

MINIMAL DISPLACEMENT AND RETRACTION PROBLEMS IN INFINITE-DIMENSIONAL HILBERT SPACES

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ABSTRACT. We give the first constructive example of a Lipschitz mapping with positive minimal displacement in an infinite-dimensional Hilbert space H . We use this construction to obtain an evaluation from below of the minimal displacement characteristic in the space H . In the second part we present a simple and constructive proof of existence of a Lipschitz retraction from a unit ball B onto a unit sphere S in the space H , and we improve an evaluation from above of a retraction constant $k_0(H)$.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with the closed unit ball B and the unit sphere S . For any $k \geq 0$, let $L(k)$ denote the class of Lipschitz mappings with constant k . If K is a bounded, closed and convex subset of X , then by d_T we will denote minimal displacement of T ,

$$d_T = \inf_{x \in K} \|x - Tx\|,$$

where $T : K \rightarrow K$. This notion for Lipschitz mappings was introduced by Goebel [7] in 1973. In this paper he gave examples of sets K and Lipschitz mappings T with positive d_T . The problem for which sets there exists a Lipschitz mapping with positive minimal displacement was studied by many authors, and a final solution was given by Lin and Sternfeld [12]. They proved that if K is bounded, closed and convex but noncompact, then there exists a Lipschitz mapping T such that $d_T > 0$. On the other hand, very little is known about qualitative properties of minimal displacement. Goebel in the above-mentioned paper introduced some functions describing the minimal displacement problem. We recall two of them. The first, denoted by φ_X , is defined as follows:

$$\varphi_X(k) = \sup \{d_T : K \subset X, r(K) = 1, T : K \rightarrow K, T \in L(k), k \geq 1\},$$

where $r(K)$ is the Chebyshev radius of K , i.e.,

$$r(K) = \inf [\sup \{\|x - y\| : x \in K\} : y \in X].$$

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The second ψ_X denotes the minimal displacement characteristic of X ,

$$\psi_X(k) = \sup \{d_T : T : B \rightarrow B, T \in L(k), k \geq 1\}.$$

It is known that for any space X ,

$$\psi_X(k) \leq \varphi_X(k) \leq 1 - \frac{1}{k}.$$

Moreover, the following holds:

$$\begin{aligned} \psi_X(1 - \alpha + \alpha k) &\geq \alpha \psi_X(k) \text{ for any } \alpha \in [0, 1], \\ \lim_{k \rightarrow \infty} \psi_X(k) &= 1 \text{ for any space } X. \end{aligned}$$

There are some “square” spaces like $c_0, C[0, 1]$ for which $\psi_X(k) = \varphi_X(k) = 1 - \frac{1}{k}$. The two functions are equal in the Hilbert space H and the following estimate holds:

$$\psi_H(k) = \varphi_H(k) \leq \left(1 - \frac{1}{k}\right) \sqrt{\frac{k}{k+1}}.$$

In 1979 Nowak [13] proved that in some infinite-dimensional Banach spaces there exist Lipschitz retractions of unit balls onto unit spheres. Four years later Benyamini and Sternfeld [1] extended the above result to all infinite-dimensional Banach spaces. This result leads directly to the definition of the so-called retraction constant $k_0(X)$, being the infimum of the set of all numbers $k > 1$ for which there exists a retraction $R : B \rightarrow S$ belonging to $L(k)$. It is known that $k_0(X) \geq 3$ for any space X . For the space H we know that $k_0(H) > 4.5$. On the other hand, Komorowski and Wośko [11] gave an example of a Lipschitz retraction with Lipschitz constant less than 64.5. Unfortunately their example is not completely constructive. For a wider discussion of the topics mentioned above we refer the reader to [3]-[6], [9]-[11], [14], and [15].

2. MINIMAL DISPLACEMENT

For more than twenty years since Goebel’s paper there have been no results on the minimal displacement problem in the space H . In 1996 the author [2] proved the following evaluation from below of the function ψ_H .

Theorem 2.1. *In an infinite-dimensional Hilbert space H ,*

$$\psi_H(k) \geq \sup_{\epsilon \in (0,2)} \left(1 - \frac{2 + \epsilon}{\sqrt{1 + \epsilon(\epsilon + 2)k^2 - 1}} - \epsilon(k + 1)\right).$$

Unfortunately this evaluation is very imprecise. Below we present a better estimation. Let us consider the Hilbert space $L^2(0, 1)$ and let S^+ denote the nonnegative part of the unit sphere S , i.e.,

$$S^+ = \{x \in S : x(t) \geq 0 \text{ for } t \in (0, 1)\}.$$

By d_T denote minimal displacement also in this setting (S^+ is not convex). We start with the following lemma.

Lemma 2.2. *For every $\alpha \geq 2$ there exists a mapping $T : S^+ \rightarrow S^+$ from the class $L\left(\alpha^{\frac{3}{2}}\right)$ such that $d_T = \frac{\sqrt{2}(\sqrt{\alpha}-1)}{\sqrt{\alpha+1}}$.*

Proof. Let $\alpha \geq 2$ and let the mapping $T : S^+ \rightarrow S^+$ be defined by an equality

$$\int_0^t [(Tx)(s)]^2 ds = \left(\int_0^t [x(s)]^2 ds \right)^\alpha \quad \text{for every } t \in (0, 1).$$

Observe that the map is well defined and that it can be written equivalently as

$$(Tx)(t) = \sqrt{\alpha} x(t) \|x\|_t^{\alpha-1},$$

where

$$\|x\|_t = \sqrt{\int_0^t [x(s)]^2 ds}.$$

First we show that T is a Lipschitz mapping.

$$\begin{aligned} \|Tx - Ty\|^2 &= \alpha \int_0^1 [x(t) \|x\|_t^{\alpha-1} - y(t) \|y\|_t^{\alpha-1}]^2 dt \\ &\leq \alpha \int_0^1 [x(t) \left| \|x\|_t^{\alpha-1} - \|y\|_t^{\alpha-1} \right| + |x(t) - y(t)| \|y\|_t^{\alpha-1}]^2 dt \\ &\leq \alpha \int_0^1 [(\alpha - 1) x(t) \left| \|x\|_t - \|y\|_t \right| + |x(t) - y(t)|]^2 dt \\ &\leq \alpha \int_0^1 [(\alpha - 1) \|x - y\| + |x(t) - y(t)|]^2 dt. \end{aligned}$$

Furthermore using standard arguments such as the triangle and Cauchy-Schwartz inequalities, we get

$$\|Tx - Ty\|^2 \leq \alpha^3 \|x - y\|^2.$$

Hence $T \in L\left(\alpha^{\frac{3}{2}}\right)$. The minimal displacement of T can be written as follows:

$$\begin{aligned} \|x - Tx\|^2 &= \|x\|^2 - 2 \langle x, Tx \rangle + \|Tx\|^2 \\ &= 2 - 2 \int_0^1 x(t) (Tx)(t) dt \\ &= 2 - 2\sqrt{\alpha} \int_0^1 [x(t)]^2 \left(\int_0^t [x(s)]^2 ds \right)^{\frac{\alpha-1}{2}} dt. \end{aligned}$$

Setting

$$z = \int_0^t [x(s)]^2 ds,$$

we get

$$\|x - Tx\|^2 = 2 - 2\sqrt{\alpha} \int_0^1 z^{\frac{\alpha-1}{2}} dz = \frac{2(\sqrt{\alpha} - 1)^2}{\alpha + 1} > 0.$$

Observe that $\|x - Tx\| = \text{const}$ for every $x \in S^+$. Moreover, when $\alpha \rightarrow \infty$, then $\|x - Tx\| \rightarrow \sqrt{2} = \text{diam } S^+$. \square

Theorem 2.3. *Let H be an infinite-dimensional Hilbert space. Then*

$$\psi_H \left(\sqrt{2}(\alpha + 1) \alpha^{\frac{3}{2}} \right) \geq \frac{\alpha - 1}{\alpha + 1} \quad \text{for } \alpha \geq 2.$$

Proof. For any $r > 0$, let $B^+(r)$ be the nonnegative part of the ball $B(r)$ of radius r , i.e., the set

$$B^+(r) = \{x \in B(r) : x(t) \geq 0 \text{ for } t \in (0, 1)\}.$$

Define a mapping $T_1 : B \rightarrow B^+(1)$ by

$$(T_1x)(t) = |x(t)|.$$

Obviously, $T_1 \in L(1)$. Let δ denote the following expression:

$$\delta = d_T \sqrt{1 - \left(\frac{d_T}{2}\right)^2} = \frac{\alpha - 1}{\alpha + 1},$$

where d_T is the minimal displacement of the map T from the previous lemma. Furthermore, we define a map \tilde{T} such that $d_{\tilde{T}} = \delta$. But now let a mapping $T_2 : B^+(1) \rightarrow B^+(1)$ be given as

$$(T_2x)(t) = \begin{cases} x(t) & \text{for } \|x\| \geq 1 - \delta, \\ 1 - \delta - \|x\| + x(t) & \text{for } \|x\| < 1 - \delta. \end{cases}$$

Obviously when $\|x\| \geq 1 - \delta$ and $\|y\| \geq 1 - \delta$, then $T_2 \in L(1)$. When $\|x\| < 1 - \delta$ and $\|y\| < 1 - \delta$, we get

$$\begin{aligned} \|T_2x - T_2y\|^2 &= \int_0^1 (x(t) - y(t) + \|y\| - \|x\|)^2 dt \\ &\leq 2 \int_0^1 (x(t) - y(t))^2 dt + 2 \int_0^1 (\|y\| - \|x\|)^2 dt \\ &\leq 4 \|x - y\|^2. \end{aligned}$$

Hence

$$\|T_2x - T_2y\| \leq 2 \|x - y\|.$$

Let us estimate $\inf_{x \in B^+} \|T_2x\|$. Observe that if $\|x\| \geq 1 - \delta$, then

$$\|T_2x\| = \|x\| \geq 1 - \delta.$$

If $\|x\| < 1 - \delta$, then

$$\begin{aligned} \|T_2x\|^2 &= \int_0^1 (1 - \delta - \|x\| + x(t))^2 dt \\ &= (1 - \delta - \|x\|)^2 + 2(1 - \delta - \|x\|) \int_0^1 x(t) dt + \int_0^1 (x(t))^2 dt \\ &\geq 2 \|x\|^2 - 2(1 - \delta) \|x\| + (1 - \delta)^2. \end{aligned}$$

The above minimum is attained for $\|x\| = \frac{1-\delta}{2}$ and is equal to $\frac{(1-\delta)^2}{2}$, which finally shows that

$$\inf_{x \in B^+} \|T_2x\| \geq \frac{1 - \delta}{\sqrt{2}}.$$

Let $C = B^+(1) \setminus B^+\left(\frac{1-\delta}{\sqrt{2}}\right)$. Suppose that a mapping $T_3 : \overline{C} \rightarrow S^+$ is the radial projection, i.e.,

$$(T_3x)(t) = \frac{x(t)}{\|x\|}.$$

From the properties of the radial projection we get that $T_3 \in L\left(\frac{\sqrt{2}}{1-\delta}\right)$. We define the above-mentioned mapping $\tilde{T} : B \rightarrow S^+$ as the composition

$$\tilde{T} = T \circ T_3 \circ T_2 \circ T_1,$$

where T is the mapping from the previous lemma. It is easy to see that

$$\tilde{T} \in L\left(\frac{2\sqrt{2}\alpha^{\frac{3}{2}}}{1-\delta}\right) \quad \text{for } \alpha \geq 2,$$

or equivalently, after simple calculations,

$$\tilde{T} \in L\left(\sqrt{2}(\alpha+1)\alpha^{\frac{3}{2}}\right) \quad \text{for } \alpha \geq 2.$$

The minimal displacement of a map \tilde{T} can be evaluated as follows:

$$d_{\tilde{T}} = \inf_{x \in B} \|x - \tilde{T}x\| \geq \inf_{x \in B^+(1)} \|x - \tilde{T}x\| = \inf_{x \in B^+(1)} \|x - T \circ T_3 \circ T_2x\|.$$

Observe that if $x \in B^+(1)$ and $\|x\| < 1 - \delta$, then

$$\|x - \tilde{T}x\| > \delta.$$

If $x \in B^+(1)$ and $\|x\| \geq 1 - \delta$, then

$$\begin{aligned} \|x - \tilde{T}x\| &= \|x - T \circ T_3 \circ T_2x\| = \|x - T \circ T_3x\| = \left\|x - T\left(\frac{x}{\|x\|}\right)\right\| \\ &\geq \left\|\frac{x}{\|x\|} - T\left(\frac{x}{\|x\|}\right)\right\| \sqrt{1 - \left(\frac{\left\|\frac{x}{\|x\|} - T\left(\frac{x}{\|x\|}\right)\right\|}{2}\right)^2} \\ &= d_T \sqrt{1 - \left(\frac{d_T}{2}\right)^2} = \delta = \frac{\alpha-1}{\alpha+1}, \end{aligned}$$

which finally shows that

$$\psi_{L^2(0,1)}\left(\sqrt{2}(\alpha+1)\alpha^{\frac{3}{2}}\right) \geq \frac{\alpha-1}{\alpha+1} \quad \text{for } \alpha \geq 2.$$

Observe that if $\alpha \rightarrow \infty$, then $\frac{\alpha-1}{\alpha+1} \rightarrow 1$, which shows that the above estimate is good for large α . □

3. RETRACTION

In this section we give the first completely constructive example of a Lipschitz retraction in the space H . Let $P : X \rightarrow B$ be the radial projection, i.e.,

$$Px = \begin{cases} x & \text{if } x \in B, \\ \frac{x}{\|x\|} & \text{if } x \notin B. \end{cases}$$

Denoting by $P(X)$ the Lipschitz constant of such a projection, it is known that $P(H) = 1$ and $1 \leq P(X) \leq 2$ for any space X [16]. We start with the following technical lemma, which holds in all infinite-dimensional Banach spaces.

Lemma 3.4. *Suppose that there exists a homotopy $G : [0, 1] \times S \rightarrow S$ such that for any $x \in S$, the following conditions hold:*

$$G(0, x) = x \quad \text{and} \quad G(1, x) \equiv x_0 \in S.$$

Assume that the homotopy G is Lipschitz with constants M and N , i.e., for every $x, y \in S$ and $c, d \in [0, 1]$, the following condition holds:

$$\|G(c, x) - G(d, y)\| \leq M|c - d| + N\|x - y\|.$$

Then there exists a retraction $R : B \rightarrow S$ of a class $R \in L\left(\frac{P(X)N}{r}\right)$, where $r \in (0, 1)$ is a solution of the equation

$$\frac{P(X)N}{r} = \frac{M - P(X)N \ln r}{1 - r}.$$

Proof. Define a retraction $R : B \rightarrow S$ for $r \in (0, 1)$ as follows:

$$Rx = \begin{cases} x_0 & \text{if } \|x\| \leq r, \\ G\left(1 - f(\|x\|), \frac{x}{\|x\|}\right) & \text{if } \|x\| > r, \end{cases}$$

where f is any increasing, convex and differentiable function on the interval $[r, 1]$ such that $f(r) = 0, f(1) = 1$. Observe that the retraction R is Lipschitz. For any $x, y \in B$ such that $\|x\| \geq r, \|y\| \geq r$ we have

$$\begin{aligned} \|Rx - Ry\| &= \left\| G\left(1 - f(\|x\|), \frac{x}{\|x\|}\right) - G\left(1 - f(\|y\|), \frac{y}{\|y\|}\right) \right\| \\ &\leq M |f(\|x\|) - f(\|y\|)| + N \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \\ &\leq M f'(\max\{\|x\|, \|y\|\}) \left| \|x\| - \|y\| \right| + \frac{P(X)N}{\max\{\|x\|, \|y\|\}} \|x - y\| \\ &\leq \max_{s \in [r, 1]} \left\{ M f'(s) + \frac{P(X)N}{s} \right\} \|x - y\|. \end{aligned}$$

To optimize the Lipschitz constant of a retraction R , observe that

$$\int_r^1 \left(M f'(t) + \frac{P(X)N}{t} \right) dt = M - P(X)N \ln r$$

does not depend on the choice of a function f . The Lipschitz constant will be minimal if the function satisfies a condition

$$M f'(t) + \frac{P(X)N}{t} = \frac{M - P(X)N \ln r}{1 - r}.$$

We get

$$\|Rx - Ry\| \leq \frac{P(X)N}{r} \|x - y\|,$$

where r is a solution of the equation

$$\frac{P(X)N}{r} = \frac{M - P(X)N \ln r}{1 - r}.$$

Observe that such a solution exists and is unique because the above functions are continuous on the interval $(0, 1)$ and one is strictly increasing while the other is strictly decreasing.

Now we can proceed to the construction of a Lipschitz retraction in the Hilbert space $L^2(0, 1)$. \square

Construction 3.5. Define a homotopy $G : [0, 1] \times S \rightarrow S$ as follows:

$$G(c, x) = \frac{(1 - c)x + cT \circ T_1 x}{\|(1 - c)x + cT \circ T_1 x\|},$$

where the map T is the map from Lemma 2.2 and the map T_1 is from the proof of Theorem 2.3. We show that the homotopy G is well defined and

$$\inf_{c \in [0, 1]} \inf_{x \in S} \|(1 - c)x + cT \circ T_1 x\| > 0.$$

Observe that

$$\inf_{c \in [0,1]} \inf_{x \in S} \|(1-c)x + cT \circ T_1x\| = \inf_{c \in [0,1]} \inf_{x \in S^+} \|(1-c)x - cTx\|$$

and

$$\begin{aligned} d_T^2 &= \inf_{x \in S^+} \|x - Tx\|^2 \\ &\leq \|x - Tx\|^2 \\ &= 2\|x\|^2 - 2\langle x, Tx \rangle + \|Tx\|^2 \\ &= 2 - 2\langle x, Tx \rangle. \end{aligned}$$

Using this estimate we get

$$\begin{aligned} \|(1-c)x - cTx\|^2 &= (1-c)^2\|x\|^2 - 2c(1-c)\langle x, Tx \rangle + c^2\|Tx\|^2 \\ &= 1 - c(1-c)(4 - d_T^2) \\ &\geq \frac{d_T^2}{4} > 0. \end{aligned}$$

The homotopy G is Lipschitz. Indeed,

$$\begin{aligned} \|G(c, x) - G(d, y)\| &\leq \|G(c, x) - G(d, x)\| + \|G(d, x) - G(d, y)\| \\ &\leq \frac{2|c-d|}{\inf_{c \in [0,1]} \inf_{x \in S} \|(1-c)x + cT \circ T_1x\|} \\ &\quad + \sup_{c \in [0,1]} \frac{1+c(k-1)}{\inf_{z \in S^+} \|(1-c)z - cTz\|} \|x-y\|, \end{aligned}$$

where $k = \alpha^{\frac{3}{2}}$. Since for $c \in [0, 1]$ we have

$$\|(1-c)x - cTx\| = \|cx - (1-c)Tx\|,$$

we obtain

$$\begin{aligned} \|H(c, x) - H(d, y)\| &\leq \frac{4}{d_T} |c-d| + \sup_{c \in [\frac{1}{2}, 1]} \frac{1+c(k-1)}{\inf_{z \in S^+} \|(1-c)z - cTz\|} \|x-y\| \\ &= M(\alpha) |c-d| + N(\alpha) \|x-y\|, \end{aligned}$$

where (after calculations)

$$M(\alpha) = \frac{2\sqrt{2(\alpha+1)}}{\sqrt{\alpha}-1} \quad \text{and} \quad N(\alpha) = \sup_{c \in [\frac{1}{2}, 1]} \frac{1+c(\alpha^{\frac{3}{2}}-1)}{\sqrt{1-2c(1-c)\frac{(\sqrt{\alpha+1})^2}{\alpha+1}}}.$$

Now define a mapping $T_2 : B^+(r) \rightarrow B^+(r)$ by

$$(T_2x)(t) = r - \|x\| + x(t).$$

From the proof of Theorem 2 we conclude that

$$T_2 \in L(2) \quad \text{and} \quad \inf_{x \in B^+(r)} \|T_2x\| \geq \frac{r}{\sqrt{2}}.$$

Now we can define a retraction $R : B \rightarrow S$ by

$$(Rx)(t) = \begin{cases} \left(T \circ \left(\frac{T_2}{\|T_2\|}\right) \circ T_1x\right)(t) & \text{if } \|x\| \leq r, \\ G\left(1 - f(\|x\|), \frac{x(t)}{\|x\|}\right) & \text{if } \|x\| > r, \end{cases}$$

where f is any increasing, convex and differentiable function defined on the interval $[r, 1]$ such that $f(r) = 0$ and $f(1) = 1$.

Observe that if $\|x\| \leq r$ and $\|y\| \leq r$ we get

$$\begin{aligned} \|Rx - Ry\| &\leq \left\| T \circ \left(\frac{T_2}{\|T_2\|} \right) \circ T_1 x - T \circ \left(\frac{T_2}{\|T_2\|} \right) \circ T_1 y \right\| \\ &\leq \frac{2\sqrt{2}k}{r} \|x - y\| = 2\sqrt{2} \frac{\alpha^{\frac{3}{2}}}{r} \|x - y\|. \end{aligned}$$

If $\|x\| > r$ and $\|y\| > r$, using the previous lemma we have

$$\|Rx - Ry\| \leq \frac{N(\alpha)}{r} \|x - y\|,$$

where r is a solution of the equation

$$\frac{N(\alpha)}{r} = \frac{M(\alpha) - N(\alpha) \ln r}{1 - r}.$$

Numerical experiments show that for $\alpha = 2.535$ we get a retraction with the minimal Lipschitz constant slightly less than 32.26. This implies the following.

Corollary 3.6. *In the infinite-dimensional Hilbert space H ,*

$$k_0(H) < 32.26.$$

The problem of exact evaluation of $k_0(X)$ for at least one space is still open.

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