results it follows that the best constants in (3.5) and (3.6) lie in $[(\frac{d-p}{p})^p, (\frac{Q-p}{p})^p]$.

Before proving our results, we fix some notation.

Let $\epsilon \geq 0$. Define

$$r_\epsilon := \left(\epsilon^2 + \sum_{i=1}^d x_i^2 \right)^{1/2},$$

$$\sigma_\epsilon := \begin{pmatrix} I_d & 0 \\ 0 & r_\epsilon^\gamma I_k \end{pmatrix}.$$

The symbol ∇_γ^ϵ stands for $\sigma_\epsilon \nabla$. For any vector field $h \in \mathcal{C}^1(\Omega, \mathbb{R}^N)$, we shall write $\operatorname{div}_\gamma^\epsilon(h) := \operatorname{div}(\sigma_\epsilon h)$.

Clearly, if $\epsilon = 0$, then $r_0 = |x|$ and

$$\nabla_\gamma = \nabla_\gamma^0 = \sigma_0 \nabla, \quad \Delta_\gamma = \operatorname{div}_\gamma^0(\nabla_\gamma).$$

Let Ω be an open subset of \mathbb{R}^N and $\hat{h} \in \mathcal{C}^1(\Omega, \mathbb{R}^N)$ be a smooth vector field with compact support. By the divergence theorem we have

$$\int_\Omega \operatorname{div}_\gamma^\epsilon \hat{h} d\xi = \int_{\partial\Omega} \hat{h} \cdot \sigma_\epsilon \nu d\Sigma = 0,$$

where ν denotes the exterior normal at point $\xi \in \partial\Omega$. Choosing $\hat{h} := |u|^p h$ with $p > 1$, $u \in \mathcal{C}_0^1(\Omega)$ and $h \in \mathcal{C}^1(\Omega, \mathbb{R}^N)$, we obtain

$$(3.7) \quad \int_\Omega |u|^p \operatorname{div}_\gamma^\epsilon h d\xi = -p \int_\Omega |u|^{p-2} u \nabla_\gamma^\epsilon u \cdot h d\xi.$$

Using the Hölder inequality and (3.7), we can prove the following.

Theorem 3.5. *Let $\epsilon \geq 0$ and $h \in \mathcal{C}^1(\Omega, \mathbb{R}^{2n+1})$ such that $\operatorname{div}_\gamma^\epsilon h > 0$. Then, for any $p > 1$ and $u \in \mathcal{C}_0^1(\Omega)$, we have*

$$(3.8) \quad \int_\Omega |u|^p \operatorname{div}_\gamma^\epsilon h d\xi \leq p^p \int_\Omega |h|^p |\operatorname{div}_\gamma^\epsilon h|^{-(p-1)} |\nabla_\gamma^\epsilon u|^p d\xi.$$

The main idea for proving Theorems 3.1 and 3.3 and the Poincaré inequality (3.14) (see Theorem 3.7 below), is to specialize the choice of the vector field h in the inequality (3.8).

Proof of Theorem 3.1. Without loss of generality, we shall consider smooth functions $u \in \mathcal{C}_0^\infty(\Omega)$. The general case will follow by a density argument.

Let $\epsilon > 0$. Define

$$\rho_\epsilon := \left(r_\epsilon^{2+2\gamma} + (1 + \gamma)^2 \sum_{i=1}^k y_i^2 \right)^{\frac{1}{2(1+\gamma)}}$$

and the vector field h_ϵ^1 as

$$(3.9) \quad h_\epsilon^1(\xi) := \frac{1}{\rho_\epsilon^\alpha} \begin{pmatrix} x r_\epsilon^\beta \\ (1 + \gamma) y r_\epsilon^{\beta-2} |x|^2 \end{pmatrix}.$$

A simple computation shows that

$$\begin{aligned} \operatorname{div}_\gamma^\epsilon h_\epsilon^1 &= \operatorname{div} \frac{1}{\rho_\epsilon^\alpha} \begin{pmatrix} x r_\epsilon^\beta \\ (1 + \gamma) y r_\epsilon^{\beta-2} |x|^2 \end{pmatrix} \\ &= \frac{r_\epsilon^\beta}{\rho_\epsilon^\alpha} \left(d + ((1 + \gamma)k + \beta - \alpha) \frac{|x|^2}{r_\epsilon^2} \right), \\ |h_\epsilon^1| &= \frac{r_\epsilon^{\beta-\gamma-2} |x|^2}{\rho_\epsilon^\alpha} \left(r_\epsilon^{2\gamma} |x|^2 + (1 + \gamma)^2 |y|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Let f_ϵ ($\epsilon > 0$) be defined by $f_\epsilon(r) := d + ((1 + \gamma)k + \beta - \alpha) \frac{r^2}{\epsilon^2 + r^2}$, $r \geq 0$. It is not difficult to see that

$$(3.10) \quad f_\epsilon(r) \geq \begin{cases} d & \text{if } (1 + \gamma)k + \beta - \alpha \geq 0, \\ d + (1 + \gamma)k + \beta - \alpha & \text{if } (1 + \gamma)k + \beta - \alpha < 0, \end{cases}$$

for every $r \geq 0$ and $\epsilon > 0$. Since $r_\epsilon \geq \epsilon$, if $d + (1 + \gamma)k > \alpha - \beta$, it follows that $\operatorname{div}_\gamma^\epsilon h_\epsilon^1 > 0$.

Thus we are in a position to apply Theorem 3.5 to $h = h_\epsilon^1$. Indeed, from (3.8) we obtain

$$(3.11) \quad \begin{aligned} & \int_\Omega |u|^p \frac{r_\epsilon^\beta}{\rho_\epsilon^\alpha} f_\epsilon(|x|) d\xi \\ & \leq p^p \int_\Omega |\nabla_\gamma^\epsilon u|^p \frac{|x|^{2p} r_\epsilon^{\beta-2p-\gamma p} (r_\epsilon^{2\gamma} |x|^2 + (1 + \gamma)|y|^2)^{p/2}}{\rho_\epsilon^\alpha f_\epsilon(|x|)^{p-1}} d\xi. \end{aligned}$$

Let $m := \min\{d, d + (1 + \gamma)k + \beta - \alpha\} > 0$. By (3.10) and $r < r_\epsilon$, the integrand in the right-hand side of (3.11), can be estimated as follows:

$$|\nabla_\gamma^\epsilon u|^p \frac{|x|^{2p} r_\epsilon^{\beta-2p-\gamma p} (r_\epsilon^{2\gamma} |x|^2 + (1 + \gamma)|y|^2)^{p/2}}{\rho_\epsilon^\alpha f_\epsilon(|x|)^{p-1}} \leq \frac{|\nabla_\gamma^\epsilon u|^p}{m^{p-1}} r_\epsilon^{\beta-\gamma p} \rho_\epsilon^{(1+\gamma)p-\alpha} \in L^1(\Omega).$$

Therefore, by our assumptions $d + (1 + \gamma)k > \alpha - \beta$ and $d > p - \beta$, we can apply the Lebesgue dominated convergence theorem to (3.11), and letting $\epsilon \rightarrow 0$, we obtain the claim.

The choices $(\alpha, \beta) = ((1 + \gamma)p, \gamma p)$, $(\alpha, \beta) = (p, 0)$ and $(\alpha, \beta) = (0, -p)$, in (3.1), yield the inequalities (3.2), (3.3) and (3.4) respectively.

It remains to show that the constant $c_{Q,p,\alpha,\beta}^p$ appearing in (3.1) is sharp.

First we consider the case $\Omega = \mathbb{R}^N$. In doing so we shall adapt the original idea of Hardy (see [15]) for the one-dimensional Euclidean case.

Given $\epsilon > 0$, consider the function

$$u(\rho) := \begin{cases} C_\epsilon & \text{if } \rho \in [0, 1], \\ C_\epsilon \rho^{-c_{Q,p,\alpha,\beta}-\epsilon} & \text{if } \rho > 1, \end{cases}$$

where $C_\epsilon := (c_{Q,p,\alpha,\beta} + \epsilon)^{-1}$. We have

$$u'(\rho) = \begin{cases} 0 & \text{if } \rho \in]0, 1[, \\ -\rho^{-\frac{Q+\beta-\alpha+p}{p}-\epsilon} & \text{if } \rho > 1, \end{cases}$$

and by computation

$$(3.12) \quad \begin{aligned} \int_{\mathbb{R}^N} |u(\llbracket \xi \rrbracket)|^p \frac{|x|^\beta}{\llbracket \xi \rrbracket^\alpha} d\xi &= C_\epsilon^p \left(\int_{B_1} \frac{|x|^\beta}{\llbracket \xi \rrbracket^\alpha} d\xi + \int_{\mathbb{R}^N \setminus B_1} |x|^\beta \llbracket \xi \rrbracket^{-Q-\beta-\epsilon p} d\xi \right) \\ &= C_\epsilon^p \left(\int_{B_1} \frac{|x|^\beta}{\llbracket \xi \rrbracket^\alpha} d\xi \right. \\ & \quad \left. + \int_{\mathbb{R}^N \setminus B_1} \frac{|x|^{\gamma p}}{\llbracket \xi \rrbracket^{\gamma p}} \llbracket \xi \rrbracket^{-\frac{Q+\beta-\alpha+p}{p}-\epsilon} |x|^{\beta-\gamma p} \llbracket \xi \rrbracket^{(1+\gamma)p-\alpha} d\xi \right) \\ &= C_\epsilon^p \left(\int_{B_1} \frac{|x|^\beta}{\llbracket \xi \rrbracket^\alpha} d\xi + \int_{\mathbb{R}^N} |\nabla_\gamma u|^p |x|^{\beta-\gamma p} \llbracket \xi \rrbracket^{(1+\gamma)p-\alpha} d\xi \right), \end{aligned}$$

where, in the last identity, we have used the relation (2.2) and the fact that u' vanish on B_1 . Since the addenda in the right-hand side of (3.12) are integrable (see Remark 2.1), by letting $\epsilon \rightarrow 0$, we easily get the claim.

In order to conclude in the general case, we proceed as follows: let $c_b(\Omega)$ be the best constant in (3.1). By invariance of (3.1) under the dilation δ_λ we have

$$c_b(B_R) = c_b(B_1) \quad \text{for any } R > 0.$$

Furthermore, we note that, if $B_R \subset \Omega \subset \mathbb{R}^N$, then

$$(3.13) \quad c_{Q,p,\alpha,\beta}^p = c_b(\mathbb{R}^N) \leq c_b(\Omega) \leq c_b(B_R) = c_b(B_1).$$

Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Since the support of φ is compact, then (3.1) holds for φ with $\Omega = B_R$, R large enough and $c = c_b(B_R) = c_b(B_1)$. Therefore, $c_b(B_1) \leq c_b(\mathbb{R}^N)$ and from (3.13) we conclude the proof. \square

Remark 3.6. Let $\phi \in \mathcal{C}^2(\Omega)$ and $\epsilon = 0$. Setting $h := \nabla_\gamma \phi$, we see that Theorem 3.3 can be reformulated as follows: if $\Delta_\gamma \phi > 0$, then for any $p > 1$ and $u \in \mathcal{C}_0^1(\Omega)$ we have

$$\int_\Omega |u|^p \Delta_\gamma \phi d\xi \leq p^p \int_\Omega |\nabla_\gamma \phi|^p |\Delta_\gamma \phi|^{-(p-1)} |\nabla_\gamma u|^p d\xi.$$

Following this idea, one can easily realize that the main problem is to decide whether such a function exists. Indeed, via a standard regularization argument, we see that (3.2) can be proved if there exists $\phi \in \mathcal{C}^2(\Omega)$ such that $\nabla_\gamma \phi = h_0^1$, where h_0^1 is the vector field defined in (3.9) with $\epsilon = 0$, $\alpha = (1 + \gamma)p$ and $\beta = \gamma p$. A simple computation shows that this is possible only when $\gamma = 0$ or $p = 2$.

In the case $\gamma = 0$ and $p > 1$, we obtain the classical Hardy inequality associated to the p -Laplacian operator on \mathbb{R}^N (see [9]), while if $\gamma > 0$ and $p = 2$, the function ϕ is given by $\phi(\xi) := \ln[\xi]$.

This show that in the framework of Hardy type inequalities associated to second-order elliptic operators with degenerate or singular coefficients, the method based on the choice of a general vector field is more efficient.

Proof of Theorem 3.3. Define h_ϵ^2 as follows:

$$h_\epsilon^2(\xi) := \frac{1}{r_\epsilon^p} \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \operatorname{div}_\gamma^\epsilon h_\epsilon^2 &= \frac{1}{r_\epsilon^p} \left(d - p \frac{|x|^2}{r_\epsilon^2} \right), \\ |h_\epsilon^2| &= \frac{|x|}{r_\epsilon^p}. \end{aligned}$$

Since $d > p$, we get $\operatorname{div}_\gamma^\epsilon h_\epsilon^2 > 0$, and, as in the previous case, by (3.8) it follows that

$$\int_\Omega \frac{|u|^p}{r_\epsilon^p} \left(d - p \frac{|x|^2}{r_\epsilon^2} \right) d\xi \leq p^p \int_\Omega \frac{|x|^p}{r_\epsilon^p \left(d - p \frac{|x|^2}{r_\epsilon^2} \right)^{p-1}} |\nabla_\gamma^\epsilon u|^p d\xi.$$

By the Lebesgue dominated convergence theorem, the last inequality implies (3.5). Finally, using the fact that $[\xi] \geq |x|$ and (3.5), we obtain (3.6). This concludes the proof. \square

We note that (3.8) also yields a Poincaré inequality for the vector field ∇_γ on domains Ω contained in a slab. More precisely, we have

Theorem 3.7. *Let Ω be an open subset on \mathbb{R}^N . Suppose that there exist $R > 0$, a real number s and an integer $1 \leq j \leq d$ such that for any $\xi = (x, y) \in \Omega$, it follows that $|x_j - s| \leq R$.*

Then, for every $u \in \mathcal{C}_0^1(\Omega)$, we have

$$(3.14) \quad c \int_{\Omega} |u|^p d\xi \leq \int_{\Omega} |\nabla_{\gamma} u|^p d\xi,$$

where $c = (\frac{1}{pR})^p$.

The proof of the theorem follows from Theorem 3.5 by using the vector field defined by

$$h := \begin{pmatrix} 0 \\ x_j - s \\ 0 \end{pmatrix}.$$

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