

SOME NUMERICAL INVARIANTS OF LOCAL RINGS

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ABSTRACT. Let R be a formal power series ring over a field of characteristic zero and $I \subseteq R$ any ideal. The aim of this work is to introduce some numerical invariants of the local rings R/I by using the theory of algebraic \mathcal{D} -modules. More precisely, we will prove that the multiplicities of the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ and $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$, where $\mathfrak{p} \subseteq R$ is any prime ideal that contains I , are invariants of R/I .

1. INTRODUCTION

Let (R, \mathfrak{m}, k) be a regular local ring of dimension n containing the field k , and let A be a local ring that admits a surjective ring homomorphism $\pi : R \rightarrow A$. Set $I = \text{Ker } \pi$. G. Lyubeznik [9] defines a new set of numerical invariants of A by means of the Bass numbers $\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H_I^{n-i}(R)) := \dim_k \text{Ext}_R^p(k, H_I^{n-i}(R))$. This invariant depends only on A , i and p , but neither on R nor on π . Completion does not change $\lambda_{p,i}(A)$; so one can assume $R = k[[x_1, \dots, x_n]]$, with x_1, \dots, x_n independent variables.

Lyubeznik numbers can be described as the multiplicities of the characteristic cycle of the local cohomology modules $H_{\mathfrak{m}}^p(H_I^{n-i}(R))$. The aim of this work is to prove that the multiplicities of the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$ and $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$, where $\mathfrak{p} \subseteq R$ is any prime ideal that contains I , are also invariants of R/I . Among these invariants we may find the Bass numbers $\mu_p(\mathfrak{p}, H_I^{n-i}(R)) := \dim_{k(\mathfrak{p})} \text{Ext}_{R_{\mathfrak{p}}}^p(k(\mathfrak{p}), H_{I_{R_{\mathfrak{p}}}}^{n-i}(R_{\mathfrak{p}}))$.

2. THE CHARACTERISTIC CYCLE

In the sequel, \mathcal{D} will denote the ring of differential operators corresponding to the formal power series ring $R = k[[x_1, \dots, x_n]]$, where k is a field of characteristic zero and x_1, \dots, x_n are independent variables. For details we refer to [4] and [5]. The ring \mathcal{D} has a natural increasing filtration given by the order such that the corresponding associated graded ring $gr(\mathcal{D})$ is isomorphic to the polynomial ring $R[\xi_1, \dots, \xi_n]$.

Let M be a finitely generated \mathcal{D} -module equipped with a good filtration, i.e., an increasing sequence of finitely generated R -submodules such that the associated graded module $gr(M)$ is a finitely generated $gr(\mathcal{D})$ -module. The characteristic ideal of M is the ideal in $gr(\mathcal{D}) = R[\xi_1, \dots, \xi_n]$ given by $J(M) :=$

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$\text{rad}(\text{Ann}_{gr(\mathcal{D})}(gr(M)))$). One may prove that $J(M)$ is independent of the good filtration on M . The characteristic variety of M is the closed algebraic set given by

$$C(M) := V(J(M)) \subseteq \text{Spec}(gr(\mathcal{D})) = \text{Spec}(R[\xi_1, \dots, \xi_n]).$$

The characteristic variety allows us to describe the support of a finitely generated \mathcal{D} -module as an R -module. Let $\pi : \text{Spec}(R[\xi_1, \dots, \xi_n]) \rightarrow \text{Spec}(R)$ be the map defined by $\pi(x, \xi) = x$. Then $\text{Supp}_R(M) = \pi(C(M))$.

The characteristic cycle of M is defined as

$$CC(M) = \sum m_i V_i$$

where the sum is taken over all the irreducible components $V_i = V(\mathfrak{q}_i)$ of the characteristic variety $C(M)$, where $\mathfrak{q}_i \in \text{Spec}(gr(\mathcal{D}))$ and m_i is the multiplicity of the module $gr(M)_{\mathfrak{q}_i}$. Notice that the contraction of \mathfrak{q}_i to R is a prime ideal; so the variety $\pi(V_i)$ is irreducible.

2.1. Bass numbers and characteristic cycle. Let $\mathfrak{p} \in \text{Spec}(R)$ be a prime ideal. The Bass numbers $\mu_p(\mathfrak{p}, H_I^{n-i}(R))$ of the local cohomology modules $H_I^{n-i}(R)$, where $I \subseteq R$ is any ideal, can be described as the multiplicities of the characteristic cycle of $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$. Namely, we have:

Proposition 2.1. *Let $I \subseteq R$ be an ideal, $\mathfrak{p} \subseteq R$ a prime ideal, and let*

$$CC(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} V_{\alpha}$$

be the characteristic cycle of the local cohomology module $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$. Then, the Bass numbers with respect to \mathfrak{p} of $H_I^{n-i}(R)$ are

$$\mu_p(\mathfrak{p}, H_I^{n-i}(R)) = \lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}},$$

where $\pi(V_{\alpha_{\mathfrak{p}}})$ is the subvariety of $X = \text{Spec}(R)$ defined by \mathfrak{p} .

Proof. Let $\widehat{R}_{\mathfrak{p}}$ be the completion with respect to the maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ of the localization $R_{\mathfrak{p}}$. Notice that $\widehat{R}_{\mathfrak{p}}$ is a formal power series ring of dimension $\text{ht } \mathfrak{p}$. Since Bass numbers are invariant by completion we have

$$\mu_p(\mathfrak{p}, H_I^{n-i}(R)) = \mu_p(\mathfrak{p}\widehat{R}_{\mathfrak{p}}, H_{I\widehat{R}_{\mathfrak{p}}}^{n-i}(\widehat{R}_{\mathfrak{p}})) = \mu_0(\mathfrak{p}\widehat{R}_{\mathfrak{p}}, H_{\mathfrak{p}\widehat{R}_{\mathfrak{p}}}^p(H_{I\widehat{R}_{\mathfrak{p}}}^{n-i}(\widehat{R}_{\mathfrak{p}}))),$$

where the last assertion follows from [9, Lemma 1.4]. By using [9, Theorem 3.4], we have

$$H_{\mathfrak{p}\widehat{R}_{\mathfrak{p}}}^p(H_{I\widehat{R}_{\mathfrak{p}}}^{n-i}(\widehat{R}_{\mathfrak{p}})) = E_{\widehat{R}_{\mathfrak{p}}}(\widehat{R}_{\mathfrak{p}}/\mathfrak{p}\widehat{R}_{\mathfrak{p}})^{\mu_0(\mathfrak{p}\widehat{R}_{\mathfrak{p}}, H_{\mathfrak{p}\widehat{R}_{\mathfrak{p}}}^p(H_{I\widehat{R}_{\mathfrak{p}}}^{n-i}(\widehat{R}_{\mathfrak{p}})))}.$$

So, its characteristic cycle is

$$CC(H_{\mathfrak{p}\widehat{R}_{\mathfrak{p}}}^p(H_{I\widehat{R}_{\mathfrak{p}}}^{n-i}(\widehat{R}_{\mathfrak{p}}))) = \mu_p(\mathfrak{p}, H_I^{n-i}(R)) V'_{\alpha_{\mathfrak{p}}},$$

where $\pi(V'_{\alpha_{\mathfrak{p}}})$ is the subvariety of $X' = \text{Spec } \widehat{R}_{\mathfrak{p}}$ defined by the ideal $\mathfrak{p}\widehat{R}_{\mathfrak{p}}$. Notice that we have used the following fact (see [9] and [1] for details):

$$CC(H_{\mathfrak{p}\widehat{R}_{\mathfrak{p}}}^{\text{ht } \mathfrak{p}}(\widehat{R}_{\mathfrak{p}})) = CC(E_{\widehat{R}_{\mathfrak{p}}}(\widehat{R}_{\mathfrak{p}}/\mathfrak{p}\widehat{R}_{\mathfrak{p}})) = V'_{\alpha_{\mathfrak{p}}}.$$

Finally, by using the flatness of the morphism $R \rightarrow \widehat{R}_{\mathfrak{p}}$, this characteristic cycle can be obtained from the characteristic cycle of $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$. Namely, if

$$CC(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} V_{\alpha}$$

is the characteristic cycle of the module $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$, then we have

$$CC(H_{\widehat{\mathfrak{p}R_{\mathfrak{p}}}}^p(H_{\widehat{IR_{\mathfrak{p}}}}^{n-i}(\widehat{R_{\mathfrak{p}}})) = \lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}} V'_{\alpha_{\mathfrak{p}}}.$$

□

2.2. Direct image. Some geometrical operations as the direct image have a key role in the theory of \mathcal{D} -modules. Our aim in this section is to give a brief survey of this operation in the particular case of the injection of \mathbb{A}_k^n in \mathbb{A}_k^{n+1} . The main references we will use in this section are [5] and [10].

Let \mathcal{D}_{n+1} and \mathcal{D}_n be the rings of differential operators corresponding to $R' = k[[x_1, \dots, x_n, t]]$ and $R = k[[x_1, \dots, x_n]]$, respectively. Let M be a \mathcal{D}_n -module. The direct image corresponding to the injection is the \mathcal{D}_{n+1} -module $i_+(M)$ defined as

$$i_+(M) = k[\partial_t] \widehat{\otimes}_k M = M[\partial_t].$$

The characteristic variety of $i_+(M)$ can be computed from the characteristic variety of M . Namely, we have

$$C(i_+(M)) = \{(\mathbf{x}, 0, \xi, \tau) \mid (\mathbf{x}, \xi) \in C(M)\} \subseteq \text{Spec}(R'[\xi_1, \dots, \xi_n, \tau]),$$

where we have considered $C(M) \subseteq \text{Spec}(R[\xi_1, \dots, \xi_n])$.

The direct image of local cohomology modules can be easily described. The following result is stated as is used in our work.

Lemma 2.2. *Let $\mathfrak{p} \subseteq R$ be a prime ideal that contains an ideal $I \subseteq R$. The direct image of the local cohomology module $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$ is*

$$i_+(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) = H_{(t)}^1(H_{\mathfrak{p}R'}^p(H_{IR'}^{n-i}(R'))).$$

Proof. Let \mathcal{D}_t be the ring of differential operators corresponding to the formal power series ring $k[[t]]$. For simplicity we will denote the local cohomology modules $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$ and $H_{\mathfrak{p}R'}^p(H_{IR'}^{n-i}(R'))$ by N and N' , respectively. Then we have

$$H_{(t)}^1(N') = H_{(t)}^1(N \widehat{\otimes}_k k[[t]]) = H_{(t)}^1(k[[t]]) \widehat{\otimes}_k N = (\mathcal{D}_t/\mathcal{D}_t \cdot (t)) \widehat{\otimes}_k N = i_+(N).$$

□

Remark 2.3. In general, let I_1, \dots, I_s be a set of ideals of R . Then, the direct image of the local cohomology module $H_{I_1}^{i_1}(\dots(H_{I_s}^{i_s}(R))\dots)$ is

$$i_+(H_{I_1}^{i_1}(\dots(H_{I_s}^{i_s}(R))\dots)) = H_{(t)}^1(H_{I_1R'}^{i_1}(\dots(H_{I_sR'}^{i_s}(R'))\dots)).$$

3. MULTIPLICITIES OF THE CHARACTERISTIC CYCLE

Let A be a ring that admits a presentation $A \cong R/I$ for a given ideal $I \subseteq R = k[[x_1, \dots, x_n]]$. Recall that we have $\text{Spec}(A) = \{\mathfrak{p} \in \text{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. Throughout this section, a prime ideal of A will also mean the corresponding prime ideal of R that contains I .

Let R/I and R'/I' be two different presentations of the local ring A . Then, for any prime ideal of A , we will denote $\mathfrak{p}' \in \text{Spec}(R')$ to be the prime ideal that corresponds to $\mathfrak{p} \in \text{Spec}(R)$ by the isomorphism $\text{Spec}(R/I) \cong \text{Spec}(R'/I')$.

Theorem 3.1. *Let A be a local ring that admits a surjective ring homomorphism $\pi : R \rightarrow A$, where $R = k[[x_1, \dots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$, let $\mathfrak{p} \subseteq A$ be a prime ideal and let*

$$CC(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} V_{\alpha}$$

be the characteristic cycle of the local cohomology modules $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$. Then the multiplicities $\lambda_{\mathfrak{p},p,i,\alpha}$ depend only on A, \mathfrak{p}, p, i and α but not on R or π .

The proof of the theorem is inspired in the proof of [9, Theorem 4.1], but here we must be careful with the behavior of the characteristic cycle. So instead of [9, Lemma 4.3] we will use the following:

Lemma 3.2. *Let $g : R' \rightarrow R$ be a surjective ring homomorphism, where R' is a formal power series ring of dimension n' . Set $I' = \ker \pi g$ and let*

$$CC(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} V_{\alpha}$$

be the characteristic cycle of the local cohomology modules $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$. Then, the characteristic cycle of $H_{\mathfrak{p}'}^p(H_{I'}^{n'-i}(R'))$ is

$$CC(H_{\mathfrak{p}'}^p(H_{I'}^{n'-i}(R'))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} V'_{\alpha},$$

where $\pi(V'_{\alpha})$ is the subvariety of $X' = \text{Spec } R'$ defined by the defining ideal of $\pi(V_{\alpha})$ contracted to R' .

Proof. R is regular, and so $\text{Ker } g$ is generated by $n' - n$ elements that form part of a minimal system of generators of the maximal ideal $\mathfrak{m}' \subseteq R'$. By induction on $n' - n$ we are reduced to the case $n' - n = 1$. So $\text{Ker } g$ is generated by one element $f \in \mathfrak{m}' \setminus \mathfrak{m}'^2$. By Cohen's structure theorem, $R' = k[[x_1, \dots, x_n, t]]$ where we assume $f = t$ by a change of variables. We identify R with the subring $k[[x_1, \dots, x_n]]$ of R' . In particular, we have to consider $I' = IR' + (t)$ and $\mathfrak{p}' = \mathfrak{p}R' + (t)$.

By using Lemma 2.2 and the degeneration of Grothendieck's spectral sequence $E_2^{p,q} = H_{(t)}^p(H_{J+(t)}^q(M)) \implies H_{J+(t)}^{p+q}(M)$ we have

$$\begin{aligned} i_+(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) &= H_{(t)}^1(H_{\mathfrak{p}R'}^p(H_{IR'}^{n-i}(R'))) = H_{\mathfrak{p}R'}^p(H_{(t)}^1(H_{IR'}^{n-i}(R'))) \\ &= H_{\mathfrak{p}R'}^p(H_{IR'+(t)}^{n+1-i}(R')) = H_{\mathfrak{p}R'}^p(H_{I'}^{n'-i}(R')) \\ &= H_{\mathfrak{p}R'+(t)}^p(H_{I'}^{n'-i}(R')) = H_{\mathfrak{p}'}^p(H_{I'}^{n'-i}(R')), \end{aligned}$$

where the second to last assertion comes from the fact that $H_{I'}^{n'-i}(R')$ is a (t) -torsion module. Then we are done by the results in Section 2.2. \square

Now we continue the proof of Theorem 3.1.

Proof. Let $\pi' : R' \rightarrow A$ and $\pi'' : R'' \rightarrow A$ be surjections with $R' = k[[y_1, \dots, y_{n'}]]$ and $R'' = k[[z_1, \dots, z_{n''}]]$. Let $I' = \ker \pi'$ and let $I'' = \ker \pi''$. Let $R''' = R' \widehat{\otimes}_k R''$ be the external tensor product, $\pi''' = \pi' \widehat{\otimes}_k \pi'' : R' \widehat{\otimes}_k R'' \rightarrow A$ and $I''' = \ker \pi'''$.

By Lemma 3.2, if the characteristic cycle of $H_{\mathfrak{p}'}^p(H_{I'}^{n'-i}(R'))$ is

$$CC(H_{\mathfrak{p}'}^p(H_{I'}^{n'-i}(R'))) = \sum \lambda'_{\mathfrak{p},p,i,\alpha} V'_{\alpha},$$

then the characteristic cycle of $H_{\mathfrak{p}'''}^p(H_{I'''}^{n'+n''-i}(R'''))$ is

$$CC(H_{\mathfrak{p}'''}^p(H_{I'''}^{n'+n''-i}(R'''))) = \sum \lambda'_{\mathfrak{p},p,i,\alpha} V'''_{\alpha},$$

where $\pi(V'''_{\alpha})$ is the subvariety of $X''' = \text{Spec } R'''$ defined by the defining ideal of $\pi(V'_{\alpha})$ contracted to R''' .

By Lemma 3.2, if the characteristic cycle of $H_{\mathfrak{p}''}^p(H_{I''}^{n''-i}(R''))$ is

$$CC(H_{\mathfrak{p}''}^p(H_{I''}^{n''-i}(R''))) = \sum \lambda''_{\mathfrak{p},p,i,\alpha} V''_{\alpha},$$

then the characteristic cycle of $H_{\mathfrak{p}'''}^p(H_{I'''}^{n'+n''-i}(R'''))$ is

$$CC(H_{\mathfrak{p}'''}^p(H_{I'''}^{n'+n''-i}(R'''))) = \sum \lambda''_{\mathfrak{p},p,i,\alpha} V''_{\alpha},$$

where $\pi(V''_{\alpha})$ is the subvariety of $X''' = \text{Spec } R'''$ defined by the defining ideal of $\pi(V''_{\alpha})$ contracted to R''' .

In particular, we have $\lambda'_{\mathfrak{p},p,i,\alpha} = \lambda''_{\mathfrak{p},p,i,\alpha}$ for all \mathfrak{p}, p, i and α . □

Remark 3.3. By the same arguments one may prove that the multiplicities of the characteristic cycle of the local cohomology modules $H_{I_1}^{i_1}(\dots(H_{I_s}^{i_s}(R))\dots)$, where I_1, \dots, I_s is a set of ideals of R containing the ideal $I = I_s$, are also invariants of R/I .

Since Bass numbers $\mu_p(\mathfrak{p}, H_I^{n-i}(R))$ are multiplicities of the characteristic cycle of $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$, we recover Lyubeznik's result:

Corollary 3.4. *Let A be a ring that admits a surjective ring homomorphism $\pi : R \rightarrow A$, where $R = k[[x_1, \dots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$ and let $\mathfrak{p} \subseteq A$ be a prime ideal. The Bass numbers $\mu_p(\mathfrak{p}, H_I^{n-i}(R))$ depend only on A, \mathfrak{p}, p and i but not on R or π .*

When \mathfrak{p} is the zero ideal, we obtain the invariance with respect to R/I of the multiplicities of the characteristic cycle of $H_I^{n-i}(R)$.

Corollary 3.5. *Let A be a local ring that admits a surjective ring homomorphism $\pi : R \rightarrow A$, where $R = k[[x_1, \dots, x_n]]$ is the formal power series ring. Set $I = \ker \pi$ and let*

$$CC(H_I^{n-i}(R)) = \sum m_{i,\alpha} V_{\alpha}$$

be the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$. Then the multiplicities $m_{i,\alpha}$ depend only on A, i and α but not on R or π .

Collecting these multiplicities by the dimension of the corresponding irreducible varieties we define the following invariants:

Definition 3.6. Let $I \subseteq R$ be an ideal. If $CC(H_I^{n-i}(R)) = \sum m_{i,\alpha} V_{\alpha}$ is the characteristic cycle of the local cohomology modules $H_I^{n-i}(R)$, then we define

$$\gamma_{p,i}(R/I) := \left\{ \sum m_{i,\alpha} \mid \dim(\pi(V_{\alpha})) = p \right\}.$$

One may prove that these invariants have the same properties as Lyubeznik numbers (see [9, Section 4]). Namely, let $d = \dim(R/I)$. Then $\gamma_{p,i}(R/I) = 0$ if $i > d$, $\gamma_{p,i}(R/I) = 0$ if $p > i$ and $\gamma_{d,d}(R/I) \neq 0$. In particular, we can collect them in a triangular matrix that we will denote by $\Gamma(R/I)$. We point out that these invariants are finer than the Lyubeznik numbers.

Example 3.7. Let $R = k[[x_1, x_2, x_3, x_4, x_5]]$. Consider the ideals

- $I_1 = (x_1, x_2, x_5) \cap (x_3, x_4, x_5)$;
- $I_2 = (x_1, x_2, x_5) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4)$.

The characteristic cycle of the corresponding local cohomology modules can be computed by means of [1, Theorem 3.8]. Collecting the multiplicities we obtain the triangular matrices

$$\Gamma(R/I_1) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}, \quad \Gamma(R/I_2) = \begin{pmatrix} 0 & 2 & 0 \\ & 1 & 0 \\ & & 2 \end{pmatrix}.$$

Computing the Lyubeznik numbers (see [1, Theorem 4.4]), we obtain the triangular matrix

$$\Lambda(R/I_1) = \Lambda(R/I_2) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}.$$

We have to point out that the quotient ring R/I_1 is Buchsbaum, but R/I_2 is not.

Remark 3.8. In order to compute the Lyubeznik numbers $\lambda_{p,i}(R/I)$ for a given ideal $I \subseteq R$ and arbitrary i, p we have to refer to U. Walther's algorithm [11]. When I is a squarefree monomial ideal, a description of these invariants is given in [1] and [13]. Some other particular computations may also be found in [6], [7], [8] and [12]. The multiplicities of the characteristic cycle of $H_{\mathfrak{p}}^p(H_I^{n-i}(R))$, where I is a squarefree monomial ideal and \mathfrak{p} is any homogeneous prime ideal, have been computed in [2].

When I is a squarefree monomial ideal (resp. the defining ideal of an arrangement of linear varieties), the multiplicities of the characteristic cycle of $H_I^{n-i}(R)$ have been computed in [1] (resp. [3]).

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