

## ON $q$ -ANALOGUES OF THE EULER CONSTANT AND LERCH'S LIMIT FORMULA

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ABSTRACT. We introduce and study a  $q$ -analogue  $\gamma(q)$  of the Euler constant via a suitably defined  $q$ -analogue of the Riemann zeta function. We show, in particular, that the value  $\gamma(2)$  is irrational. We also present a  $q$ -analogue of the Hurwitz zeta function and establish an analogue of the limit formula of Lerch in 1894 for the gamma function. This limit formula can be regarded as a natural generalization of the formula of  $\gamma(q)$ .

### 1. INTRODUCTION

The Euler constant

$$\gamma = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right) = 0.5772156649 \cdots$$

is a famous mysterious constant. Actually, the true nature of the Euler constant concerning the irrationality and the transcendency is not known. The most natural appearance of the Euler constant seems to be

$$(1.1) \quad \gamma = \lim_{s \rightarrow 1} \left( \zeta(s) - \frac{1}{s-1} \right),$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re}(s) > 1,$$

is the Riemann zeta function. More precisely, the Laurent expansion of  $\zeta(s)$  around  $s = 1$  is given by

$$\zeta(s) = \frac{1}{s-1} + \gamma + c_1(s-1) + c_2(s-1)^2 + \cdots.$$

Keeping the expression (1.1) in mind, in 1894, Lerch [L] found the remarkable limit formula:

$$(1.2) \quad \lim_{s \rightarrow 1} \left( \zeta(s, x) - \frac{1}{s-1} \right) = -\frac{\Gamma'}{\Gamma}(x).$$

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Here  $\zeta(s, x)$  is the Hurwitz zeta function defined by the series

$$(1.3) \quad \zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \operatorname{Re}(s) > 1,$$

for  $x \notin \mathbb{Z}_{\leq 0}$  and  $\Gamma(x)$  is the gamma function. Since we know the facts  $\Gamma(1) = 1$ ,  $\Gamma'(1) = -\gamma$ , the formula (1.2) is considered as a natural generalization of (1.1). This (1.2) is also a prototype of many important limit formulas such as Kronecker's limit formulas.

The aim of the present paper is to investigate a  $q$ -analogue of the Euler constant and further to establish the  $q$ -analogue of Lerch's limit formula (1.2) using Jackson's  $q$ -gamma function  $\Gamma_q(x)$  (cf. Moak [M]). Throughout the present paper we assume that  $q > 1$  and put  $[x]_q = \frac{q^x - 1}{q - 1}$  for  $x \in \mathbb{C}$ .

### 2. A $q$ -EULER CONSTANT

In order to introduce a  $q$ -Euler constant properly, we study the following  $q$ -analogue of the Riemann zeta function  $\zeta(s)$ :

$$(2.1) \quad \zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s}, \quad \operatorname{Re}(s) > 1.$$

First we look at the analytic nature of  $\zeta_q(s)$ .

**Theorem 2.1.** *Suppose  $q > 1$ . We have*

- (1)  $\zeta_q(s)$  is meromorphic for  $s \in \mathbb{C}$ .
- (2) Around  $s = 1$  we have the Laurent expansion

$$(2.2) \quad \zeta_q(s) = \frac{q-1}{\log q} \cdot \frac{1}{s-1} + \gamma(q) + c_1(q)(s-1) + \dots$$

with

$$(2.3) \quad \gamma(q) = \sum_{n=1}^{\infty} \frac{1}{[n]_q} + \frac{(q-1) \log(q-1)}{\log q} - \frac{q-1}{2}.$$

*Proof.* Using the binomial expansion, one calculates

$$\begin{aligned} \zeta_q(s) &= (q-1)^s \sum_{n=1}^{\infty} q^n (q^n - 1)^{-s} \\ &= (q-1)^s \sum_{n=1}^{\infty} q^{n(1-s)} (1 - q^{-n})^{-s} \\ &= (q-1)^s \sum_{n=1}^{\infty} q^{n(1-s)} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^k q^{-nk} \\ &= (q-1)^s \sum_{k=0}^{\infty} \frac{s(s+1) \cdots (s+k-1)}{k!} \sum_{n=1}^{\infty} q^{-n(s+k-1)} \\ &= (q-1)^s \sum_{k=0}^{\infty} \frac{s(s+1) \cdots (s+k-1)}{k!} \frac{1}{q^{s+k-1} - 1}. \end{aligned}$$

This shows that  $\zeta_q(s)$  is meromorphic for  $s \in \mathbb{C}$ .

To prove the second assertion, let us look at the expression above around  $s = 1$ :

$$\zeta_q(s) = (q - 1)^s \left\{ \frac{1}{q^{s-1} - 1} + s \cdot \frac{1}{q^s - 1} + \frac{s(s + 1)}{2} \cdot \frac{1}{q^{s+1} - 1} + \dots \right\}.$$

Then, since

$$(q - 1)^s = (q - 1) + \{(q - 1) \log(q - 1)\}(s - 1) + a_2(s - 1)^2 + \dots,$$

$$\frac{1}{q^{s-1} - 1} = \frac{1}{\log q} \cdot \frac{1}{s - 1} - \frac{1}{2} + b_1(s - 1) + \dots,$$

we find

$$\zeta_q(s) = \frac{q - 1}{\log q} \cdot \frac{1}{s - 1} + \left[ -\frac{q - 1}{2} + \frac{(q - 1) \log(q - 1)}{\log q} \right. \\ \left. + (q - 1) \left\{ \frac{1}{q - 1} + \frac{1}{q^2 - 1} + \frac{1}{q^3 - 1} + \dots \right\} \right] + c_1(s - 1) + \dots.$$

Hence we observe that the constant term is given by

$$\gamma(q) = \sum_{n=1}^{\infty} \frac{1}{[n]_q} + \frac{(q - 1) \log(q - 1)}{\log q} - \frac{q - 1}{2}.$$

This completes the proof of the theorem. □

In view of the formula (1.1) for representing the Euler constant, the formula (2.2) yields the following definition.

**Definition.** We call  $\gamma(q)$  the  $q$ -Euler constant.

**Example.** We list a few examples:

$$\begin{aligned} \gamma(2) &= \sum_{n=1}^{\infty} \frac{1}{2^n - 1} - \frac{1}{2}, \\ \gamma(3) &= 2 \sum_{n=1}^{\infty} \frac{1}{3^n - 1} - 1 + \frac{2 \log 2}{\log 3}, \\ \gamma(4) &= 3 \sum_{n=1}^{\infty} \frac{1}{4^n - 1} - \frac{3}{2} + \frac{3 \log 3}{2 \log 2}, \\ \gamma(5) &= 4 \sum_{n=1}^{\infty} \frac{1}{5^n - 1} - 2 + \frac{8 \log 2}{\log 5}, \\ \gamma(6) &= 5 \sum_{n=1}^{\infty} \frac{1}{6^n - 1} - \frac{5}{2} + \frac{5 \log 5}{\log 6}, \quad \text{etc.} \end{aligned}$$

We recall the  $q$ -gamma function  $\Gamma_q(x)$  of Jackson [J] constructed as

$$(2.4) \quad \Gamma_q(x)^{-1} = q^{-\binom{x}{2}} (q - 1)^{x-1} \prod_{n=0}^{\infty} \frac{1 - q^{-x-n}}{1 - q^{-1-n}}$$

$$(2.5) \quad = q^{-x(x+1)/2} (q - 1)^x [x]_q \prod_{n=1}^{\infty} \left( 1 + q^{-x} \frac{[x]_q}{[n]_q} \right).$$

We refer to Moak [M] concerning the  $q$ -gamma function (also cf. [A]).

**Theorem 2.2.** *We have*

$$(2.6) \quad \gamma(q) = -\Gamma'_q(1) \cdot \frac{q-1}{\log q}.$$

*Proof.* By logarithmic differentiation we have

$$-\frac{\Gamma'_q(1)}{\Gamma_q(1)} = -\frac{1}{2} \log q + \log(q-1) + (\log q) \sum_{n=0}^{\infty} \frac{1}{q^{n+1}-1}.$$

Thus, using the fact that  $\Gamma_q(1) = 1$ , we obtain

$$-\Gamma'_q(1) = \frac{\log q}{q-1} \cdot \gamma(q).$$

This proves the theorem.  $\square$

We show next that the  $q$ -Euler constant tends to the Euler constant for the limit  $q \downarrow 1$  as follows.

**Theorem 2.3.** *We have*

$$(2.7) \quad \lim_{q \downarrow 1} \gamma(q) = \gamma.$$

*Proof.* From the fact (see Moak [M])

$$\lim_{q \downarrow 1} \Gamma_q(x) = \Gamma(x),$$

where  $\Gamma(x)$  is the usual gamma function given by

$$\Gamma(x)^{-1} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}},$$

we have

$$\lim_{q \downarrow 1} \Gamma'_q(1) = \Gamma'(1) = -\gamma.$$

Since  $\lim_{q \downarrow 1} \frac{q-1}{\log q} = 1$ , the result follows from Theorem 2.2 immediately.  $\square$

*Remark 2.1.* In [KKW] we have shown that

$$(2.8) \quad \lim_{q \downarrow 1} \zeta_q(s) = \zeta(s) \quad (\forall s \in \mathbb{C}, s \neq 1).$$

Note that we are assuming  $q > 1$  here, but in [KKW]  $q$  is taken as  $0 < q < 1$ . Thus, in view of the definition of  $\gamma(q)$  in terms of  $\zeta_q(s)$ , the assertion in the theorem above seems quite consistent with this equation (2.8). (See also Remark 2.3 below.)

As to the question concerning the irrationality of the  $q$ -Euler constant  $\gamma(q)$  we have the following partial answer.

**Theorem 2.4.** *Let  $q \geq 2$  be an integer. Then*

$$\gamma(q) - \frac{(q-1) \log(q-1)}{\log q}$$

*is an irrational number. In particular,  $\gamma(2)$  is irrational.*

*Proof.* By Theorem 2.1 we have

$$\begin{aligned} \gamma(q) - \frac{(q-1)\log(q-1)}{\log q} &= (q-1) \sum_{n=1}^{\infty} \frac{1}{q^n - 1} - \frac{q-1}{2} \\ &= (q-1) \sum_{n=1}^{\infty} \frac{d(n)}{q^n} - \frac{q-1}{2}, \end{aligned}$$

where  $d(n)$  denotes the number of the divisors of  $n$ . Hence the irrationality of the left-hand side follows directly from the result of Erdős [E].  $\square$

*Remark 2.2.* It is interesting to judge the irrationality of  $\gamma(q)$  for  $q = 3, 4, 5, \dots$ . One may also ask the transcendency of  $\gamma(q)$ . The point is, of course, the existence of the log-term. As to the series expression for the Euler constant  $\gamma = \lim_{q \downarrow 1} \gamma(q)$  involving a log-term, the following is known (for instance, see [S]):

$$\gamma = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left[ \frac{\log n}{\log 2} \right],$$

where the bracket  $[x]$  denotes the integer part of  $x \in \mathbb{R}$ .

*Remark 2.3.* The  $q$ -analogue  $\zeta_q(s)$  of  $\zeta(s)$  that we are treating here is a special case of  $f_q(s, t)$  considered in [KKW] and defined by

$$f_q(s, t) = \sum_{n=1}^{\infty} q^{-nt} [n]_q^{-s}.$$

In fact,  $\zeta_q(s) = f_q(s, -1)$ . It is known in [KKW] that  $f_q(s, t)$  is meromorphically extended via the formula (the proof is the same as our assertion (1) of Theorem 2.1)

$$f_q(s, t) = (q-1)^{s-1} \sum_{r=0}^{\infty} \binom{s+r-1}{r} [t+s+r]_q^{-1}.$$

Let  $\ell$  be a positive integer. Writing

$$\begin{aligned} f_q(s, -\ell) &= (q-1)^s \left\{ \frac{1}{q^{s-\ell} - 1} + s \frac{1}{q^{s-\ell+1} - 1} + \frac{s(s+1)}{2} \frac{1}{q^{s-\ell+2} - 1} \right. \\ &\quad + \frac{s(s+1)(s+2)}{3!} \frac{1}{q^{s-\ell+3} - 1} + \dots \\ &\quad \left. + \frac{s(s+1)(s+2) \cdots (s+\ell-2)}{(\ell-1)!} \frac{1}{q^{s-1} - 1} + \dots \right\}, \end{aligned}$$

if we define  $\gamma(q, -\ell)$ , the  $q$ -Euler constant of level  $-\ell$ , by the constant term of the Laurent expansion of  $f_q(s, -\ell)$  around  $s = 1$ , we have similarly

$$\begin{aligned} \gamma(q, -\ell) &= \sum_{n=1-\ell, n \neq 0}^{\infty} \frac{1}{[n]_q} + \frac{q-1}{\log q} \left\{ \sum_{n=1}^{\ell-1} \frac{1}{n} + \log(q-1) \right\} - \frac{q-1}{2} \\ &= \gamma(q) + \frac{q-1}{\log q} \cdot \sum_{n=1}^{\ell-1} \frac{1}{n} + \sum_{n=1}^{\ell-1} \frac{1}{[-n]_q}. \end{aligned}$$

In particular, one has also  $\lim_{q \downarrow 1} \gamma(q, -\ell) = \gamma$  by Theorem 2.3. It would also be interesting to study a  $q$ -analogue of the higher version of Euler's constant developed

in [HIKW]. (See also [KW] for some natural appearance of the higher Euler’s constant.)

3. A  $q$ -ANALOGUE OF THE LIMIT FORMULA OF LERCH

Define a  $q$ -analogue of the Hurwitz zeta function  $\zeta(s, x)$  by

$$(3.1) \quad \zeta_q(s, x) = \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]_q^s}, \quad \text{Re}(s) > 1,$$

for  $x \notin \mathbb{Z}_{\leq 0}$ . In this section, we present the following theorem, a  $q$ -analogue of Lerch’s limit formula using Jackson’s  $q$ -gamma function  $\Gamma_q(x)$  (see (2.4)). We shall give two proofs of this limit formula. The first proof is a direct one and is given here. We shall give another proof of the theorem using a different  $q$ -analogue of  $\zeta(s, x)$  in the next section.

**Theorem 3.1.** *Let  $q > 1$ . Then  $\zeta_q(s, x)$  is meromorphic in  $s \in \mathbb{C}$ . Moreover,  $\zeta_q(s, x)$  has a simple pole at  $s = 1$ , and we have the limit formula*

$$(3.2) \quad \lim_{s \rightarrow 1} \left( \zeta_q(s, x) - \frac{q-1}{\log q} \cdot \frac{1}{s-1} \right) = -\frac{q-1}{\log q} \cdot \frac{\Gamma'_q(x)}{\Gamma_q(x)}.$$

*Proof.* Similar to the proof of Theorem 2.1, the classical binomial expansion asserts

$$\begin{aligned} \zeta_q(s, x) &= (q-1)^s \sum_{k=0}^{\infty} \frac{s(s+1) \cdots (s+k-1)}{k!} \cdot \frac{q^{(s+k-1)(1-x)}}{q^{s+k-1} - 1} \\ &= (q-1)^s \left\{ \frac{q^{(s-1)(1-x)}}{q^{s-1} - 1} + s \frac{q^{s(1-x)}}{q^s - 1} + \frac{s(s+1)}{2} \frac{q^{(s+1)(1-x)}}{q^{s+1} - 1} + \cdots \right\}. \end{aligned}$$

This shows that  $\zeta_q(s, x)$  is meromorphic in  $s \in \mathbb{C}$ . We now look at the Laurent expansion around  $s = 1$ . Actually, since

$$\begin{aligned} (q-1)^s &= (q-1) + \{(q-1) \log(q-1)\}(s-1) + a_2(s-1)^2 + \cdots, \\ q^{(s-1)(1-x)} &= 1 + \{(1-x) \log q\}(s-1) + b_2(s-1)^2 + \cdots, \\ \frac{1}{q^{s-1} - 1} &= \frac{1}{\log q} \cdot \frac{1}{s-1} - \frac{1}{2} + c_1(s-1) + \cdots, \end{aligned}$$

we find without difficulty that

$$\begin{aligned} \zeta_q(s, x) &= \frac{q-1}{\log q} \cdot \frac{1}{s-1} + \left[ -\frac{q-1}{2} + \frac{(q-1) \log(q-1)}{\log q} \right. \\ &\quad \left. + (q-1)(1-x) + (q-1) \sum_{k=1}^{\infty} \frac{q^{k(1-x)}}{q^k - 1} \right] + d_1(s-1) + \cdots. \end{aligned}$$

On the other hand, from the definition (2.4) of  $\Gamma_q(x)$ , we have

$$(3.3) \quad \begin{aligned} -\log \Gamma_q(x) &= -\frac{x(x-1)}{2} \log q + (x-1) \log(q-1) \\ &\quad - \log \left( \prod_{n=1}^{\infty} (1 - q^{-n}) \right) - \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{q^{k(1-x)}}{q^k - 1}. \end{aligned}$$

It follows immediately that

$$(3.4) \quad -\frac{\Gamma'_q}{\Gamma_q}(x) = \left(-x + \frac{1}{2}\right) \log q + \log(q - 1) + (\log q) \sum_{k=1}^{\infty} \frac{q^{k(1-x)}}{q^k - 1}.$$

Hence we conclude that

$$\zeta_q(s, x) = \frac{q-1}{\log q} \cdot \frac{1}{s-1} + \left(-\frac{q-1}{\log q} \cdot \frac{\Gamma'_q}{\Gamma_q}(x)\right) + d_1(s-1) + \dots.$$

This completes the proof of the theorem. □

*Remark 3.1.* Since  $\zeta_q(s) = \zeta_q(s, 1)$  and  $\Gamma_q(1) = 1$ , the theorem above implies, in particular, that

$$(3.5) \quad \lim_{s \rightarrow 1} \left( \zeta_q(s) - \frac{q-1}{\log q} \cdot \frac{1}{s-1} \right) = -\frac{q-1}{\log q} \cdot \Gamma'_q(1).$$

This clearly shows again the relation (2.6).

#### 4. ANOTHER PROOF OF THE LIMIT FORMULA

We define a different  $q$ -analogue  $\zeta_q^\circ(s, x)$  of the Hurwitz zeta function given by

$$(4.1) \quad \zeta_q^\circ(s, x) = \sum_{n=0}^{\infty} \frac{1}{[n+x]_q^s}, \quad \operatorname{Re}(s) > 0,$$

for  $x \notin \mathbb{Z}_{\leq 0}$ . We first show the following theorem to give another proof of the limit formula (3.2).

**Theorem 4.1.** *Let  $q > 1$ . Then  $\zeta_q^\circ(s, x)$  is meromorphic in  $s \in \mathbb{C}$ . Moreover,  $\zeta_q^\circ(s, x)$  has the following Laurent expansion at  $s = 0$ :*

$$(4.2) \quad \zeta_q^\circ(s, x) = \frac{1}{\log q} \cdot \frac{1}{s} + \left\{ \frac{1}{2} - x + \frac{\log(q-1)}{\log q} \right\} + \log \left( \frac{\Gamma_q(x)}{C_q} \right) \cdot s + \dots,$$

where

$$(4.3) \quad C_q = q^{-\frac{1}{12}}(q-1)^{\frac{1}{2} - \frac{\log(q-1)}{2 \log q}} \prod_{n=1}^{\infty} (1 - q^n).$$

*Proof.* The meromorphic continuation is exactly similar to the one in Theorem 3.1. In fact, we have

$$\begin{aligned} \zeta_q^\circ(s, x) &= (q-1)^s \sum_{k=0}^{\infty} \frac{s(s+1) \cdots (s+k-1)}{k!} \cdot \frac{q^{(s+k)(1-x)}}{q^{s+k} - 1} \\ &= (q-1)^s \left\{ \frac{q^{s(1-x)}}{q^s - 1} + s \frac{q^{(s+1)(1-x)}}{q^{s+1} - 1} + \frac{s(s+1)}{2} \frac{q^{(s+2)(1-x)}}{q^{s+2} - 1} + \dots \right\}. \end{aligned}$$

Hence, using the expansions

$$\begin{aligned} (q-1)^s &= 1 + (\log(q-1))s + \frac{(\log(q-1))^2}{2} s^2 + \dots, \\ \frac{q^{s(1-x)}}{q^s - 1} &= \frac{1}{\log q} \cdot \frac{1}{s} + \left( \frac{1}{2} - x \right) + \left( \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12} \right) (\log q) \cdot s + \dots, \end{aligned}$$

we obtain

$$\begin{aligned} \zeta_q^\circ(s, x) &= \frac{1}{\log q} \cdot \frac{1}{s} + \left\{ \frac{1}{2} - x + \frac{\log(q-1)}{\log q} \right\} \\ &+ \left\{ \frac{(\log(q-1))^2}{2 \log q} + \left( \frac{1}{2} - x \right) \log(q-1) \right. \\ &\left. + \left( \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12} \right) \log q + \sum_{k=1}^\infty \frac{1}{k} \cdot \frac{q^{k(1-x)}}{q^k - 1} \right\} s + \dots \end{aligned}$$

The rest of the proof follows immediately from the expression (3.3) of  $\log \Gamma_q(x)$ .  $\square$

The following simple relation between  $\zeta_q(s, x)$  and  $\zeta_q^\circ(s, x)$  holds.

**Proposition 4.2.** *Let  $q > 1$ ,  $\zeta_q(s, x)$  and  $\zeta_q^\circ(s, x)$  be as above. Then the following formula holds for  $\text{Re}(s) > 0$ :*

$$(4.4) \quad \frac{\partial}{\partial x} \zeta_q^\circ(s, x) = -s \frac{\log q}{q-1} \zeta_q(s+1, x).$$

*Proof.* Since

$$[n+x]_q^{-s} = \left( \frac{q^{n+x} - 1}{q-1} \right)^{-s},$$

we have easily

$$\begin{aligned} \frac{\partial}{\partial x} [n+x]_q^{-s} &= -s \cdot (\log q) \cdot \frac{q^{n+x}}{q-1} \left( \frac{q^{n+x} - 1}{q-1} \right)^{-s-1} \\ &= -s \cdot \frac{\log q}{q-1} \cdot q^{n+x} \cdot [n+x]_q^{-s-1}. \end{aligned}$$

It follows hence that

$$\begin{aligned} \frac{\partial}{\partial x} \zeta_q^\circ(s, x) &= -s \cdot \frac{\log q}{q-1} \sum_{n=0}^\infty q^{n+x} [n+x]_q^{-s-1} \\ &= -s \cdot \frac{\log q}{q-1} \zeta_q(s+1, x). \end{aligned}$$

This shows the proposition.  $\square$

We now give the second proof of Theorem 3.1, the  $q$ -analogue of Lerch’s formula (3.2). In view of the relation (4.4), Theorem 4.1 provides also the meromorphic extension of  $\zeta_q(s, x)$  around  $s = 1$ . Hence, write the Laurent expansions of  $\zeta_q^\circ(s, x)$  at  $s = 0$  and  $\zeta_q(s, x)$  at  $s = 1$  respectively as

$$\begin{aligned} \zeta_q^\circ(s, x) &= \frac{a_{-1}(x)}{s} + a_0(x) + a_1(x)s + \dots, \\ \zeta_q(s, x) &= \frac{b_{-1}(x)}{s-1} + b_0(x) + b_1(x)(s-1) + \dots. \end{aligned}$$

Then, by Proposition 4.2 we have

$$a'_m(x) = -\frac{\log q}{q-1} b_{m-1}(x)$$

for  $m = 0, 1, 2, \dots$ . In particular, we obtain

$$b_0(x) = -\frac{q-1}{\log q} a_1'(x).$$

Thus, using the result of Theorem 4.1 described as

$$a_1(x) = \log \left( \frac{\Gamma_q(x)}{C_q} \right),$$

we have

$$b_0(x) = -\frac{q-1}{\log q} \frac{\Gamma_q'}{\Gamma_q}(x).$$

This certainly gives (3.2).

*Remark 4.1.* One can prove an expected relation of a  $q$ -analogue of the Hurwitz zeta function ( $\lim_{q \downarrow 1} \zeta_q(s, x) = \zeta(s, x)$  for all  $s \in \mathbb{C}$ ) by the same way exactly as in [KKW] and similar to the relation that the  $q$ -analogue of the Riemann zeta function possesses. However, it should be remarked that the relation  $\lim_{q \downarrow 1} \zeta_q^c(s, x) = \zeta(s, x)$  does not hold for  $\text{Re}(s) \leq 1$  in general.

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