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LUZIN'S THEOREM FOR CHARGES

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Abstract. A charge in the Euclidean space $\mathbb{R}^m$ is an additive function defined on the family of all bounded BV sets equipped with a suitable topology. We define derivatives of charges and show that each measurable function defined on $\mathbb{R}^m$ is equal almost everywhere to the derivative of a charge.

A well-known theorem of Luzin [2, Théorème Fondamental, p. 90] says that for each real-valued measurable function f defined on the real line $\mathbb{R}$ there is a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F' = f$ almost everywhere. Replacing F by an additive function of compact intervals in $\mathbb{R}^m$ whose distribution function is continuous, and using the derivation base of regular compact intervals, yields a straightforward generalization of Luzin's theorem for real-valued measurable functions defined on $\mathbb{R}^m$. Recent studies, aimed toward obtaining Stokes' theorem for non-Lipschitzian forms, rely on additive functions that satisfy a more stringent continuity condition, and whose derivatives are calculated with respect to a coordinate free derivation base. Whether Luzin's theorem still holds in this context is a nontrivial question, to which we give an affirmative answer.

The ambient space of this note is $\mathbb{R}^m$ where $m \geq 1$ is a fixed integer. In $\mathbb{R}^m$ we shall use exclusively the *Euclidean norm* $|\cdot|$ induced by the usual inner product $x \cdot y$. Given $r > 0$, we let $B(r) := \big\{x \in \mathbb{R}^m : |x| < r\big\}$. A *cube* is a compact nondegenerate cube in $\mathbb{R}^m$. For $E \subset \mathbb{R}^m$, we denote by $d(E)$, $\operatorname{int} E$, and χ_E the *diameter, interior* and *indicator* of E, respectively.

We shall consider only two measures (i.e., outer measures) in $\mathbb{R}^m$: Lebesgue measure $\mathcal{L} := \mathcal{L}^m$ and the Hausdorff measure $\mathcal{H} := \mathcal{H}^{m-1}$. For $E \subset \mathbb{R}^m$, we write $|E|$ instead of $\mathcal{L}(E)$. The words "measure", "measurable", and "negligible", as well as the expressions "almost everywhere" and "almost all" always refer to Lebesgue measure $\mathcal{L}$.

The collection of all bounded BV sets in $\mathbb{R}^m$ [1, Chapter 5] is denoted by $\boldsymbol{\mathcal{BV}}$. The *regularity* of $A \in \boldsymbol{\mathcal{BV}}$ is the number

$$r(A) := \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } |A| > 0, \\ 0 & \text{if } |A| = 0 \end{cases}$$

where $\|A\|$ denotes the *perimeter* of A. The *unit exterior normal* of a BV set A, denoted by ν_A, is defined $\mathcal{H}$-almost everywhere on the *measure-theoretic boundary* $\partial_* A$ of A [1, Sections 5.7 and 5.8].

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We say that a sequence $\{A_i\}$ of BV sets *converges* to $A \subset \mathbb{R}^m$ whenever each A_i is contained in a fixed compact set $K \subset \mathbb{R}^m$,

$$\sup \|A_i\| < \infty, \quad \text{and} \quad \lim \bigl|(A - A_i) \cup (A_i - A)\bigr| = 0;$$

in this case A is necessarily in $\boldsymbol{\mathcal{BV}}$ [1, Section 5.2.1]. The convergence of BV sets induces a nonmetrizable sequential Hausdorff topology $\mathfrak{T}$ in $\boldsymbol{\mathcal{BV}}$ [3, Proposition 1.7.7].

Throughout, by a function we mean a real-valued function. A *charge* is a function F defined on $\boldsymbol{\mathcal{BV}}$ that is additive and $\mathfrak{T}$-continuous. Explicitly, a function F defined on $\boldsymbol{\mathcal{BV}}$ is a charge whenever the following conditions are satisfied.

Additivity: $F(A \cup B) = F(A) + F(B)$ for each pair of disjoint sets $A, B \in \boldsymbol{\mathcal{BV}}$.

Continuity: Given $\varepsilon > 0$, there is a $\theta > 0$ such that $\bigl|F(A)\bigr| < \varepsilon$ for each BV set $A \subset B(1/\varepsilon)$ with $\|A\| < 1/\varepsilon$ and $|A| < \theta$.

If F is a charge, then $F(B) = 0$ for each bounded negligible set $B \subset \mathbb{R}^m$. We mention two essentially distinct examples of charges [3, Examples 2.1.3 and 2.1.4]:

- if $f \in L^1_{\text{loc}}(\mathbb{R}^m)$, then $A \mapsto \int_A f \, d\mathcal{L} : \boldsymbol{\mathcal{BV}} \to \mathbb{R}$ is a charge called the *Lebesgue primitive* of f;
- if $v : \mathbb{R}^m \to \mathbb{R}^m$ is continuous, then $A \mapsto \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H} : \boldsymbol{\mathcal{BV}} \to \mathbb{R}$ is a charge called the *flux* of v.

Given a charge F and a locally BV set $E \subset \mathbb{R}^m$, define a charge $F \llcorner E$ by the formula

$$(F \llcorner E)(A) := F(A \cap E)$$

for each bounded BV set A; if $F = F \llcorner E$, we say that F is a *charge in* E. With no danger of confusion, the charge $\mathcal{L} \restriction \boldsymbol{\mathcal{BV}}$ is denoted by $\mathcal{L}$.

For a function g defined on a locally BV set E, let $\overline{g}(x) = g(x)$ if $x \in E$ and $\overline{g}(x) = 0$ if $x \in \mathbb{R}^m - E$. The resulting function $\overline{g}$ on $\mathbb{R}^m$ is called the *zero extension* of g. When $\overline{g} \in L^1_{\text{loc}}(\mathbb{R}^m)$, we call the above-defined Lebesgue primitive G of $\overline{g}$ the *Lebesgue primitive* of g. Clearly G is a charge in E.

Let F be a charge and $x \in \mathbb{R}^m$. If a finite limit

$$\lim \frac{F(B_i)}{|B_i|}$$

exists for each sequence $\{B_i\}$ in $\boldsymbol{\mathcal{BV}}$ with

$$\lim d\bigl(B_i \cup \{x\}\bigr) = 0 \quad \text{and} \quad \inf r\bigl(B_i \cup \{x\}\bigr) > 0,$$

then all these limits have the same value, denoted by $DF(x)$ and called the *derivative* of F at x.

Remark. If F is the flux of a continuous $v : \mathbb{R}^m \to \mathbb{R}^m$, then $DF(x) = \operatorname{div} v(x)$ exists at each x at which v is differentiable [3, Example 2.3.2]. This simple observation leads to a very general Gauss-Green theorem — the main reason for studying charges and their derivatives.

Let $E \subset \mathbb{R}^m$ be a locally BV set. A function f defined on E is called a *derivative* if there is a charge F in E such that $DF(x) = f(x)$ for almost all $x \in E$; in this case we say that f is *the derivative* of F, and F is *a primitive* of f. Each primitive of the zero function in $\mathbb{R}^m$ is called a *singular* charge. If a function g defined on E has the Lebesgue primitive G, then G is a primitive of g according to the previous definition [3, Theorem 3.1.9].

Let F be a charge, and let E be a locally BV set. The *oscillation* of F on E, denoted by $O(F,E)$, is the infimum of all $\varepsilon > 0$ such that $\left|F(B)\right| < \varepsilon$ for each bounded BV set $B \subset E$ with $\|B\| < 1/\varepsilon$. We call $O(F) := O(F, \mathbb{R}^m)$ the *total oscillation* of F. In general $0 \leq O(F,E) \leq \infty$, but observe $O(F,E) = 0$ if and only if $F \llcorner E = 0$.

Observation 1. *Let F be a charge. Then $O(F \llcorner E) = O(F,E)$ for each convex locally BV set E, and $O(F,A) < \infty$ for each $A \in \mathcal{BV}$.*

Proof. If E is a convex locally BV set, then $\|B \cap E\| \leq \|B\|$ by [3, Corollary 1.9.4], and the first claim follows. Next choose $A \in \mathcal{BV}$, find a cube K containing A, and denote by $K_1, \dots, K_{p^m}$ the nonoverlapping congruent cubes whose union is K. For $i = 1, \dots, p^m$ and p sufficiently large, $|F(B)| < 1$ whenever $B \subset K_i$ is a BV set with $\|B\| < 1$. If $B \subset A$ is any BV set with $\|B\| < 1/p^m$, then $\|B \cap K_i\| \leq \|B\| < 1$. Hence $\left|F(B)\right| \leq \sum_{i=1}^{p^m} \left|F(B \cap K_i)\right| < p^m$, which implies $O(F,A) \leq p^m < \infty$. □

It follows from [3, Proposition 2.3.5] that each derivative in a locally BV set E is a measurable function. Our goal is to show that the converse is also true. Specifically, we prove the following theorem.

Main Theorem. *Let f be a measurable function defined on a locally BV set E. Given $\varepsilon > 0$, there is a primitive F of f with $O(F,E) < \varepsilon$.*

Remark. If $m = 1$, the theorem follows directly from the classical Luzin theorem [4, Chapter VII, Theorem (2.3)]. However, for $m \geq 2$ we have a *new result*, which differs from the generalizations mentioned at the end of [4, Chapter VII, Section 2] in small print. Notwithstanding, our proof still follows the main idea of Luzin's original argument.

Lemma 2. *Let K be a cube, and let $G := \mathcal{L} \llcorner K$. Given $\varepsilon > 0$, there is a nonnegative singular charge S in K such that $O(G - S) < \varepsilon$.*

Proof. With no loss of generality, we may assume $K = J^m$ where $J := [0, h]$. Choose an integer $p > h/\varepsilon^2$, let $c_0 := 0$, and for $i = 1, \dots, p$, let $c_i := i(h/p)$ and $J_i := [c_{i-1}, c_i]$. In each J_i select a negligible perfect set C_i, and using [4, Chapter III, Section (13.4)], construct a continuous increasing function $s_i : J_i \to \mathbb{R}$ so that $s_i(c_{i-1}) = c_{i-1}$, $s_i(c_i) = c_i$, and s_i is constant on each connected component of $J_i - C_i$. Define a continuous increasing function s in $\mathbb{R}$ by letting $s(t) := s_i(t)$ for $t \in J_i$, $s(t) := 0$ for $t < 0$, and $s(t) := h$ for $t > h$. If $g(t)$ equals t, or 0, or h, according to whether $t \in J$, or $t < 0$, or $t > h$, respectively, then $g : t \mapsto g(t)$ is a continuous increasing function in $\mathbb{R}$ such that $|s - g| < h/p$. Let

$$v_s(x) = \big(s(\xi_1), 0, \dots, 0\big) \quad \text{and} \quad v_g(x) = \big(g(\xi_1), 0, \dots, 0\big)$$

for each $x = (\xi_1, \dots, \xi_m)$ in $\mathbb{R}^m$, and denote by S_1 and G_1 the flux of v_s and v_g, respectively. The Gauss-Green theorem [3, Corollary 3.2.10] yields $G = G_1 \llcorner K$. Observe that $C := \bigcup_{i=1}^{p} C_i$ is a perfect negligible subset of J, and the function s is constant on each connected component of $\mathbb{R} - C$. Hence $S_1(B) = 0$ for every bounded BV set B whose closure is contained in a connected component of $\mathbb{R}^m - (C \times \mathbb{R}^{m-1})$. It follows that S_1 is a nonnegative singular charge, and so is

$S := S_1 \llcorner K$. If $B \subset K$ is a BV set with $\|B\| < 1/\varepsilon$, then

$$\begin{aligned}\bigl|G(B) - S(B)\bigr| &= \bigl|G_1(B) - S_1(B)\bigr| \\ &= \left|\int_{\partial_* B} (v_g - v_s) \cdot \nu_B \, d\mathcal{H}\right| \le \frac{h}{p}\|B\| < \frac{h}{p\varepsilon} < \varepsilon.\end{aligned}$$

This and Observation 1 imply that $O(G - S) = O(G - S, K) < \varepsilon$. ☐

Corollary 3. *Let g be a bounded measurable function defined on a cube K. Given $\varepsilon > 0$, there is a primitive F of g with $O(F) < \varepsilon$.*

Proof. We denote by G the Lebesgue primitive of g, and split the proof into four steps.

If $g = \chi_U$ where $U \subset K$ is an open set, find nonoverlapping cubes $K_1, \dots, K_p$ contained in U so that the measure of $V := U - \bigcup_{i=1}^{p} K_i$ is less than $\varepsilon/2$. For $i = 1, \dots, p$, denote by S_i a singular charge associated with $G_i = \mathcal{L} \llcorner K_i$ and $\varepsilon/(2p)$ according to Lemma 2. Observe that $G = \mathcal{L} \llcorner V + \sum_{i=1}^{p} G_i$, and let $S = \sum_{i=1}^{p} S_i$. By Lemma 2,

$$\bigl|G(B) - S(B)\bigr| \le |B \cap V| + \sum_{i=1}^{p} \Bigl|G_i(B) - S_i(B)\Bigr| < \varepsilon$$

for each $B \in \mathcal{BV}$ with $\|B\| < 1/\varepsilon$.

Let $g = \chi_E$ where $E \subset K$ is a measurable set. With no loss of generality, we may assume $E \subset \operatorname{int} K$, and find an open set $U \subset K$ so that $E \subset U$ and $|U - E| < \varepsilon/2$. If S is a singular charge associated with χ_U and $\varepsilon/2$ according to the first paragraph, then

$$\bigl|G(B) - S(B)\bigr| \le \bigl|B \cap (U - E)\bigr| + \Bigl||B \cap U| - S(B)\Bigr| < \varepsilon$$

for each $B \in \mathcal{BV}$ with $\|B\| < 1/\varepsilon$.

If g is a simple function, then $g = \sum_{i=1}^{p} c_i \chi_{E_i}$ where $c_1, \dots, c_p$ are real numbers and $E_1, \dots, E_p$ are measurable subsets of K. Thus $G = \sum_{i=1}^{p} c_i G_i$ where G_i is the Lebesgue primitive of χ_{E_i}. Let $c := 1 + \sum_{i=1}^{p} |c_i|$, and denote by S_i a singular charge associated with χ_{E_i} and ε/c according to the second paragraph. Letting $S = \sum_{i=1}^{p} c_i S_i$, we obtain

$$\bigl|G(B) - S(B)\bigr| \le \sum_{i=1}^{p} |c_i| \cdot \bigl|G_i(B) - S_i(B)\bigr| < \varepsilon$$

for each $B \in \mathcal{BV}$ with $\|B\| < 1/\varepsilon$.

In the general case, find a simple measurable function g_1 so that $\int_K |g - g_1| \, d\mathcal{L} < \varepsilon/2$. If S is a singular charge associated with g_1 and $\varepsilon/2$ according to the third paragraph, then

$$\bigl|G(B) - S(B)\bigr| \le \int_{B \cap K} |g - g_1| \, d\mathcal{L} + \left|\int_B g_1 \, d\mathcal{L} - S(B)\right| < \varepsilon$$

for each $B \in \mathcal{BV}$ with $\|B\| < 1/\varepsilon$. Now $F := G - S$ has the desired properties. ☐

Lemma 4. *Let $\{F_i\}$ be a sequence of charges in a locally BV set E, and let $\sum_{i=1}^{\infty} O(F_i) < \infty$. Then $F := \sum_{i=1}^{\infty} F_i$ is a charge in E and $O(F) \le \sum_{i=1}^{\infty} O(F_i)$.*

Proof. Let $\eta > \sum_{i=1}^{\infty} O(F_i)$, and select $\eta_i > O(F_i)$ so that $\sum_{i=1}^{\infty} \eta_i < \eta$. Given $B \in \mathcal{BV}$, there is an integer $n \geq 1$ with $\|B\| < 1/\eta_i$ for each $i > n$. Thus

$$\sum_{i=1}^{\infty} \bigl|F_i(B)\bigr| = \sum_{i=1}^{n} \bigl|F_i(B)\bigr| + \sum_{i=n+1}^{\infty} \bigl|F_i(B)\bigr| < \sum_{i=1}^{n} \bigl|F_i(B)\bigr| + \sum_{i=n+1}^{\infty} \eta_i < \infty,$$

and we see that the function $F := \sum_{i=1}^{\infty} F_i$ is well defined and additive on $\mathcal{BV}$. Moreover, $F(B) = F(B \cap E)$ for each $B \in \mathcal{BV}$.

Choose an $\varepsilon > 0$ and find $\theta_i > 0$ so that $\bigl|F_i(B)\bigr| < (\varepsilon/2)2^{-i}$ for each BV set $B \subset B(1/\varepsilon)$ with $\|B\| < 1/\varepsilon$ and $|B| < \theta_i$. Select a positive integer n for which $\sum_{i=n+1}^{\infty} \eta_i < \varepsilon/2$, and let $\theta := \min\{\theta_1, \dots, \theta_n\}$. If $B \subset B(1/\varepsilon)$ is a BV set with $\|B\| < 1/\varepsilon$ and $|B| < \theta$, then

$$\bigl|F(B)\bigr| \leq \sum_{i=1}^{n} \bigl|F_i(B)\bigr| + \sum_{i=n+1}^{\infty} \bigl|F_i(B)\bigr| < \frac{\varepsilon}{2} \sum_{i=1}^{n} 2^{-i} + \sum_{i=n+1}^{\infty} \eta_i < \varepsilon,$$

since $\|B\| < 1/\eta_i$ for $i > n$. Hence F is a charge in E.

Finally, let $B \in \mathcal{BV}$ and $\|B\| < 1/\eta$. Since $1/\eta < 1/\eta_i$ for each i,

$$\bigl|F(B)\bigr| \leq \sum_{i=1}^{\infty} \bigl|F_i(B)\bigr| < \sum_{i=1}^{\infty} \eta_i < \eta.$$

The inequality $O(F) \leq \sum_{i=1}^{\infty} O(F_i)$ follows from the arbitrariness of η. □

Remark. Lemma 4 implies that $\sigma : (F, G) \mapsto O(F - G)$ is a *complete invariant metric* in the linear space CH_0 of all charges whose total oscillation is finite. By Observation 1, the linear space $CH(A)$ of all charges in a bounded BV set A is a closed subspace of (CH_0, σ), and it is easy to verify that the topology of $\bigl(CH(A), \sigma\bigr)$ is the same as that of the Banach space $\bigl(CH(A), \|\cdot\|_k\bigr)$ defined in [3, Proposition 2.2.4].

Let $\kappa := m^{-m}\bigl|B(1)\bigr|^{-1}$, and recall that if $m \geq 2$, then for each $B \in \mathcal{BV}$ the *isoperimetric inequality* [3, Theorem 1.8.7] asserts that

$$(*) \qquad |B|^{m-1} \leq \kappa \|B\|^m.$$

Lemma 5. *Let g be a bounded measurable function defined on a cube K. Given $\varepsilon > 0$ and a closed set $C \subset K$, there is a charge F in K satisfying the following conditions.*

1. *$O(F) < \varepsilon$.*
2. *$DF(x) = g(x)$ for almost all $x \in K - C$.*
3. *If $x \in C$ and $\eta > 0$, then $\bigl|F(B)\bigr| < (\kappa\varepsilon/\eta^m)|B|$ for each BV set B with $r\bigl(B \cup \{x\}\bigr) > \eta$ and $d\bigl(B \cup \{x\}\bigr) \leq 1$.*
4. *$DF(x) = 0$ for all x in $C \cup (\mathbb{R}^m - K)$.*

Proof. The open set $\operatorname{int} K - C$ is the union of nonoverlapping cubes $K_1, K_2, \dots$, and $0 < d_i < d(K)$ for each distance d_i between K_i and C. Let $\varepsilon' := \varepsilon/\bigl[1 + d(K)^m\bigr]$, and let F_i be a primitive of $g \restriction K_i$ associated with $\varepsilon_i := \varepsilon' d_i^m 2^{-i}$ according to Corollary 3. Since $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon' d(K)^m < \varepsilon$, Corollary 3 and Lemma 4 imply that $F := \sum_{i=1}^{\infty} F_i$ is a charge in K that satisfies condition (1).

Since $F(B) = F_i(B)$ for every BV set $B \subset \operatorname{int} K_i$, it is clear that $DF(x) = DF_i(x) = g(x)$ for almost all $x \in \operatorname{int} K_i$. Since $\bigcup_{i=1}^{\infty} \operatorname{int} K_i$ differs from $K - C$ by a negligible set, property (2) holds.

Choose an $x \in C$ and $\eta > 0$, and find a positive integer n so that $\varepsilon_i < \eta$ for $i > n$. Select a BV set B with $r\bigl(B \cup \{x\}\bigr) > \eta$ and $d := d\bigl(B \cup \{x\}\bigr) \leq 1$. The inequality

$$\eta < \frac{|B|}{d\|B\|} \leq \frac{d^{m-1}}{\|B\|} \leq \frac{1}{\|B\|}$$

implies that $\|B\| < 1/\varepsilon_i$ for $i = 1, 2, \ldots$. If $B \cap K_i = \emptyset$, then $F_i(B) = 0$. If $B \cap K_i \neq \emptyset$, then $\bigl|F_i(B)\bigr| < \varepsilon_i \leq \varepsilon' d^m 2^{-i}$ since $d_i \leq d$. Thus

$$(**) \qquad \bigl|F(B)\bigr| \leq \sum_{i=1}^{\infty} \bigl|F_i(B)\bigr| < \varepsilon' d^m \sum_{i=1}^{\infty} 2^{-i} < \varepsilon d^m.$$

With the help of inequality $(*)$, observe that $\eta^m < \kappa|B|/d^m$ [3, Inequality (2.3.1)]. Hence $\bigl|F(B)\bigr| < (\kappa\varepsilon/\eta^m)|B|$, which establishes condition (3).

Using the same x, η, and n as in the previous paragraph, select a BV set B so that $r\bigl(B \cup \{x\}\bigr) > \eta$ and $d := d\bigl(B \cup \{x\}\bigr) < \delta$ where $\delta := \min\{1, d_1, \ldots, d_n\}$. Since this implies that $F_i(B) = 0$ for $i = 1, \ldots, n$, inequality $(**)$ is replaced by

$$\bigl|F(B)\bigr| \leq \sum_{i=n+1}^{\infty} \bigl|F_i(B)\bigr| < \varepsilon' d^m \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n} \varepsilon d^m.$$

Consequently, $\bigl|F(B)\bigr|/|B| < 2^{-n}(\kappa\varepsilon/\eta^m)$, which means that the quotient $\bigl|F(B)\bigr|/|B|$ can be made as small as we wish by taking n sufficiently large. In other words, $DF(x) = 0$, and condition (4) follows. Indeed, $DF(x) = 0$ for each $x \in \mathbb{R}^m - K$, because K is closed and F is a charge in K. □

Observation 6. *Let h be a measurable function defined almost everywhere on a measurable set E of finite measure. For $\beta > 0$, there is a measurable set $A \subset E$ such that $|E - A| < \beta$ and h is defined and bounded on A.*

Proof. Denote by E_n the set of all $x \in E$ for which $h(x)$ is defined and $h(x) < n$, and observe that $\lim |E - E_n| = 0$. □

Lemma 7. *Let $C_0 := \emptyset$, and let g be a measurable function defined on $\mathbb{R}^m$. Given $\varepsilon > 0$, there are charges F_i in $K_i := [-i, i]^m$ and closed sets $C_i \subset K_i$ such that the following conditions are met for $i = 1, 2, \ldots$.*

1. *$C_{i-1} \subset C_i$ and $|K_i - C_i| < 1/i$.*
2. *$O(F_i) < \varepsilon/2^i$, and $DF_i(x)$ exists for almost all $x \in \mathbb{R}^m$.*
3. *If $G_i = \sum_{j=1}^{i} F_j$, then $DG_i(x) = g(x)$ for each $x \in C_i$.*
4. *If $x \in C_{i-1}$ and $\eta > 0$, then $\bigl|F(B)\bigr| < (\kappa\varepsilon/2^i\eta^m)|B|$ for each BV set B with $r\bigl(B \cup \{x\}\bigr) > \eta$ and $d\bigl(B \cup \{x\}\bigr) \leq 1$.*

Proof. According to Observation 6, there is a measurable set $E_1 \subset K_1$ such that $|K_1 - E_1| < 1$ and the zero extension g_1 of $g \restriction E_1$ is bounded. In view of Corollary 3, the restriction $g_1 \restriction K_1$ has a primitive F_1 with $O(F_1) < \varepsilon/2$. In particular, $DF_1(x)$ exists for almost all $x \in \mathbb{R}^m$, and $DF_1(x) = g(x)$ for almost all $x \in E_1$. We can find a closed set $C_1 \subset E_1$ so that $|K_1 - C_1| < 1$ and $DF_1(x) = g(x)$ exists for each $x \in C_1$. Since $C_0 = \emptyset$ and $G_1 = F_1$, the charge F_1 satisfies conditions (1)–(4).

Proceeding inductively, suppose that charges F_i in K_i and closed sets $C_i \subset K_i$ satisfying conditions (1)–(4) have been constructed for each positive integer i smaller than an integer $n \geq 2$. By Observation 6, there is a measurable set $E_n \subset K_n - C_{n-1}$ such that $\bigl|(K_n - C_{n-1}) - E_n\bigr| < 1/n$ and the function $x \mapsto$

$g(x) - DG_{n-1}(x)$ is defined and bounded on E_n. Denote by g_n the zero extension of this function, and by F_n the charge associated with K_n, $g_n \restriction K_n$, $\varepsilon/2^n$, and C_{n-1} according to Lemma 5. Clearly F_n satisfies conditions (2) and (4), and $DF_n(x) = 0$ for each $x \in C_{n-1}$. Since $DF_n(x) = g_n(x)$ for almost all $x \in E_n$, there is a closed set $D_n \subset E_n$ such that $\bigl|(K_n - C_{n-1}) - D_n\bigr| < 1/n$ and $DF_n(x) = g_n(x)$ exists for each $x \in D_n$. Letting $C_n := C_{n-1} \cup D_n$ satisfies condition (1). Since $G_n = G_{n-1} + F_n$, we have

$$DG_n(x) = DG_{n-1}(x) + g_n(x) = DG_{n-1}(x) + \bigl[g(x) - DG_{n-1}(x)\bigr] = g(x)$$

for each $x \in D_n$, and by the induction hypothesis,

$$DG_n(x) = DG_{n-1}(x) = g(x)$$

for each $x \in C_{n-1}$. Consequently, condition (3) is also satisfied. □

Proof of the Main Theorem. Denote by g the zero extension of f, and choose an $\varepsilon > 0$. For $i = 1, 2, \ldots$, let $K_i = [-i, i]^m$, C_i, and F_i be associated with g and ε according to Lemma 7. Lemma 4 shows that $G := \sum_{i=1}^{\infty} F_i = \lim G_i$ is a charge with $O(G) < \varepsilon$. If $C := \bigcup_{i=1}^{\infty} C_i$, then $|K_i - C| \le |K_i - C_i| < 1/i$ for $i = 1, 2, \ldots$. Thus each $K_i - C$ is a negligible set, and so is $\mathbb{R}^m - C = \bigcup_{i=1}^{\infty}(K_i - C)$. Choose an $x \in C$ and a sequence $\{B_k\}$ of bounded BV sets such that $\lim d\bigl(B_k \cup \{x\}\bigr) = 0$ and $\inf r\bigl(B_k \cup \{x\}\bigr) > \eta > 0$. With no loss of generality, we may assume $d\bigl(B_k \cup \{x\}\bigr) \le 1$ for each k. Since $\{C_i\}$ is an increasing sequence, there is an integer $p \ge 1$ such that $x \in C_n$ whenever $n > p$. Given an integer $n > p$, condition (4) of Lemma 7 yields

$$\begin{aligned} \left|\frac{G(B_k)}{|B_k|} - g(x)\right| &\le \left|\frac{G_n(B_k)}{|B_k|} - g(x)\right| + \sum_{n+1}^{\infty}\left|\frac{F_i(B_k)}{|B_k|}\right| \\ &< \left|\frac{G_n(B_k)}{|B_k|} - g(x)\right| + \frac{\kappa\varepsilon}{\eta^m}\sum_{i=n+1}^{\infty} 2^{-i}. \end{aligned}$$

By condition (3) of Lemma 7, $\limsup\bigl|G(B_k)/|B_k| - g(x)\bigr| \le 2^{-n}(\kappa\varepsilon/\eta^m)$. Since $n > p$ is arbitrary, $DG(x) = g(x)$. If $F := G \llcorner E$, then $O(F, E) = O(G, E) \le O(G) < \varepsilon$. Moreover, $DF(x) = DG(x) = f(x)$ exists for almost all $x \in E$ by [3, Corollary 2.5.4]. □

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