

## PARAMETRIC DECOMPOSITION OF POWERS OF IDEALS VERSUS REGULARITY OF SEQUENCES

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ABSTRACT. Let  $Q = (a_1, a_2, \dots, a_s)$  ( $\subsetneq A$ ) be an ideal in a Noetherian local ring  $A$ . Then the sequence  $a_1, a_2, \dots, a_s$  is  $A$ -regular if every  $a_i$  is a non-zerodivisor in  $A$  and if  $Q^n = \bigcap_{\alpha} (a_1^{\alpha_1}, a_2^{\alpha_2}, \dots, a_s^{\alpha_s})$  for all integers  $n \geq 1$ , where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$  runs over the elements of the set  $\Lambda_{s,n} = \{(\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s \mid \alpha_i \geq 1 \text{ for all } 1 \leq i \leq s \text{ and } \sum_{i=1}^s \alpha_i = s + n - 1\}$ .

### 1. INTRODUCTION

Throughout this note let  $A$  denote a commutative ring with the non-zero multiplicative identity. Let  $a_1, a_2, \dots, a_s \in A$  ( $s \geq 1$ ) and  $Q = (a_1, a_2, \dots, a_s)$  in  $A$ . For each integer  $n \geq 1$  we put

$$\Lambda_{s,n} = \{(\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s \mid \alpha_i \geq 1 \text{ for all } 1 \leq i \leq s \text{ and } \sum_{i=1}^s \alpha_i = s + n - 1\}.$$

Let  $Q(\alpha) = (a_1^{\alpha_1}, a_2^{\alpha_2}, \dots, a_s^{\alpha_s})$  for each  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \Lambda_{s,n}$ . Then  $Q^n \subseteq \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$ , and W. Heinzer, L. J. Ratliff, Jr. and K. Shah ([HRS, Theorem 2.4]) proved, among other things, that the equality holds true for all  $n \geq 1$  if the sequence  $a_1, a_2, \dots, a_s$  is  $A$ -regular. The purpose of this note is to study the question of when the converse holds true, and our conclusion is stated as follows.

**Theorem (1.1).** *Let  $A$  be a Noetherian local ring and assume that  $Q \subsetneq A$ . Then the following two conditions are equivalent:*

- (1) *The sequence  $a_1, a_2, \dots, a_s$  is  $A$ -regular.*
- (2) *Every  $a_i$  is a non-zerodivisor in  $A$  and the equality*

$$Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$$

*holds true for all integers  $n \geq 1$ .*

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The condition in Theorem (1.1) (2) that every  $a_i$  is a non-zero-divisor in  $A$  is not superfluous. (Consider the case  $s = 1$  or the case where  $a_i = 0$  for all  $1 \leq i \leq s$ . See Example (3.6) for a more non-trivial example.)

The condition in Theorem (1.1) (2) is a local condition. The following result gives a global version of Theorem (1.1).

**Corollary (1.2).** *Let  $A$  be a Noetherian ring and  $Q \subsetneq A$ . Assume that every  $a_i$  is a non-zero-divisor in  $A$ . Then the following two conditions are equivalent:*

- (1)  $\text{grade}(Q, A) = s$ .
- (2)  $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$  for all  $n \geq 1$ .

Our proof of Theorem (1.1) is based on the induction on  $s$ , which we will discuss in Section 3 (Proposition (3.4)). For this purpose we need some preliminary steps, including a brief proof of part of [HRS, Theorem 2.4], which we shall summarize in Section 2.

## 2. SOME LEMMATA

We put  $Q_i = (a_1, a_2, \dots, a_i)$  for each  $0 \leq i \leq s$ . Let us begin with the following.

**Lemma (2.1).** *Suppose that  $s \geq 2$  and  $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$  for all  $n \geq 1$ . Then  $(a_s^\ell) \cap Q_{s-1}^m \subseteq Q^{\ell+m}$  for all  $\ell, m \geq 1$ .*

*Proof.* Let  $x \in (a_s^\ell) \cap Q_{s-1}^m$  and assume that  $x \notin Q^{\ell+m} = \bigcap_{\alpha \in \Lambda_{s,\ell+m}} Q(\alpha)$ . We choose  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \Lambda_{s,\ell+m}$  so that  $x \notin Q(\alpha)$ . Then  $\alpha_s \geq \ell + 1$ , since  $x \in (a_s^\ell)$ . Therefore,  $\sum_{i=1}^{s-1} \alpha_i = [s + (\ell + m) - 1] - \alpha_s \leq s + m - 2$ . Let  $\delta = (s + m - 2) - \sum_{i=1}^{s-1} \alpha_i$ . Then, since  $(\alpha_1 + \delta, \alpha_2, \dots, \alpha_{s-1}) \in \Lambda_{s-1,m}$ , we get

$$x \in Q_{s-1}^m \subseteq (a_1^{\alpha_1+\delta}, a_2^{\alpha_2}, \dots, a_{s-1}^{\alpha_{s-1}}) \subseteq Q(\alpha),$$

which is impossible. □

The following result (2.2) is due to [HRS]. We give a brief proof for the sake of completeness. Our proof might be of some interest, because it is totally different from the one given in [HRS].

**Proposition (2.2).** *Suppose the sequence  $a_1, a_2, \dots, a_s$  is  $A$ -regular. Then*

$$Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$$

for all  $n \geq 1$ .

*Proof.* Assume  $Q^n \subsetneq \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$  and choose  $a \in \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$  so that  $a \notin Q^n$ . Let  $\ell$  be the largest integer satisfying the condition  $a \in Q^\ell$ . Then  $0 \leq \ell < n$ . Let  $G = \bigoplus_{n \geq 0} Q^n / Q^{n+1}$  be the associated graded ring of  $Q$  and put  $X_i = a_i \text{ mod } Q^2$  ( $1 \leq i \leq s$ ). Then  $X_1, X_2, \dots, X_s$  are algebraically independent over  $k = A/Q$  and  $G = k[X_1, X_2, \dots, X_s]$ . We put  $Y = a \text{ mod } Q^{\ell+1}$ . Then  $Y \neq 0$  and  $\text{deg } Y = \ell < n$ . Let  $\alpha \in \Lambda_{s,n}$ . Then, since  $a \in Q(\alpha)$  and the sequence  $X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_s^{\alpha_s}$  is  $G$ -regular, we get from [VV, Proposition 2.1] that  $Y \in (X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_s^{\alpha_s})$ . (The result [VV, Proposition 2.1] holds true with no extra assumption on base rings  $A$ .)

Therefore, in order to produce a contradiction, we have only to check the parametric decomposition

$$(X_1, X_2, \dots, X_s)^n = \bigcap_{\alpha \in \Lambda_{s,n}} (X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_s^{\alpha_s})$$

of ideals  $(X_1, X_2, \dots, X_s)^n$  in the polynomial ring  $G = k[X_1, X_2, \dots, X_s]$ .

Let  $0 \leq \beta_i \in \mathbb{Z}$  ( $1 \leq i \leq s$ ) and  $c \in k$ . Let  $M = c \prod_{i=1}^s X_i^{\beta_i}$  and assume that  $M \in \bigcap_{\alpha \in \Lambda_{s,n}} (X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_s^{\alpha_s})$  but  $M \notin (X_1, X_2, \dots, X_s)^n$ . We put  $m = \sum_{i=1}^s \beta_i$ . Then, since  $m \leq n-1$ , letting  $\delta = (n-1) - m$ , we get  $(\beta_1 + \delta + 1, \beta_2 + 1, \dots, \beta_s + 1) \in \Lambda_{s,n}$ . Consequently,

$$0 \neq c \prod_{i=1}^s X_i^{\beta_i} \in \bigcap_{\alpha \in \Lambda_{s,n}} (X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_s^{\alpha_s}) \subseteq (X_1^{\beta_1 + \delta + 1}, X_2^{\beta_2 + 1}, \dots, X_s^{\beta_s + 1}),$$

which is impossible. Hence  $(X_1, X_2, \dots, X_s)^n = \bigcap_{\alpha \in \Lambda_{s,n}} (X_1^{\alpha_1}, X_2^{\alpha_2}, \dots, X_s^{\alpha_s})$ .  $\square$

### 3. PROOF OF THEOREM (1.1)

Let  $A$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$  and assume that  $Q = (a_1, a_2, \dots, a_s) \subseteq \mathfrak{m}$ . To prove Theorem (1.1) we need the following. Let us note a quick proof for the sake of completeness.

**Lemma (3.1).** *Let  $a \in A$  and assume that  $(0) : a^\ell \subseteq (a)$  for all  $\ell \geq 1$ . Then  $a$  is a non-zero-divisor in  $A$ .*

*Proof.* We may assume that  $a \in \mathfrak{m}$ . Let  $x \in (0) : a^\ell$  and write  $x = ay$  ( $y \in A$ ). Then  $y \in (0) : a^{\ell+1}$ , whence  $(0) : a^\ell \subseteq a[(0) : a^{\ell+1}]$ . Take  $\ell \gg 0$  so that  $(0) : a^\ell = (0) : a^{\ell+1}$ . Then, since  $(0) : a^{\ell+1} = a[(0) : a^{\ell+1}]$ , by Nakayama's lemma we get  $(0) : a^{\ell+1} = (0)$ , whence  $a$  is a non-zero-divisor in  $A$ , as is  $a^{\ell+1}$ .  $\square$

The following two results are the key for our proof of Theorem (1.1).

**Lemma (3.2).** *Suppose that  $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$  for all  $n \geq 1$ . Then  $Q_{s-1} : a_s = Q_{s-1}$  if  $a_s$  is a non-zero-divisor in  $A$ .*

*Proof.* We may assume that  $s \geq 2$ . By Lemma (3.1) it suffices to show that

$$Q_{s-1} : a_s^\ell \subseteq Q$$

for all  $\ell \geq 1$ . Since  $a_s$  is a non-zero-divisor in  $A$ , it is enough to check that  $(a_s^\ell) \cap Q_{s-1} \subseteq a_s^\ell Q$ . Assume the contrary and choose  $m \geq 1$  so that  $(a_s^\ell) \cap Q_{s-1} \subseteq a_s^\ell Q + Q_{s-1}^m$ , but  $(a_s^\ell) \cap Q_{s-1} \not\subseteq a_s^\ell Q + Q_{s-1}^{m+1}$ . Then, since

$$(a_s^\ell) \cap Q_{s-1} \subseteq (a_s^\ell Q + Q_{s-1}^m) \cap (a_s^\ell) = a_s^\ell Q + [(a_s^\ell) \cap Q_{s-1}^m],$$

by Lemma (2.1) we get  $(a_s^\ell) \cap Q_{s-1} \subseteq a_s^\ell Q + Q_{s-1}^{\ell+m}$ . The following claim now readily shows  $(a_s^\ell) \cap Q_{s-1} \subseteq a_s^\ell Q + Q_{s-1}^{m+1}$ , which is impossible.

**Claim (3.3).**  $Q^{\ell+m} \subseteq a_s^\ell Q + Q_{s-1}^{m+1}$ .

*Proof of Claim (3.3).* Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s$  ( $\alpha_i \geq 0$ ) and assume that  $\sum_{i=1}^s \alpha_i = \ell + m$ . Then, if  $\alpha_s \geq \ell$ , we certainly have

$$a_1^{\alpha_1} a_2^{\alpha_2} \dots a_s^{\alpha_s} = a_s^\ell (a_1^{\alpha_1} a_2^{\alpha_2} \dots a_{s-1}^{\alpha_{s-1}} a_s^{\alpha_s - \ell}) \in a_s^\ell Q,$$

because  $\alpha_1 + \dots + \alpha_{s-1} + (\alpha_s - \ell) = \sum_{i=1}^s \alpha_i - \ell = m \geq 1$ . Suppose that  $\alpha_s \leq \ell - 1$ . Then  $a_1^{\alpha_1} \dots a_{s-1}^{\alpha_{s-1}} \in Q_{s-1}^{m+1}$ , because  $\sum_{i=1}^{s-1} \alpha_i = \sum_{i=1}^s \alpha_i - \alpha_s \geq (\ell + m) - (\ell - 1) =$

$m+1$ . Hence  $a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_s^{\alpha_s} \in a_s^\ell Q + Q_{s-1}^{m+1}$  in any case, so that  $Q^{\ell+m} \subseteq a_s^\ell Q + Q_{s-1}^{m+1}$  as claimed.  $\square$

**Proposition (3.4).** *Assume that  $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$  for all  $n \geq 1$ . Then  $Q_i^n = \bigcap_{\alpha \in \Lambda_{i,n}} Q_i(\alpha)$  for all integers  $1 \leq i \leq s$  and  $n \geq 1$ .*

*Proof.* We may assume that  $s \geq 2$  and  $i = s - 1$ . Let  $x \in \bigcap_{\alpha \in \Lambda_{s-1,n}} Q_{s-1}(\alpha)$  and assume  $x \notin Q_{s-1}^n$ . Choose the integer  $\ell \geq 0$  so that  $x \in Q_{s-1}^n + (a_s^\ell)$  but  $x \notin Q_{s-1}^n + (a_s^{\ell+1})$ . We write  $x = y + a_s^\ell z$  ( $y \in Q_{s-1}^n, z \in A$ ). We then have the following.

**Claim (3.5).**  $x - y \in Q^{\ell+n} = \bigcap_{\alpha \in \Lambda_{s,\ell+n}} Q(\alpha)$ .

*Proof of Claim (3.5).* Let  $\alpha \in \Lambda_{s,\ell+n}$ . Then, since  $x - y = a_s^\ell z \in (a_s^\ell)$ , to see that  $x - y \in Q(\alpha)$ , we may assume  $\alpha_s \geq \ell + 1$ . Let  $\delta = (s + n - 2) - \sum_{i=1}^{s-1} \alpha_i$ . Then, since  $\alpha_s \geq \ell + 1$  and  $\sum_{i=1}^s \alpha_i = s + (\ell + n) - 1$ , we have  $\delta \geq 0$ . Because  $(\alpha_1 + \delta, \alpha_2, \dots, \alpha_{s-1}) \in \Lambda_{s-1,n}$  and  $y \in Q_{s-1}^n$ , we get that  $x - y \in (a_1^{\alpha_1 + \delta}, a_2^{\alpha_2}, \dots, a_{s-1}^{\alpha_{s-1}}) \subseteq Q(\alpha)$  (recall that  $x \in \bigcap_{\alpha \in \Lambda_{s-1,n}} Q_{s-1}(\alpha)$ ). Thus  $x - y \in Q(\alpha)$  for all  $\alpha \in \Lambda_{s,\ell+n}$ , so that we have  $x - y \in Q^{\ell+n}$ .  $\square$

Consequently, we get  $x \in Q_{s-1}^n + Q^{\ell+n} \subseteq Q_{s-1}^n + (a_s^{\ell+1})$ , which is impossible. Hence  $Q_{s-1}^n = \bigcap_{\alpha \in \Lambda_{s-1,n}} Q_{s-1}(\alpha)$  for all  $n \geq 1$ .  $\square$

We are now in a position to finish the proof of Theorem (1.1).

*Proof of Theorem (1.1).* (2)  $\Rightarrow$  (1) We may assume that  $s \geq 2$  and our assertion holds true for  $s - 1$ . Then by Lemma (3.2) and Proposition (3.4) the sequence  $a_1, a_2, \dots, a_{s-1}$  is  $A$ -regular and  $Q_{s-1} : a_s = Q_{s-1}$ , that is, the sequence  $a_1, a_2, \dots, a_s$  is  $A$ -regular.

(1)  $\Rightarrow$  (2) This follows directly from Proposition (2.2) and is due to [HRS].  $\square$

The condition in Theorem (1.1) (2) that every  $a_i$  is a non-zero-divisor in  $A$  is not superfluous. Let us explore one example.

**Example (3.6).** Let  $B = k[[X, Y, Z]]$  be the formal power series ring over a field  $k$  and put  $A = B/(XY)$ . Let  $x, y$ , and  $z$  denote respectively the reduction of  $X, Y$ , and  $Z \pmod{(XY)}$ . Let  $a_1 = z, a_2 = x$ , and  $Q = (a_1, a_2)$ . Then the sequence  $a_1, a_2$  is not  $A$ -regular, but the equality  $Q^n = \bigcap_{\alpha \in \Lambda_{2,n}} Q(\alpha)$  holds true for all  $n \geq 1$ .

*Proof.* Since  $a_2 y = 0$ , the sequence  $a_1, a_2$  is not  $A$ -regular. Let  $C = A/[(0) : a_2]$ . Then  $C = k[[X, Z]]$ , since  $(0) : a_2 = (y)$ , so that the sequence  $a_1 = z, a_2 = x$  is  $C$ -regular. Hence  $(a_1, a_2)^n C = \bigcap_{\alpha \in \Lambda_{2,n}} (a_1^{\alpha_1}, a_2^{\alpha_2}) C$  for all  $n \geq 1$  (cf. Proposition (2.2)). Therefore, to see that  $Q^n = \bigcap_{\alpha \in \Lambda_{2,n}} Q(\alpha)$ , it is enough to show  $[(0) : a_2] \cap (a_1^n, a_2) \subseteq Q^n$ . We will check that  $[(0) : a_2] \cap (a_1^n, a_2) \subseteq a_1^n A$ . Let  $f \in [(0) : a_2] \cap (a_1^n, a_2)$  and write  $f = a_1^n g + a_2 h$  ( $g, h \in A$ ). Let  $\bar{\phantom{x}}$  stand for the reduction mod  $(0) : a_2$ . Then  $\overline{a_1^n g} + \overline{a_2 h} = 0$ , so that  $\bar{h} \in a_1^n C$ . Hence  $f = a_1^n g + a_2 h \in a_1^n A$ , because  $h \in a_1^n A + [(0) : a_2]$ .  $\square$

Before closing this note, let us add one more result, which is an immediate consequence of Theorem (1.1).

**Corollary (3.7).** *Let  $A$  be a Noetherian local ring with  $s = \dim A \geq 1$ . Then the following two conditions are equivalent:*

- (1)  $A$  is a Cohen-Macaulay ring.  
 (2)  $\dim A/\mathfrak{p} = s$  for all  $\mathfrak{p} \in \text{Ass } A$  and the ring  $A$  contains a system  $a_1, a_2, \dots, a_s$  of parameters, satisfying the equality

$$Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$$

for all  $n \geq 1$ .

When this is the case, one has the equality  $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$  for all integers  $n \geq 1$  and for every system  $a_1, a_2, \dots, a_s$  of parameters.

In the forthcoming paper [GS] we shall give a characterization of Noetherian local rings  $A$  with  $s = \dim A \geq 2$ , in which the equality  $Q^n = \bigcap_{\alpha \in \Lambda_{s,n}} Q(\alpha)$  holds true for all integers  $n \geq 1$  and for every parameter ideal  $Q = (a_1, a_2, \dots, a_s)$ .

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