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HECKE ALGEBRAS FOR THE BASIC CHARACTERS OF THE UNITRIANGULAR GROUP

CARLOS A. M. ANDRÉ

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ABSTRACT. Let $U_n(q)$ denote the unitriangular group of degree n over the finite field with q elements. In a previous paper we obtained a decomposition of the regular character of $U_n(q)$ as an orthogonal sum of basic characters. In this paper, we study the irreducible constituents of an arbitrary basic character $\xi_{\mathcal{D}}(\varphi)$ of $U_n(q)$. We prove that $\xi_{\mathcal{D}}(\varphi)$ is induced from a linear character of an algebra subgroup of $U_n(q)$, and we use the Hecke algebra associated with this linear character to describe the irreducible constituents of $\xi_{\mathcal{D}}(\varphi)$ as characters induced from an algebra subgroup of $U_n(q)$. Finally, we identify a special irreducible constituent of $\xi_{\mathcal{D}}(\varphi)$, which is also induced from a linear character of an algebra subgroup. In particular, we extend a previous result (proved under the assumption $p \geq n$ where p is the characteristic of the field) that gives a necessary and sufficient condition for $\xi_{\mathcal{D}}(\varphi)$ to have a unique irreducible constituent.

Let p be a prime number, let $q = p^e$ $(e \geq 1)$ be a power of p, and let $\mathbb{F}_q$ denote the finite field with q elements. Throughout the paper, we will denote by U the the unitriangular group $U_n(q)$ of degree n over $\mathbb{F}_q$; by definition, U consists of all unipotent uppertriangular $n \times n$ matrices with coefficients in $\mathbb{F}_q$. We clearly have $U = 1+\mathfrak{u}$ where $\mathfrak{u} = \mathfrak{u}_n(q)$ is the $\mathbb{F}_q$-space consisting of all nilpotent uppertriangular $n \times n$ matrices over $\mathbb{F}_q$; in particular, the p-group U is an $\mathbb{F}_q$-*algebra group* (in the sense of [5]; see also [4]). Moreover, let $\mathfrak{u}^*$ denote the dual $\mathbb{F}_q$-space of $\mathfrak{u}$.

For simplicity, we write $\Phi = \{(i,j) \colon 1 \leq i < j \leq n\}$ and we refer to an element of Φ as a *root*. For any $(i,j) \in \Phi$, let $e_{ij} \in \mathfrak{u}_n(q)$ be the $n \times n$ matrix $e_{ij} = (\delta_{ri}\delta_{sj})_{1 \leq r,s \leq n}$ where δ denotes the usual Kronecker symbol. Then, $(e_{ij} \colon (i,j) \in \Phi)$ is an $\mathbb{F}_q$-basis of $\mathfrak{u}$ to which we will refer as the *standard basis* of $\mathfrak{u}$. On the other hand, for any $(i,j) \in \Phi$, let $e_{ij}^* \in \mathfrak{u}^*$ be defined by $e_{ij}^*(a) = a_{ij}$ for all $a \in \mathfrak{u}$ (given a matrix x, we will denote by x_{ij} the (i,j)-th coefficient of x). Then, $(e_{ij}^* \colon (i,j) \in \Phi)$ is an $\mathbb{F}_q$-basis of $\mathfrak{u}^*$, dual to the standard basis of $\mathfrak{u}$.

Let $\psi \colon \mathbb{F}_q^+ \to \mathbb{C}$ be an arbitrary nontrivial character of the additive group $\mathbb{F}_q^+$ of $\mathbb{F}_q$ (this character will be kept fixed throughout the paper) and, for any $f \in \mathfrak{u}^*$, let $\psi_f \colon \mathfrak{u} \to \mathbb{C}$ be the function defined by $\psi_f(a) = \psi(f(a))$ for all $a \in \mathfrak{u}$. It is clear that this function is a linear character of the additive group $\mathfrak{u}^+$ of $\mathfrak{u}$ and that the mapping $f \mapsto \psi_f$ defines a one-to-one correspondence between $\mathfrak{u}^*$ and the set of all

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irreducible characters of $\mathfrak{u}^+$. (Throughout the article, all characters are taken over the complex field.)

The group U acts on $\mathfrak{u}^*$ via the *coadjoint representation*: for any $x \in U$ and any $f \in \mathfrak{u}^*$, we define the linear map $x \cdot f \in \mathfrak{u}^*$ by $(x \cdot f)(a) = f(x^{-1}ax)$ for all $a \in \mathfrak{u}$. Let $\mathcal{O} \subseteq \mathfrak{u}^*$ be an arbitrary U-orbit. By [2, Lemma 1], we know that the cardinality $|\mathcal{O}|$ of $\mathcal{O}$ is a power of q^2. Let $\phi_{\mathcal{O}}\colon U \to \mathbb{C}$ be the class function defined by

$$\phi_{\mathcal{O}}(1+a) = \frac{1}{\sqrt{|\mathcal{O}|}} \sum_{f \in \mathcal{O}} \psi(f(a))$$

for all $a \in \mathfrak{u}$. In general, $\phi_{\mathcal{O}}$ is not a character (see [6]). However, there are some examples where they are, in fact, irreducible characters of U. A particular family consists of the *elementary characters* of U, which are defined as follows. Let $(i,j) \in \Phi$ be any root and let $\alpha \in \mathbb{F}_q$ be any nonzero element. (Throughout the paper, we will denote by $\mathbb{F}_q^\times$ the subset of $\mathbb{F}_q$ consisting of all nonzero elements.) Let $\mathcal{O}_{ij}^*(\alpha) \subseteq \mathfrak{u}^*$ be the U-orbit that contains the element $\alpha e_{ij}^* \in \mathfrak{u}^*$, and let $\xi_{ij}(\alpha)$ denote the class function $\phi_{\mathcal{O}_{ij}^*(\alpha)}$ that corresponds to $\mathcal{O}_{ij}^*(\alpha)$. By [2, Lemma 2], we know that this class function is, in fact, an irreducible character of U. We will refer to $\xi_{ij}(\alpha)$ as the (i,j)-th *elementary character* of U associated with α.

Now, a subset $\mathcal{D} \subseteq \Phi$ is called a *basic subset* if $|\mathcal{D} \cap \{(i,j)\colon i < j \leq n\}| \leq 1$ for all $1 \leq i < n$, and if $|\mathcal{D} \cap \{(i,j)\colon 1 \leq i < j\}| \leq 1$ for all $1 < j \leq n$. In particular, the empty set is a basic subset of Φ. Given an arbitrary nonempty basic subset $\mathcal{D}$ of Φ and given an arbitrary map $\varphi\colon \mathcal{D} \to \mathbb{F}_q^\times$, we define the *basic character* $\xi_{\mathcal{D}}(\varphi)$ of U to be the product of elementary characters

$$\xi_{\mathcal{D}}(\varphi) = \prod_{(i,j) \in \mathcal{D}} \xi_{ij}(\varphi(i,j)).$$

For our purposes, it is convenient to consider the trivial character 1_U of U as the basic character $\xi_{\mathcal{D}}(\varphi)$ corresponding to the empty subset of Φ and to the empty function $\varphi\colon \mathcal{D} \to \mathbb{F}_q^\times$. By [2, Theorem 1], we know that every irreducible character χ of U is a constituent of $\xi_{\mathcal{D}}(\varphi)$ for a unique basic subset $\mathcal{D} \subseteq \Phi$ and a unique map $\varphi\colon \mathcal{D} \to \mathbb{F}_q^\times$.

The purpose of this paper is to study the decomposition of an arbitrary basic character $\xi_{\mathcal{D}}(\varphi)$ of U. Throughout the paper, the basic subset $\mathcal{D} \subseteq \Phi$ and the map $\varphi\colon \mathcal{D} \to \mathbb{F}_q^\times$ will be kept fixed. Moreover, we will simplify the notation and write ξ to denote the basic character $\xi_{\mathcal{D}}(\varphi)$. We begin by proving that ξ is induced from a linear character of a certain algebra subgroup of U. (Following [5], a subgroup of $U = 1 + \mathfrak{u}$ is called an *algebra subgroup* if it is of the form $1 + J$ for some multiplicatively closed $\mathbb{F}_q$-subspace J of $\mathfrak{u}$.) In order to construct this subgroup, we consider the U-action on $\mathfrak{u}^*$ given by *left translation*: for any $x \in U$ and any $f \in \mathfrak{u}^*$, we define the linear map $xf \in \mathfrak{u}^*$ by $(xf)(a) = f(x^{-1}a)$ for all $a \in \mathfrak{u}$. For any $f \in \mathfrak{u}^*$, let $U(f) = \{x \in U\colon xf = f\}$ be the centralizer of f in U. Therefore, we have

$$U(f) = \{x \in U\colon f(xb) = f(b) \text{ for all } b \in \mathfrak{u}\}.$$

On the other hand, let

$$\mathfrak{u}(f) = \{a \in \mathfrak{u}\colon f(ab) = 0 \text{ for all } b \in \mathfrak{u}\}.$$

It is easy to see that $\mathfrak{u}(f)$ is a multiplicatively closed $\mathbb{F}_q$-subspace of $\mathfrak{u}$. In fact, the following (easy) result holds.

Lemma 1. *For any $f \in \mathfrak{u}^*$, we have $U(f) = 1 + \mathfrak{u}(f)$; hence $U(f)$ is an algebra subgroup of U.*

Proof. Let $a \in \mathfrak{u}(f)$ be arbitrary and let $x = 1 + a$. Then, $f(xb) = f(b + ab) = f(b)+f(ab) = f(b)$ for all $b \in \mathfrak{u}$, and this implies that $x^{-1} \in U(f)$ (hence, $x \in U(f)$). Conversely, let $x \in U(f)$. Then, $f(x^{-1}b) = f(b)$ for all $b \in \mathfrak{u}$. Replacing b by xb, we deduce that $f(b) = f(xb)$ for all $b \in \mathfrak{u}$, and so $a = x - 1 \in \mathfrak{u}$ satisfies $f(ab) = 0$ for all $b \in \mathfrak{u}$. This means that $a \in \mathfrak{u}(f)$, and so the equality $U(f) = 1+\mathfrak{u}(f)$ holds. □

As a special case, let $(i, j) \in \Phi$ and consider the element $e^*_{ij} \in \mathfrak{u}^*$. It is not difficult to show that $\mathfrak{u}(e^*_{ij}) = \{a \in \mathfrak{u} \colon a_{ik} = 0 \text{ for all } i < k < j\}$; hence,

$$U(e^*_{ij}) = 1 + \mathfrak{u}(e^*_{ij}) = \{x \in U \colon x_{ik} = 0 \text{ for all } i < k < j\}.$$

For simplicity, we write $\mathfrak{n}_{ij} = \mathfrak{u}(e^*_{ij})$ and $N_{ij} = U(e^*_{ij})$. Let $\alpha \in \mathbb{F}_q^\times$ be arbitrary and let $\lambda_{ij}(\alpha) \colon N_{ij} \to \mathbb{C}$ be defined by $\lambda_{ij}(\alpha)(x) = \psi(\alpha x_{ij})$ for all $x \in U_{ij}(q)$. Then, by [2, Lemma 2], $\lambda_{ij}(\alpha)$ is a linear character of N_{ij} and the (i, j)-th elementary character $\xi_{ij}(\alpha)$ is the induced character $\lambda_{ij}(\alpha)^U$. More generally, for the (arbitrarily) given basic subset $\mathcal{D}$ and for the map $\varphi \colon \mathcal{D} \to \mathbb{F}_q^\times$, let $e^* \in \mathfrak{u}^*$ denote the element

$$e^* = \sum_{(i,j)\in\mathcal{D}} \varphi(i, j)e^*_{ij}$$

and consider the centralizer $U(e^*)$ of e^* in U. Moreover, let $\lambda \colon U(e^*) \to \mathbb{C}$ be the map defined by

$$\lambda(1 + a) = \psi(e^*(a))$$

for all $a \in \mathfrak{u}(e^*)$ (we recall that $U(e^*) = 1 + \mathfrak{u}(e^*)$). Then, λ is a linear character of $U(e^*)$. In fact, let $x, y \in U(e^*)$ be arbitrary and let $a, b \in \mathfrak{u}(e^*)$ be such that $x = 1 + a$ and $y = 1 + b$. Then, $xy = 1 + a + b + ab$ and so

$$\lambda(xy) = \lambda(x)\lambda(y)\psi(e^*(ab)) = \lambda(x)\lambda(y)$$

(because $a, b \in \mathfrak{u}(e^*)$, hence $e^*(ab) = 0$). We note that

$$\lambda(x) = \prod_{(i,j)\in\mathcal{D}} \psi(\varphi(i, j)x_{ij})$$

for all $x \in U(e^*)$. In order to prove that $\xi = \lambda^U$, it is very useful to describe the centralizer $U(e^*)$ as follows. Let

$$\mathcal{S} = \bigcup_{(i,j)\in\mathcal{D}} \{(i, k) \colon i < k < j\} \subseteq \Phi,$$

and let $\mathcal{R} = \Phi - \mathcal{S}$. Let $\mathfrak{n}$ be the $\mathbb{F}_q$-subspace of $\mathfrak{u}$ spanned by the vectors e_{ij} for $(i, j) \in \mathcal{R}$. Then, $\mathfrak{n} = \{a \in \mathfrak{u} \colon a_{rs} = 0 \text{ for all } (r, s) \in \mathcal{S}\}$, and so $\mathfrak{n} = \bigcap_{(i,j)\in\mathcal{D}} \mathfrak{n}_{ij}$. In particular, we deduce that $\mathfrak{n}$ is a multiplicatively closed $\mathbb{F}_q$-subspace of $\mathfrak{u}$. Therefore, we may consider the algebra subroup $N = 1 + \mathfrak{n}$ of U. Then,

$$N = \bigcap_{(i,j)\in\mathcal{D}} N_{ij} = \{x \in U \colon x_{rs} = 0 \text{ for all } (r, s) \in \mathcal{S}\}.$$

We have the following result.

Lemma 2. *The subgroup N is the centralizer $U(e^*)$ of e^* in U.*

Proof. We consider the standard basis $(e_{ij}\colon (i,j)\in\Phi)$ of $\mathfrak{u}$. By Lemma 1, it is enough to prove that $\mathfrak{n}$ consists of all matrices $a\in\mathfrak{u}$ that satisfy $e^*(ae_{ij})=0$ for all $(i,j)\in\Phi$. Given an arbitrary element $a\in\mathfrak{u}$, we have $ae_{ij}=\sum_{1\leq r<i}a_{ri}e_{rj}$. Therefore, $e^*(ae_{ij})$ can be nonzero only if $(r,j)\in\mathcal{D}$ for some $1\leq r<i$; and, if this is the case, we have $e^*(ae_{ij})=a_{ri}\varphi(r,j)$. Now, let $a\in\mathfrak{n}$, let $(i,j)\in\Phi$ and suppose that $(r,j)\in\mathcal{D}$ for some $1\leq r<i$. Then, $a_{ri}=0$ and so $e^*(ae_{ij})=0$. It follows that $a\in\mathfrak{u}(e^*)$. Conversely, suppose that $a\in\mathfrak{u}(e^*)$ and let $(i,j)\in\mathcal{D}$. Then, for all $i<k<j$, we have $a_{ik}\varphi(i,j)=e^*(ae_{kj})=0$ and so $a_{ik}=0$. Thus, $a\in\mathfrak{n}$ and the proof is complete. □

We are now able to prove the following result.

Theorem 1. *The basic character ξ of U is induced by the linear character λ of N.*

Proof. We proceed by induction on the cardinality d of the set $\mathcal{D}$. The result is trivial if $d=0$ and, as we mentioned before, the case $d=1$ is given by [2, Lemma 2]. Now, suppose that $d>1$ and assume that the result is true for all the basic characters that correspond to the basic subsets $\mathcal{D}_0\subseteq\Phi$ with less than d elements. Let $(i,j)\in\mathcal{D}$, let $\mathcal{D}_0=\mathcal{D}-\{(i,j)\}$, and let $\varphi_0\colon\mathcal{D}_0\to\mathbb{F}_q^\times$ be the restriction of φ to $\mathcal{D}_0$. Moreover, let $\alpha=\varphi(i,j)$, let $e_0^*=e^*-\alpha e_{ij}^*$, and let $N_0=U(e_0^*)$. Then, $N=N_0\cap N_{ij}$ and $U=N_0N_{ij}$. Let $\lambda_0\colon N_0\to\mathbb{C}$ be the linear character defined by $\lambda_0(1+a)=\psi(e_0^*(a))$ for all $a\in\mathfrak{u}(e_0^*)$ (we recall that $U(e_0^*)=1+\mathfrak{u}(e_0^*)$). By induction, we know that $(\lambda_0)^U$ is the basic character $\xi_0=\xi_{\mathcal{D}_0}(\varphi_0)$ and so $\xi=\xi_0\zeta=(\lambda_0)^U\mu^U$ where $\zeta=\xi_{ij}(\alpha)$ and $\mu=\lambda_{ij}(\alpha)$. By Mackey's Subgroup Theorem (see [3, Theorem 10.13]), we have $\zeta_{N_0}=(\mu^U)_{N_0}=(\mu_N)^{N_0}$ and so

$$\xi=(\lambda_0)^U\zeta=(\lambda_0\zeta_{N_0})^U=(\lambda_0(\mu_N)^{N_0})^U.$$

Since $\lambda_0(\mu_N)^{N_0}=((\lambda_0)_N\mu_N)^{N_0}$ and since $\lambda=(\lambda_0)_N\mu_N$ (as we observed above), we conclude that $\xi=(\lambda^{N_0})^U=\lambda^U$. □

Now, let $\mathbb{C}[U]$ (resp., $\mathbb{C}[N]$) be the group algebra of U (resp., of N). As usual, we consider $\mathbb{C}[N]$ as a subalgebra of $\mathbb{C}[U]$. Let

$$\varepsilon=\frac{1}{|N|}\sum_{x\in N}\overline{\lambda(x)}x\in\mathbb{C}[N]$$

be the central primitive idempotent that corresponds to the linear character λ of N (hence, the left ideal $\mathbb{C}[N]\varepsilon$ of $\mathbb{C}[N]$ affords the character λ of N; see [3, Proposition 9.21]). By [3, Proposition 11.21], the left ideal $\mathbb{C}[U]\varepsilon$ of $\mathbb{C}[U]$ affords the induced character λ^U of U, and the multiplicity of an arbitrary irreducible character χ of U as a constituent of λ^U is given by the value $\chi(\varepsilon)$. Let $\mathcal{H}=\varepsilon\mathbb{C}[U]\varepsilon$ be the Hecke algebra associated with the linear character λ of N. Since $\mathbb{C}[U]$ is a semisimple algebra, the Hecke algebra $\mathcal{H}$ is also semisimple (by Proposition 5.13 and Theorem 5.18 of [3]). Moreover, by [3, Theorem 11.25], the mapping $\chi\mapsto\chi_{\mathcal{H}}$ defines a bijection between the set of all irreducible constituents of λ^U and the set of all irreducible characters of $\mathcal{H}$. In the following result, we describe a $\mathbb{C}$-basis of $\mathcal{H}$. First, we introduce some notation. Let $\mathcal{S}'\subseteq\Phi$ be the subset consisting of all roots $(i,j)\in\Phi$ for which there exist $j<k<l\leq n$ with $(i,k),(j,l)\in\mathcal{D}$. It is clear that $\mathcal{S}'\subseteq\mathcal{S}$. Let $\mathfrak{x}$ be the $\mathbb{F}_q$-subspace of $\mathfrak{u}$ spanned by the vectors e_{ij} for $(i,j)\in\mathcal{S}'$ and let $X=1+\mathfrak{x}\subseteq U$.

Proposition 1. *For each $x\in U$, let $\operatorname{ind}x=|N:N\cap x^{-1}Nx|$ be the index of x and let $a_x=(\operatorname{ind}x)\varepsilon x\varepsilon\in\mathcal{H}$. Then, $(a_x\colon x\in X)$ is a $\mathbb{C}$-basis of $\mathcal{H}$.*

Proof. First, we observe that, for an arbitrary element $x \in U$, the intersection $xNx^{-1} \cap N$ is the algebra subgroup $1 + (x\mathfrak{n}x^{-1} \cap \mathfrak{n})$ of U. On the other hand, for any $b \in \mathfrak{n}$, we have $xbx^{-1} \in \mathfrak{n}$ if and only if $xb \in \mathfrak{n}$. In fact, since $\mathfrak{n} = \mathfrak{u}(e^*)$ (by Lemma 2), we have $xbx^{-1} \in \mathfrak{n}$ if and only if $e^*(xbx^{-1}a) = 0$ for all $a \in \mathfrak{u}$. Replacing a by xa, we conclude that $xbx^{-1} \in \mathfrak{n}$ if and only if $e^*(xba) = 0$ for all $a \in \mathfrak{u}$. It follows that $xbx^{-1} \in \mathfrak{n}$ if and only if $xb \in \mathfrak{u}(e^*) = \mathfrak{n}$, as required.

Next, we observe that each double coset of N in U may be represented by an element $x \in U$ with the form $x = 1 + a$ where $a \in \sum_{(i,j)\in\mathcal{S}} \mathbb{F}_q e_{ij}$. In fact, let us denote by T the subset of U consisting of all these elements. Then, $|T| = |U : N|$ and, in fact, T is a complete set of representatives for the right cosets of N in U (hence, it contains a set of representatives for the double cosets of N in U). To see this, let $x, y \in T$ be such that $x = zy$ for some $z \in N$. Moreover, let $a \in \mathfrak{n}$ be such that $z = 1 + a$, so that $x = y + ay$. Since $a \in \mathfrak{n} = \mathfrak{u}(e^*)$, we have $e^*(ab) = 0$ for all $b \in \mathfrak{u}$. Therefore, we also have $e^*(ayb) = 0$ for all $b \in \mathfrak{u}$, and so $ay \in \mathfrak{u}(e^*) = \mathfrak{n}$. It follows that $x - y = ay \in \mathfrak{n}$, and this clearly implies that $x = y$.

Now, by [3, Proposition 11.30], the Hecke algebra $\mathcal{H}$ has a $\mathbb{C}$-basis formed by some of the elements a_x for $x \in T$ satisfying $\lambda(x^{-1}yx) = \lambda(y)$ for all $y \in xNx^{-1} \cap N$. We claim that such an element x lies in X. Suppose that this is not the case. Then, there exists $(i,k) \in \Phi$ with $(i,k) \notin \mathcal{S}'$ and $x_{ik} \neq 0$. Since $x \in T$, we must have $(i,k) \in \mathcal{S}$, and so there exists $k < j \leq n$ with $(i,j) \in \mathcal{D}$. By the definition of $\mathcal{S}'$, we have $(k,l) \notin \mathcal{D}$ for all $j < l \leq n$. Moreover, we may choose the root $(i,j) \in \mathcal{D}$ such that, for any $(r,s) \in \mathcal{D}$ with $j < s \leq n$, we have $(r,t) \in \mathcal{S}'$ whenever $r < t < s$ is such that $x_{rt} \neq 0$. Now, we claim that $x(1 + e_{kj})x^{-1} \in xNx^{-1} \cap N$ (we note that $1 + e_{kj} \in N$, because $(k,j) \notin \mathcal{S}$). To prove this, it is enough to show that $xe_{kj} \in \mathfrak{n}$. In fact, let $(r,s) \in \mathcal{S}$ be arbitrary. Then, the (r,s)-th coefficient of xe_{kj} can be nonzero only if $s = j$ and $r \leq k$; and, if this is the case, that coefficient is x_{rk}. Since $(r,j) \in \mathcal{S}$ (by our choice), there exists $j < t \leq n$ with $(r,t) \in \mathcal{D}$ and so, by the choice of j, we must have $x_{rk} = 0$ (because $(r,k) \notin \mathcal{S}'$). It follows that $xe_{kj} \in \mathfrak{n}$, as required. Now, let $\alpha \in \mathbb{F}_q$ be arbitrary and consider the $y_\alpha = x(1 + \alpha e_{kj})x^{-1} = 1 + \alpha x e_{kj} x^{-1}$. We note that $y_\alpha \in xNx^{-1} \cap N$ because $xe_{kj}x^{-1} \in \mathfrak{n}$ (hence $\alpha x e_{kj} x^{-1} \in \mathfrak{n}$). Therefore, by the definition of T, we have $\lambda(x^{-1}y_\alpha x) = \lambda(y_\alpha)$ and so

$$\psi(\alpha e^*(e_{kj})) = \psi(\alpha e^*(xe_{kj}x^{-1}))$$

(by the definition of λ). On the one hand, we have $e^*(e_{kj}) = 0$ (because $(k,j) \notin \mathcal{D}$) and, on the other hand, we know that $xe_{kj} \in \mathfrak{n}$; hence $e^*(xe_{kj}b) = 0$ for all $b \in \mathfrak{u}$ and this implies that $e^*(xe_{kj}z) = e^*(xe_{kj})$ for all $z \in U$. In particular, we deduce that

$$\psi(\alpha e^*(xe_{kj})) = \psi(\alpha e^*(xe_{kj}x^{-1})) = \psi(\alpha e^*(e_{kj})) = 1.$$

Now, suppose that $e^*(xe_{kj}) \neq 0$. Then, the mapping $\alpha \mapsto \alpha e^*(xe_{kj})$ defines a permutation of $\mathbb{F}_q$, and so the equality $\psi(\alpha e^*(xe_{kj})) = 1$ (which holds for any $\alpha \in \mathbb{F}_q$) implies that ψ is the trivial character of $\mathbb{F}_q^+$, contrary to the choice of ψ. Therefore, $e^*(xe_{kj}) = 0$. However, $xe_{kj} = e_{kj} + \sum_{1\leq r<k} x_{rk}e_{rj}$, and so

$$e^*(xe_{kj}) = x_{ik}e^*(e_{ij}) = x_{ik}\varphi(i,j).$$

Since $\varphi(i,j) \neq 0$, we get a contradiction and so $x \in X$, as claimed.

By the result mentioned above, we conclude that the set $\{a_x : x \in X\}$ contains a $\mathbb{C}$-basis of $\mathcal{H}$. By the same result, to complete the proof we must show that, for $x, x' \in X$ with $x \neq x'$, the double cosets NxN and $Nx'N$ are distinct. To see this, suppose that $x' = yxz$ for some $y, z \in N$. Let $(i,j) \in \mathcal{S}'$ be the largest root such that $x'_{ij} \neq x_{ij}$; this means that $x'_{rs} = x_{rs}$ whenever $(r,s) \in X$ is such that, either $j < s$, or $j = s$ and $r < i$. Since $(i,j) \in \mathcal{S}$ and since $y \in N$, we have $y_{ir} = 0$ for all $i < r < j$, and so $x'_{ij} = \sum_{i \leq s \leq j} x_{is} z_{sj}$. Now, suppose that $x_{is} \neq 0$ for some $i < s < j$. Then, $(i,s) \in \mathcal{S}'$ and so there exists $j+1 < k \leq n$ with $(s,k) \in \mathcal{D}$. Therefore, $(s,j) \in \mathcal{S}$ and this implies that $z_{sj} = 0$ (because $z \in N$). It follows that $x'_{ij} = x_{ij} + z_{ij}$. However, $(i,j) \in \mathcal{S}$; hence $z_{ij} = 0$. Therefore, $x'_{ij} = x_{ij}$ and this contradiction implies that the double cosets NxN and $Nx'N$ are distinct. □

In the next result, we show that the $\mathbb{C}$-basis $(a_x \colon x \in X)$ is (in certain sense) a "group basis" of $\mathcal{H}$.

Proposition 2. *The $\mathbb{F}_q$-subspace $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{x}$ of $\mathfrak{u}$ is multiplicatively closed, and so $S = 1 + \mathfrak{s}$ is an algebra subgroup of U. Moreover, N is a normal subgroup of S and X is a complete set of representatives of the elements of the quotient group S/N.*

Proof. We recall that $\mathfrak{n}$ is spanned by the vectors e_{ij} for $(i,j) \in \mathcal{R}$ where $\mathcal{R} = \Phi - \mathcal{S}$. Therefore, the first assertion of the proposition will follow once we prove that the (disjoint) union $\mathcal{R} \cup \mathcal{S}'$ is a closed subset of Φ, i.e., we have $(i,k) \in \mathcal{R} \cup \mathcal{S}'$ whenever $(i,j), (j,k) \in \mathcal{R} \cup \mathcal{S}'$. This is clear in the case where $(i,j), (j,k) \in \mathcal{R}$. It is also clear that $(i,k) \in \mathcal{R}$ in the case where $(i,j) \in \mathcal{R}$ and $(j,k) \in \mathcal{S}'$. Now, suppose that $(i,j) \in \mathcal{S}'$ and that $(j,k) \in \mathcal{R}$. By definition of $\mathcal{S}'$, there exist $(i,r), (j,s) \in \mathcal{D}$ with $j < r$. Moreover, we must have $s \leq k$ because $(j,k) \in \mathcal{R}$. Therefore, $r < k$ and so $(i,k) \in \mathcal{R}$. Finally, suppose that $(i,j), (j,k) \in \mathcal{S}'$. Then, there exist $(i,r), (j,s), (k,t) \in \mathcal{D}$ with $j < r$ and $k < s$. We have two cases: on the one hand, if $r \leq k$, then $(i,k) \in \mathcal{R}$; on the other hand, if $k < r$, then $(i,k) \in \mathcal{S}'$ because $(i,r), (k,t) \in \mathcal{D}$.

For the second assertion, we note that, since $N = 1 + \mathfrak{n}$ and since $x(1+a)x^{-1} = 1 + xax^{-1}$ for all $x \in U$ and all $a \in \mathfrak{u}$, it is enough to prove that $xax^{-1} \in \mathfrak{n}$ for all $x \in S$ and all $a \in \mathfrak{n}$. Let $x \in S$ and let $a \in \mathfrak{n}$ be arbitrary. Then, by Lemma 2, we have $xax^{-1} \in \mathfrak{n}$ if and only if $e^*(xax^{-1}b) = 0$ for all $b \in \mathfrak{u}$. Replacing b by xb, we conclude that $xax^{-1} \in \mathfrak{n}$ if and only if $e^*(xab) = 0$ for all $b \in \mathfrak{u}$. Therefore, we have $xax^{-1} \in \mathfrak{n}$ if and only if $xa \in \mathfrak{n}$. Now, let $(r,s) \in \mathcal{S}$. Then, $(xa)_{rs} = \sum_{r \leq t \leq s} x_{rt} a_{ts}$. If $a_{ts} \neq 0$, we must have $(t,s) \in \mathcal{R}$. On the other hand, x_{rt} can be nonzero only if $(r,t) \in \mathcal{S}'$. Therefore, there exist $(r,u), (t,v) \in \mathcal{D}$ with $u < v$. Since $(r,s) \in \mathcal{S}$, we must have $s < u < v$ and so $(t,s) \in \mathcal{S}$. This contradiction implies that $(r,t) \notin \mathcal{S}'$ and so $x_{rt} = 0$. It follows that $(xa)_{rs} = 0$, and this implies that $xa \in \mathfrak{n}$. □

Corollary 1. *For any $x \in X$, we have $a_x = \varepsilon x = x\varepsilon$. In particular, $(x\varepsilon \colon x \in X)$ is a $\mathbb{C}$-basis of $\mathcal{H}$.*

Proof. Let $x \in X$ be arbitrary. Then, $x \in S$ and so $\operatorname{ind} x = 1$ (because N is normal in S). Moreover, since λ is S-invariant (by definition of X), we deduce that

$$x\varepsilon = \frac{1}{|N|} \sum_{y \in N} \lambda(y^{-1}) xy = \frac{1}{|N|} \sum_{z \in N} \lambda(x^{-1}z^{-1}x) zx = \varepsilon x.$$

The result follows because ε is an idempotent. □

Now, let $\mathbb{C}[S]$ be the group algebra of S. Then, $\mathbb{C}[S]$ is a subalgebra of $\mathbb{C}[U]$ and so $\mathcal{H}_0 = \varepsilon\mathbb{C}[S]\varepsilon$ is a subalgebra of $\mathcal{H}$. Since $a_x \in \mathcal{H}_0$ for all $x \in X$, we conclude that $\mathcal{H}_0 = \mathcal{H}$. Since $\mathcal{H}_0$ is the Hecke algebra associated with the (normal) subgroup

N of S and with the linear character λ of N, we may use [3, Theorem 11.25] to deduce the following result.

Theorem 2. *The mapping $\phi \mapsto \phi^U$ defines a bijection between the set of all irreducible constituents of the induced character λ^S and the set of all irreducible constituents of the basic character ξ. Moreover, this bijection preserves multiplicities, i.e., $\langle \phi^U, \xi\rangle_U = \langle \phi, \lambda^S\rangle_S$ for all irreducible constituents ϕ of λ^S. (Given any finite group G, we denote by $\langle\cdot,\cdot\rangle_G$ the usual Frobenius scalar product on the $\mathbb{C}$-space of all class functions of G.)*

Proof. By [3, Theorem 11.25], the mapping $\chi \mapsto \chi_{\mathcal{H}}$ defines a bijection between the set of all irreducible constituents of λ^U and the set of all irreducible characters of $\mathcal{H}$. By the same result (and by the paragraph above), the mapping $\phi \mapsto \phi_{\mathcal{H}}$ defines a bijection between the set of irreducible constituents of λ^S and the set of irreducible characters of $\mathcal{H}$. Therefore, the irreducible constituents of λ^U are in one-to-one correspondence with the irreducible constituents of λ^S.

Now, let $\chi \in \mathrm{Irr}(U)$ be a constituent of λ^U and let $\theta = \chi_{\mathcal{H}}$ be the irreducible character of $\mathcal{H}$ that corresponds to χ. (Given any finite group G, we denote by $\mathrm{Irr}(G)$ the set of all irreducible characters of G.) On the other hand, let $\phi \in \mathrm{Irr}(S)$ be the (unique) constituent of λ^S such that $\phi_{\mathcal{H}} = \theta$. We claim that $\chi = \phi^U$. To see this, let $\chi' \in \mathrm{Irr}(U)$ be any irreducible constituent of ϕ^U. Then, χ' is a constituent of λ^U and so $\chi'_{\mathcal{H}}$ is an irreducible character of $\mathcal{H}$. Since ϕ is a constituent of χ'_S (by Frobenius reciprocity), we conclude that θ is a constituent of $\chi'_{\mathcal{H}}$ and so $\theta = \chi'_{\mathcal{H}}$ (because θ and $\chi'_{\mathcal{H}}$ are irreducible). Therefore, χ' is the unique irreducible constituent of λ^U with $\chi'_{\mathcal{H}} = \theta$. Thus, $\chi' = \chi$ and so χ is the unique irreducible constituent of ϕ^U. It follows that $\phi^U = m\chi$ where $m = \langle \phi^U, \chi\rangle_U$. In particular, we have $\phi^U(1) = m\chi(1)$ and so $|U : S|\phi(1) = m\chi(1)$. Finally, for each $x \in X$, let $\widehat{a}_x = \varepsilon x^{-1}\varepsilon \in \mathcal{H}$. Then, by [3, Theorem 11.32], we have

$$c\chi(1) = |U : N|\langle \chi, \lambda^U\rangle_U$$

where $c = \sum_{x\in X} \theta(\widehat{a}_x)\theta(a_x)$ (we recall that $\operatorname{ind} x = 1$ for all $x \in X$). Similarly,

$$c\phi(1) = |S : N|\langle \phi, \lambda^S\rangle_S.$$

Since $\langle \chi, \lambda^U\rangle_U = \theta(\varepsilon) = \langle \phi, \lambda^S\rangle_S$ (by [3, Theorem 11.25]), we deduce that

$$\chi(1) = |U : S|\phi(1) = m\chi(1)$$

and so $m = 1$. It follows that $\phi^U = \chi$ is an irreducible constituent of $\lambda^U = \xi$. □

Next, we identify a distinguished irreducible constituent of the arbitrary basic character ξ of U. In particular, we generalize [1, Corollary 5] for an arbitary prime, proving that ξ has a unique irreducible constituent if and only if the *derived set* $\mathcal{D}'$ of $\mathcal{D}$ is empty. We start by recalling the definition of $\mathcal{D}'$. A *chain* in Φ (of length $r-1$) is a subset $\mathcal{C} \subseteq \Phi$ with the form $\mathcal{C} = \{(i_1, i_2), (i_2, i_3), \dots, (i_{r-1}, i_r)\}$. Given two chains $\mathcal{C}_1 = \{(i_1, i_2), (i_2, i_3), \dots, (i_{r-1}, i_r)\}$ and $\mathcal{C}_2 = \{(j_1, j_2), (j_2, j_3), \dots, (j_{s-1}, j_s)\}$ in Φ, we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ *intertwine* if $r = s$ and if $i_t < j_t < i_{t+1} < j_{t+1}$ for all $1 \le t \le r-1$. Finally, given a basic subset $\mathcal{D} \subseteq \Phi$, we say that a root $(i, j) \in \Phi$ is $\mathcal{D}$-*derived* if there exist two intertwining chains $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{D}$ (of length $r-1$) with $i = i_1$ and $j = j_1$ (where the notation is as above) satisfying the following two conditions: (1) if $(i_0, i_1) \in \mathcal{D}$ for some $1 \le i_0 < i_1$, then $j_1 < i_0$; (2) if $(j_r, j_{r+1}) \in \mathcal{D}$ for some $j_r < j_{r+1} \le n$, then $j_{r+1} < i_r$. We denote by $\mathcal{D}'$ the set of all $\mathcal{D}$-derived roots and call it the *derived set* of $\mathcal{D}$. Now, the set $\mathcal{S}'$ can be decomposed as a disjoint union of maximal chains. Let $\mathcal{C} = \{(i_1, i_2), (i_2, i_3), \ldots, (i_r, i_{r+1})\} \subseteq \mathcal{S}'$ be a chain. Then, by the definition of $\mathcal{S}'$, the sets $\mathcal{C}_1 = \{(i_1, i_3), (i_3, i_5), \ldots\}$ and $\mathcal{C}_2 = \{(i_2, i_4), (i_4, i_6), \ldots\}$ are intertwinning chains in $\mathcal{D}$. On the other hand, it is clear that the chain $\mathcal{C}$ is maximal in $\mathcal{S}'$ if and only if $\mathcal{S}'$ does not contain roots $(i, i_1), (i_{r+1}, j) \in \Phi$ (for some $1 \leq i < i_1$ and some $i_{r+1} < j \leq n$). We note that, if $\mathcal{C}$ is maximal in $\mathcal{S}'$, the root $(i_1, i_2) \in \mathcal{C}$ is $\mathcal{D}$-derived if and only if the length r of $\mathcal{C}$ is odd. Moreover, every $\mathcal{D}$-derived root (if it exists) must appear as the initial root of a unique maximal chain in $\mathcal{S}'$. Hence, the derived set $\mathcal{D}'$ is empty if and only if all maximal chains in $\mathcal{S}'$ have even length.

Now, we define the subset $\mathcal{S}'_0$ of $\mathcal{S}'$ as follows. Let $(i, j) \in \mathcal{S}'$ be arbitrary and let $\mathcal{C} = \{(i_1, i_2), (i_2, i_3), \ldots, (i_r, i_{r+1})\}$ be the maximal chain in $\mathcal{S}'$ that contains the root (i, j). Let $1 \leq s \leq r$ be such that $(i, j) = (i_s, i_{s+1})$. Then, $(i, j) \in \mathcal{S}'_0$ if and only if the subchain $\mathcal{C}_{ij} = \{(i_s, i_{s+1}), \ldots, (i_r, i_{r+1})\}$ of $\mathcal{C}$ has odd length. Therefore, we have $\mathcal{S}'_0 \cap \mathcal{C} = \{(i_r, i_{r+1}), (i_{r-2}, i_{r-1}), \ldots\}$. We have the following result.

Lemma 3. *Let $\mathfrak{x}_0$ be the $\mathbb{F}_q$-subspace of $\mathfrak{u}$ spanned by the vectors e_{ij} for $(i, j) \in \mathcal{S}'_0$. Then, the $\mathbb{F}_q$-subspace $\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{x}_0$ of $\mathfrak{u}$ is multiplicatively closed and so $P = 1 + \mathfrak{p}$ is an algebra subgroup of S.*

Proof. Suppose that $\mathfrak{p}$ is not multiplicatively closed. Then, we may choose the largest $1 \leq k \leq n$ with the property that there exist $(i, j), (j, k) \in \mathcal{R} \cup \mathcal{S}'_0$ with $(i, k) \notin \mathcal{R} \cup \mathcal{S}'_0$ (we recall that $\mathcal{R} = \Phi - \mathcal{S}$). If $(i, j) \in \mathcal{R}$, then $(i, k) \in \mathcal{R}$. Therefore, we must have $(i, j) \in \mathcal{S}'_0$; hence $(i, j) \in \mathcal{S}'$ and so there exist $(i, r), (j, s) \in \mathcal{D}$ with $r < s$. If $(j, k) \in \mathcal{R}$, then $s \leq k$ and so $(i, k) \in \mathcal{R}$ (because $r < s \leq k$ and $(i, r) \in \mathcal{D}$). Since this cannot happen, we must have $(j, k) \in \mathcal{S}'_0$ and so $k < s$ (because $\mathcal{S}'_0 \subseteq \mathcal{S}' \subseteq \mathcal{S}$). If $r \leq k$, then $(i, k) \in \mathcal{R}$, which cannot happen. Therefore, we have $k < r$. Since $(j, k) \in \mathcal{S}'_0 \subseteq \mathcal{S}'$, there exists $(k, t) \in \mathcal{D}$ with $s < t$. Now, since $(i, k) \notin \mathcal{S}'_0$, there exists $(r, u) \in \mathcal{D}$ with $t < u$ (otherwise $(r, t) \in \mathcal{R}$ and this implies that $(i, k) \in \mathcal{S}'_0$). It follows that $(k, r) \in \mathcal{S}'$ and so $(k, r) \in \mathcal{S}'_0$ (otherwise, $(i, k) \in \mathcal{S}'_0$). Hence, we have $(j, k), (k, r) \in \mathcal{S}'_0$. Since $(j, s), (r, u) \in \mathcal{D}$ and since $s < u$, we have $(j, r) \in \mathcal{S}'$. On the other hand, since $(i, j) \in \mathcal{S}'_0$, the root (j, r) does not lie in $\mathcal{S}'_0$. Since $k < r$, this contradicts the choice of k. The proof is complete. □

Let $\mu\colon P \to \mathbb{C}$ be the map defined by

$$\mu(1 + a) = \psi(e^*(a))$$

for all $a \in \mathfrak{p}$. We claim that μ is a linear character of P. To see this, let $a, b \in \mathfrak{p}$ be arbitrary and let $x = 1 + a$ and $y = 1 + b$. Then, $xy = 1 + a + b + ab$ and so $\mu(xy) = \mu(x)\mu(y)\psi(e^*(ab))$. Since $e^*(e_{ik}) \neq 0$ if and only if $(i, k) \in \mathcal{D}$, we clearly have $e^*(e_{ij}e_{jk}) = e^*(e_{ik}) = 0$ for all $(i, j), (j, k) \in \mathcal{R} \cup \mathcal{S}'_0$. It follows that $e^*(ab) = 0$ and so $\mu(xy) = \mu(x)\mu(y)$. Therefore, μ is in fact a linear character of P. (Moreover, it is clear that $\mu_N = \lambda$ and so, by Gallagher's Theorem (see [3, Theorem 11.5]), we have $\lambda^P = \sum_\omega \omega(1)(\omega\mu)$ where the sum extends over all the irreducible characters of the quotient group P/N (viewed as characters of P).) In the following, we prove that the induced character μ^U is irreducible and that, in fact, $\mu^U = \phi_{\mathcal{O}}$ where $\mathcal{O} \subseteq \mathfrak{u}^*$ is the coadjoint U-orbit that contains the element e^*. We start by proving the following result.

Proposition 3. *The induced character μ^S is an irreducible constituent of λ^S. Moreover, let $\mathcal{O} \subseteq \mathfrak{s}^*$ be the coadjoint S-orbit that contains the element $e^* \in \mathfrak{s}^*$.*

Then, we have

$$\mu^S(1+a) = \frac{1}{\sqrt{|\mathcal{O}|}} \sum_{f \in \mathcal{O}} \psi(f(a))$$

for all $a \in \mathfrak{s}$.

Proof. By [2, Proposition 1]), the induced character μ^S is a linear combination of the class functions $\phi_{\mathcal{O}'}$ corresponding to the coadjoint S-orbits $\mathcal{O}' \subseteq \mathfrak{s}^*$. Let $\pi\colon \mathfrak{s}^* \to \mathfrak{p}^*$ be the natural projection (given by the restriction of functions) and let Ω^* denote the set of all coadjoint S-orbits $\mathcal{O}' \subseteq \mathfrak{s}^*$ such that $e^* \in \pi(\mathcal{O}')$ (here, we abuse the notation and write e^* instead of $\pi(e^*)$). Since $e^*([a,b]) = 0$ for all $a, b \in \mathfrak{p}$, the group P centralizes the element $e^* \in \mathfrak{p}^*$ and so $\{e^*\}$ is a single coadjoint P-orbit on $\mathfrak{p}^*$. Therefore, by [2, Proposition 2] (and by Frobenius reciprocity), μ^S is a linear combination of the class functions $\phi_{\mathcal{O}'}$ for $\mathcal{O}' \in \Omega^*$; moreover, for each $\mathcal{O}' \in \Omega^*$, the multiplicity $m_{\mathcal{O}'} = \langle \mu^S, \phi_{\mathcal{O}'} \rangle_S$ is a positive integer.

Now, let $B\colon \mathfrak{s} \times \mathfrak{s} \to \mathbb{F}_q$ be the skew-symmetric $\mathbb{F}_q$-bilinear form defined by $B(a,b) = f([a,b])$ for all $a, b \in \mathfrak{s}$ and let $\mathfrak{r} = \{a \in \mathfrak{s}\colon e^*([a,b]) = 0 \text{ for all } b \in \mathfrak{s}\}$ be the radical of B. It is well known that $m = \dim \mathfrak{s} - \dim \mathfrak{r}$ is an even number. Moreover, by [2, Lemma 1], we have $|\mathcal{O}| = q^m$. On the other hand, let M be the matrix with entries $e^*([e_{ij}, e_{kl}])$ for $(i,j),(k,l) \in \mathcal{S}' \cup \mathcal{R}$. Then, since $\operatorname{rank} M = \dim \mathfrak{s} - \dim \mathfrak{r}$, we have $|\mathcal{O}| = q^{\operatorname{rank} M}$. Let $\mathcal{C}_1, \dots, \mathcal{C}_t$ be the distinct maximal chains in $\mathcal{S}'$ and, for each $1 \leq s \leq t$, let M_s be the submatrix of M defined by the roots $(i,j) \in \mathcal{C}_s$. It is easy to see that $\operatorname{rank} M = \operatorname{rank} M_1 + \cdots + \operatorname{rank} M_t$ and that, for each $1 \leq s \leq t$,

$$\operatorname{rank} M_s = \begin{cases} |\mathcal{C}_s|, & \text{if } |\mathcal{C}_s| \text{ is even,} \\ |\mathcal{C}_s| - 1, & \text{if } |\mathcal{C}_s| \text{ is odd} \end{cases}$$

(see the proof of [1, Theorem 3]). It follows that $\operatorname{rank} M = |\mathcal{S}'| - |\mathcal{D}'|$. Since $\dim \mathfrak{s} - \dim \mathfrak{p} = \frac{1}{2}(|\mathcal{S}'| - |\mathcal{D}'|)$, we conclude that $|\mathcal{O}| = q^{\operatorname{rank} M} = q^{2(\dim \mathfrak{s} - \dim \mathfrak{p})} = |S : P|^2$. Finally, since

$$|S : P| = \mu^S(1) = \sum_{\mathcal{O}' \in \Omega^*} m_{\mathcal{O}'} \phi_{\mathcal{O}'}(1) = \sum_{\mathcal{O}' \in \Omega^*} m_{\mathcal{O}'} \sqrt{|\mathcal{O}'|}$$

(and since $\mathcal{O} \in \Omega^*$), we conclude that $\Omega^* = \{\mathcal{O}\}$, that $m_{\mathcal{O}} = 1$ and that $\mu^S = \phi_{\mathcal{O}}$. Since $\langle \phi_{\mathcal{O}}, \phi_{\mathcal{O}} \rangle_S = 1$ (by [2, Proposition 1]), the induced character μ^S is irreducible. Moreover, μ^S is a constituent of λ^S because μ is a constituent of λ^P. □

We now may apply Theorem 2 to justify the first assertion of the following corollary.

Theorem 3. *The induced character μ^U is an irreducible constituent of ξ. Moreover, μ^U is the class function $\phi_{\mathcal{O}}$ that corresponds to the coadjoint U-orbit $\mathcal{O} \subseteq \mathfrak{u}^*$ that contains the element $e^* \in \mathfrak{u}^*$.*

Proof. It remains to show that $\mu^U = \phi_{\mathcal{O}}$. Let $\mathcal{O}_0 \subseteq \mathfrak{s}^*$ be the coadjoint S-orbit that ontains the element $e^* \in \mathfrak{s}^*$. By the previous proposition, we have $\mu^S = \phi_{\mathcal{O}_0}$ and so $\mu^U = (\phi_{\mathcal{O}_0})^U$. Therefore, we must prove that $(\phi_{\mathcal{O}_0})^U = \phi_{\mathcal{O}}$. Let $\pi\colon \mathfrak{u}^* \to \mathfrak{s}^*$ be the natural projection. Since $\mathcal{O}_0 \subseteq \pi(\mathcal{O})$, the class function $\phi_{\mathcal{O}}$ occurs as a constituent of $(\phi_{\mathcal{O}_0})^U = \mu^U$ with positive integer multiplicity (by [2, Proposition 2] and by Frobenius reciprocity). Let M be the matrix with entries $e^*([e_{ij}, e_{kl}])$ for $(i,j),(k,l) \in \Phi$. It is easy to prove that $\operatorname{rank} M = \operatorname{rank} M_0 + 2(|\mathcal{S}| - |\mathcal{S}'|)$ where M_0

is the submatrix of M defined by the roots $(i,j) \in \mathcal{R} \cup \mathcal{S}'$. As in the proof of the previous proposition, we conclude that

$$|\mathcal{O}| = q^{\operatorname{rank} M} = q^{\operatorname{rank} M_0} = q^{2(|\mathcal{S}|-|\mathcal{S}'|)} = |\mathcal{O}_0|\,|U:S|^2$$

and so

$$\phi_{\mathcal{O}}(1) = \sqrt{|\mathcal{O}|} = |U:S|\sqrt{|\mathcal{O}_0|} = |U:S|\phi_{\mathcal{O}_0}(1) = (\phi_{\mathcal{O}_0})^U(1).$$

It follows that $\phi_{\mathcal{O}} = (\phi_{\mathcal{O}_0})^U$, and this completes the proof. □

Finally, [2, Theorems 1 and 2] imply that

$$\langle \mu^U, \xi \rangle_U = q^{s'-s}\mu^U(1)$$

where $s = |\mathcal{S}|$ and $s' = |\mathcal{S}'|$; we note that, since $\xi = \lambda^U$ (by Theorem 1), we have $\xi(1) = |U:N| = q^s$). On the other hand, it is easy to see that $q^{s'} = |S:N|$. Since $\mu^U(1) = |U:P|$, we conclude that $\langle \mu^U, \xi\rangle_U = |S:P|$. Now, suppose that the derived set $\mathcal{D}'$ of $\mathcal{D}$ is empty. Then, all maximal chains in $\mathcal{S}'$ have even length and so $|S:P| = \sqrt{q^{s'}} = \mu^S(1)$ (see the proof of Proposition 3). On the other hand, by Theorem 2, we know that

$$\langle \mu^S, \lambda^S \rangle_S = \langle (\mu^S)^U, \xi \rangle_U = \langle \mu^U, \xi \rangle_U = |S:P|.$$

Since $\lambda^S(1) = |S:N|$ and since $|S:N| = |S:P|^2$ (because $\mathcal{D}' = \emptyset$), we conclude that $\lambda^S = |S:P|\mu^S$. Therefore, we deduce the following result (see [1, Corollary 5] for the case where $p \geq n$).

Theorem 4. *The basic character ξ has a unique irreducible constituent if and only if the derived set $\mathcal{D}'$ of $\mathcal{D}$ is empty. If this is the case, we have $\xi = m\mu^U$ where $m = |S:P|$ (and $m^2 = |S:N|$).*

Proof. If $\mathcal{D}'$ is empty, we deduce that $\xi = \lambda^U = (\lambda^S)^U = m(\mu^S)^U = m\mu^U$. In the general situation, we know that μ^S is an irreducible constituent of λ^S with multiplicity $m = |S:P|$. On the other hand, it is easy to see that $|P:N| = q^{|\mathcal{D}'|}|S:P|$ and so $|S:N| = q^{|\mathcal{D}'|}|S:P|^2$. Therefore, $\lambda^S(1) = mq^{|\mathcal{D}'|}\mu^S(1)$ and this implies that $\lambda^S = m\mu^S$ if and only if $\mathcal{D}'$ is empty. The result follows (using Theorem 2). □

References

[1] C. A. M. André, *Basic sums of coadjoint orbits of the unitriangular group*, J. Algebra **176** (1995), 959–1000. MR **96h:**20081b

[2] C. A. M. André, *Basic characters of the unitriangular group (for arbitrary primes)*, Proc. Amer. Math. Soc. **130**, no. 7, (2002), 1943–1954.

[3] C. W. Curtis and I. Reiner, *Methods of representation theory (with applications to finite groups and orders)*, Vol. 1, Wiley-Interscience, New York, 1981. MR **82i:**20001

[4] B. Huppert, *Character theory of finite groups*, Walter de Gruyter, Berlin, 1998. MR **99j:**20011

[5] I. M. Isaacs, *Characters of groups associated with finite algebras*, J. Algebra **177** (1995) 708–730. MR **96k:**20011

[6] I. M. Isaacs and D. Karagueuzian, *Conjugacy in groups of upper triangular matrices*, J. Algebra **202** (1998), 704–711. MR **99b:**20011

Departamento de Matemática e Centro de Estruturas Lineares e Combinatórias, Faculdade de Ciências da Universidade de Lisboa, Rua Ernesto de Vasconcelos, Edifício C1, Piso 3, 1749-016 Lisboa, Portugal

E-mail address: candre@fc.ul.pt