MORAVA K-THEORY OF EXTRASPECIAL 2-GROUPS

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Abstract. We compute the Morava K-theory of some extraspecial 2-groups and associated compact groups.

1. Introduction

Let $G$ be a finite group and $BG$ denote its classifying space. Not that many computations for the Morava K-theory of $BG$ have been carried out, the most notable exception being Kriz’s article [5] and its successor [6], where he calculates just enough about the 3-primary second Morava K-theory of the 3-Sylow subgroup of $GL_4(F_3)$ to conclude that it cannot be concentrated in even degrees, the first such example known. Other computations can be found in [1], [3], [4], [8], [9], [10], and [11].

In this paper we present a few more calculations concerning extraspecial 2-groups. We mainly work with integral Morava K-theory for the prime 2, which shall be denoted $\tilde{K}(n)$. This is a complex oriented cohomology theory with coefficients $\tilde{K}(n)^* \cong WF_{2^n}[v_n,v_n^{-1}]$, the ring of Laurent polynomials over the Witt ring $WF_{2^n}$, with $v_n$ of degree $-2(2^n - 1)$. It has a complex orientation $x$ such that the 2-series of the associated formal group law takes the form $[2](x) = 2x - v_n x^{2^n}$. Sometimes we switch to the mod 2 reduction $K(n)$.

In Section 2 we describe the groups we want to study and recall Quillen’s computation of their mod 2 cohomology. As a corollary we consider a slight modification serving as motivation for our calculational approach. Section 3 contains the main technical result, Lemma 3.1, which under favourable circumstances computes the spectral sequence of an extension of $\mathbb{Z}/2 \times \mathbb{Z}/2$ by a “good” group in the sense of Hopkins-Kuhn-Ravenel, i.e., whose Morava K-theory is generated by transfers of Euler classes. The next two sections contain applications to extraspecials of order 8 and 32. Section 4 is a rehash of the already known computations for $D_8$ and $Q_8$ and serves mainly to set up notation for the next section, where we deal with the central products $D_8 \circ D_8$ and $D_8 \circ Q_8$. We need some of the multiplicative structure for $D_8$, and make repeated use of generalized characters à la Hopkins-Kuhn-Ravenel [3]. We also consider the associated compact groups that arise by replacing the centre $\mathbb{Z}/2$ by the circle group $S^1$. The last section contains calculations of the Euler
characteristics of extraspecial groups (for any prime), due also to Brunetti [2]. We omit proofs, since they are now available in [2].

2. Extraspecial 2-groups

There are three types of (almost) extraspecial 2-groups, the so-called real, complex and quaternion types. These may be described as central products. Let $D_8$ and $Q_8$ denote the dihedral, respectively quaternion, group of order 8. The extraspecials of real type have order $2^{2m+1}$ for some $m > 0$ and correspond to $m$-fold central products of $D_8$ (for the quaternion type replace one of copy $D_8$ with a $Q_8$) whereas the complex type is obtained as the central product of a real extraspecial with a cyclic group of order four.

In this section we try to motivate our subsequent computations, and thus concentrate on the real case only. So let $D(m) := D_8 \circ \cdots \circ D_8$ ($m$ copies); in Hall-Senior notation this group is known as $2^{1+2m}$. Its mod 2 cohomology was computed by Quillen [7]: one has a central extension

$$1 \to \mathbb{Z}/2 \to D(m) \to E \to 1$$

where $E \cong (\mathbb{Z}/2)^{3m}$ is a $2m$-dimensional vector space over $\mathbb{F}_2$. The Serre spectral sequence associated to this extension takes the form

$$E_2 = H^*(BE; H^*(B\mathbb{Z}/2)) \cong F_2[u] \otimes F_2[x_1, \ldots, x_{2m}]$$

with $u$ and $x_i$ in degree one; the extension class is $q := x_1x_2 + \cdots + x_{2m-1}x_{2m}$. Quillen’s computation can be summarised as follows:

**Theorem 2.1** (Quillen [7]). The only differentials in the spectral sequence (2.2) are $d_2u = q, d_{2k+1}u^{2^k} = Q_{k-1}q$ for $1 \leq k < m$, where $Q_i$ stands for Milnor’s primitive operation in the Steenrod algebra. The sequence $(q, Q_0q, \ldots, Q_{m-2}q)$ is regular, and $u^{2^m}$ is a permanent cycle since it represents the Euler class $w_{2m}$ of the spin representation $\Delta$. Thus

$$H^*(D(m); \mathbb{F}_2) \cong F_2[w_{2m}] \otimes F_2[x_1, \ldots, x_{2m}]/(q, Q_0q, \ldots, Q_{m-2}q).$$

The nontrivial Stiefel-Whitney classes of $\Delta$ are $w_{2i}$ and $w_{2m-2i}$, $0 \leq i \leq m$. □

Knowing the result, one can slightly rearrange the computation. $D(m+1)$ contains $D(m)$ as a normal subgroup with quotient $\mathbb{Z}/2 \times \mathbb{Z}/2$, i.e., one has an extension

$$1 \to D(m) \to D(m+1) \to V \to 1$$

with $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ acting trivially on the kernel. The Serre spectral sequence corresponding to (2.3) has $E_2$-term given by

$$E_2 = H^*(BV; H^*(BD(m)) \cong F_2[x_{2m+1}, x_{2m+2}] \otimes H^*(BD(m)).$$

**Corollary 2.2.** The spectral sequence (2.4) collapses on the $E_3$-page. The only nontrivial differential is $d_2w_{2m} = x_{2m+1}x_{2m+2} \otimes w_{2m-1}$.

**Proof.** Since the cohomology of extraspecial 2-groups of real type is detected on maximal elementary abelian subgroups, the action of $d_2$ can be worked out by looking at the restrictions to those subgroups. Each maximal elementary abelian $W$ is of the form $C \times U$ where $C$ is the centre and $U$ a maximal isotropic subspace.
of the central quotient $E$. (Recall from [7] that $q$ may be regarded as a quadratic form on $E$.) The corresponding extension is of the form

$$1 \rightarrow C \times U \rightarrow D_8 \times U \rightarrow V \rightarrow 1,$$

and the only differential is $d_2u = x_{2m+1}x_{2m+2}$. Quillen tells us that $\Delta$ restricts to $W$ as $\chi \otimes \text{reg}(U)$, where $\chi$ is the nontrivial character of $C$ and $\text{reg}(U)$ the regular representation of $U$. Applying the formula expressing $w.(\chi \otimes \text{reg}(U))$ in terms of $w.(\chi)$ and $w.(\text{reg}(U))$ we obtain

$$w_i(\chi \otimes \text{reg}(U)) = \sum_{j=0}^{i} \binom{2m-i+j}{j} w_1(\chi)^j w_{i-j}(\text{reg}(U)).$$

So $w_{2m}$ restricts to $\sum_{k=0}^{m} u^2 w_{2m-2k}(\text{reg}(U))$, since other Stiefel-Whitney classes of $\text{reg}(U)$ are zero, and $w_{2m-1}$ restricts to $w_{2m-1}(\text{reg}(U))$. Thus $d_2$ is as claimed; the rest follows from a Poincaré series calculation.

Note that $w_{2m}^2$ represents the Euler class of the spin representation of $D(m+1)$. Furthermore, there are extension problems in the $E_\infty$-term. Let $q_m = x_1x_2 + \cdots + x_{2m-1}x_{2m}$ denote the extension class of $D(m)$. Then $q_m$ drops in filtration to $x_{2m+1}x_{2m+2}$ (so we get the relation $q_{m+1} = 0$), and the other relations follow as solutions to extension problems related to $q_m q_m = 0$ and $x_{2m+1}x_{2m+2}w_{2m-1} = 0$.

The (additive) simplicity of the spectral sequence of this extension is what lets us believe it to be possible to emulate this computation in Morava $K$-theory. In the subsequent sections we shall try to prove that the Atiyah-Hirzebruch-Serre spectral sequence of $[2,3]$ behaves analogously, meaning it has only two differentials (the second being $v_n \otimes Q_n$, see below).

### 3. Spectral sequence calculations

In this section we consider the Atiyah-Hirzebruch-Serre spectral sequence associated to extensions

$$1 \rightarrow G' \rightarrow G \rightarrow V \rightarrow 0$$

with $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, acting trivially on $G'$. The spectral sequence has $E_2$-term given by

$$(3.1) \quad E_2^{s,t} = H^s(\mathbb{Z}/2 \otimes \mathbb{Z}/2; \tilde{K}(n)^*(BG')) \Rightarrow \tilde{K}(n)^*(BG').$$

**Lemma 3.1.** Let $G$ be as above. Suppose $K(n)^{\text{odd}}(BG') = 0$ for all $n \geq 1$, and moreover that all elements in $E_4^{0,*}$ are permanent cycles. Then $\tilde{K}(n)^{\text{odd}}(BG) = 0$ and $\tilde{K}(n)^*(BG)$ has no $p$-torsion, and $K(n)^{\text{odd}}(BG) = 0$.

**Proof.** $K(n)^{\text{odd}}(BG') = 0$ implies $\tilde{K}(n)^{\text{odd}}(BG') = 0$ and $\tilde{K}(n)^*(BG')$ is $p$-torsion free. One has $H^*(BV; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2]$; setting $y_i = x_i^2$ and $\alpha = x_1^2x_2 + x_1x_2^2$, the $E_2$-page of the spectral sequence is

$$E_2^{s,t} \cong \begin{cases} \tilde{K}(n)^*(BG') & \text{for } * = 0, \\ \tilde{K}(n)^*(BG') \otimes \mathbb{F}_2[y_1, y_2, \alpha]/(\alpha^2 = y_1^2y_2 + y_1y_2^2) & \text{for } * > 0. \end{cases}$$

We shall write $\pi$ for the element $y_1^2y_2 + y_1y_2^2$. The first potentially nontrivial differential is $d_4$. Any even (respectively odd) degree element in $E_2^{s,t}$ is of the form $x \otimes f (x \otimes f(x))$ for some $x \in \tilde{K}(n)^*(BG')$ and $f \in \mathbb{F}_2[y_1, y_2]$. We shall first consider the case $n \geq 2$, the argument for $n = 1$ being similar (see the remark at
the end). Note that $d_3$ is zero on any element of $\mathbb{F}_3[y_1, y_2, \alpha]/(\alpha^2 = y_1^2 y_2 + y_1 y_2^2)$ by comparison to the Atiyah-Hirzebruch spectral sequence for $V$ and $n \geq 2$. Hence $d_3(x \otimes f) = x' \otimes f \alpha$ and $d_3(x \otimes f \pi) = x' \otimes f \pi$ for some $x' \in K(n)^*(BG')$. Thus we obtain additive isomorphisms

$$
\begin{align*}
E_4^{0, \ast} & \cong \bar{K}, \\
E_4^{0, \ast} & \cong K \otimes \mathbb{F}_2[y_1, y_2]/(\pi) \oplus H \otimes \mathbb{F}_2[y_1, y_2]/\{\alpha, \pi\}
\end{align*}
$$

where $\bar{K} = \text{Ker}(d_3|_{\bar{K}(n)^*(BG')})$, $K = \text{Ker}(d_3|_{K(n)^*(BG')}) = \bar{K}/(\bar{K} \cap 2E_2^{0, \ast})$, and $H = \text{H}(K(n)^*(BG'); d_3 \otimes \alpha^{-1})$. As a $\bar{K}(n)^*$-algebra, the $E_4$-page is generated by $\alpha$, $y_1$, and the generators in $\bar{K}$. By hypothesis, all but $\alpha$ are permanent cycles. So the next nonzero differential is

$$
d_{2n+1}^e(\alpha) = v_n \otimes Q_n \alpha = v_n \otimes (y_1^{2n} y_2 + y_1 y_2^{2n}) = v_n \otimes q \pi
$$

where $q = (y_1^{2n} y_2 + y_1 y_2^{2n})/\pi = (y_1^{2n-2} + y_1^{2n-3} y_2 + \cdots + y_2^{2n-2})$. Thus we get

$$
\begin{align*}
E_2^{0, \ast} & \cong \bar{K}, \\
E_{2n+1}^{0, \ast} & \cong K \otimes \mathbb{F}_2[y_1, y_2]/(\pi) \oplus H \otimes \mathbb{F}_2[y_1, y_2]/\{y\}.
\end{align*}
$$

This is concentrated in even degrees, whence $E_{2n+1}^{0, \ast} \cong E_\infty$ and $\bar{K}(n)^{\text{odd}}(BG) = 0$. It remains to prove that $\bar{K}(n)^*(BG)$ has no 2-torsion. Let $0 \neq x \in \bar{K}(n)^*(BG)$. Represent $x$ by $x' \in E_\infty$. If $x' \in E_\infty^{0, \ast}$ then it cannot be 2-torsion, since $\bar{K}(n)^*(BG)$ is 2-torsion free. If $x'$ is in $K \otimes \mathbb{F}_2[y_1, y_2]/(\pi)$, we may write $x' = \bar{x} \otimes f$ with $\bar{x} \in \bar{K}$, $f \in \mathbb{F}_2[y_1, y_2]/(\pi)$. Rewrite $f$ as $y_1 f_1 + \lambda y_2$, $\lambda \in \mathbb{F}_2$. Since $2y_1 = v_n y_1^{2n}$ in $\bar{K}(n)^*(BG)$ (this is immediate from the calculation for cyclic groups), $2x$ can be represented by

$$(2x)' = \sum v_n \bar{x} \otimes (y_1^{2n} f_1 + \lambda y_2^{2n+s-1}).$$

We claim that the right-hand side of this expression is nonzero: if $\lambda \neq 0$, it does not lie in the ideal $(y_1 y_2) \subseteq (\pi)$, and if $\lambda = 0$, then $y_1^{2n} f \in (\pi)$ implies $y_1 f \in (\pi)$. Lastly suppose $x' \in H \otimes \mathbb{F}_2[y_1, y_2]/(y_1^{2n} y_2 + \cdots + y_2^{2n-2}) \{\pi\} \subset H \otimes \mathbb{F}_2[y_1, y_2]/(Q_n \alpha)$. Write $x' = \sum \bar{x} \otimes f \pi$ and $f \pi = y_1 f_1$. Then $(2x)' \neq 0$ if $v_n \otimes y_1 f_1 \neq 0$. But $y_1 f_1 \in (Q_n \alpha)$ implies $f_1 y_1 \in (Q_n \alpha)$: tensoring up with the finite field of $2^n$ elements $\mathbb{F}_{2^n}$ yields

$$Q_n \alpha = y_1^{2n} y_2 + y_1 y_2^{2n} = \prod_{\mu \in \mathbb{F}_{2^n}} (y_1 + \mu y_2).$$

Finally, for $n = 1$ the differential $d_3$ is given by $v_1 \pi$; the claim follows by filtering $E_2^{0, \ast}$ by powers of $\pi$ and setting $q = 1$.

Since $\bar{K}$ is 2-torsion free and the map defined by

$$a y_1 \mapsto a y_1^{s+2n-1} \text{ and } y_2 \mapsto y_2^{s+2n-1}$$

on $E_2^{0, \ast}$ is injective, one easily sees

**Corollary 3.2.** Suppose $G$ is as in Lemma 3.1. Then there is an additive isomorphism

$$
\begin{align*}
K(n)^*(BG) & \cong E_\infty^{0, \ast}/2 \oplus E_\infty^{0, \ast}/(y_1^{2n}, y_2^{2n}) \\
& \cong \bar{K}/2 \oplus K \otimes \mathbb{Z}/2[y_1, y_2]/(y_1^{2n}, y_2^{2n}, \pi) \oplus H \otimes \mathbb{Z}/2[y_1, y_2]/(y_1^{2n-1}, y_2^{2n-1}, q)\{\pi\}.
\end{align*}
$$
4. The cases $D_8$ and $Q_8$

The groups $D_8$ and $Q_8$ have presentations
\[
D_8 = \langle a_1, a \mid a_1^2 = a^4 = 1, [a_1, a] = a^2 \rangle, \\
Q_8 = \langle a_1, a_2 \mid a_1^2 = a_2^4 = 1, [a_1, a_2] = a_1^2 = a_2^2 \rangle,
\]
respectively. Thus there are central extensions of the form $\mathbb{Z}/2 \to G \to V$ for $G$ either $D_8$ or $Q_8$, i.e., we have $G' = \mathbb{Z}/2$ in the setup of Section 3. Setting $a_2 = aa_1$ in the case of $D_8$, the quotient $V$ is generated by the cosets $\tilde{a}_i$ for each group; let $x_i \in H^*(BV; \mathbb{F}_2)$ be dual to $\tilde{a}_i$. Recall that $\tilde{K}(n)^*(BG) \cong \tilde{K}(n)^*[u]/(2u - v_nu^{2n})$ where $u$ is the Euler class of the nontrivial linear character $\eta$ of $\mathbb{Z}/2$. In the spectral sequence $\mathbb{E}_2^{p,q}$, we get $d_3u = \alpha$. Hence $H = \text{Ker}(d_3)/\text{Im}(d_3 \otimes \alpha^{-1}) = 0$, and $u^2$ is a permanent cycle, since it is the restriction of the Euler class of the irreducible two-dimensional complex representation $\rho$ of $G$ to the fibre. Thus
\[
E_\infty^{0,*} \cong \tilde{K}(n)^*[u]/((2u - v_nu^{2n}) \cap \tilde{K}(n)^*[u^2]) \cong \tilde{K}(n)^*[u^2][1,2u]
\]
\[
E_\infty^{0,*} \cong \tilde{K}(n)^*[u^2]/(v_nu^{2n}) \otimes \mathbb{F}_2[y_1,y_2]/(\pi).
\]
It follows that $\tilde{K}(n)^*(BG)$ is concentrated in even degrees and has no 2-torsion, whence $K(n)^*(BG) \cong \tilde{K}(n)^*(BG)/(2)$. Choosing an element $\tilde{c}_2 \in \tilde{K}(n)^*(BG)$ represented by $u^2$, one obtains

**Theorem 4.1** ([9], [8]). Let $G$ be either $D_8$ or $Q_8$. Then there is an additive isomorphism
\[
K(n)^*(BG) \cong (K(n)^*[\tilde{c}_1] \oplus K(n)^*[y_1,y_2]/(\pi,y_1^{2n},y_2^{2n})[\tilde{c}_2]/(\tilde{c}_2^{n-1})).
\]

The multiplicative structure is given by
\[
\tilde{c}_1y_1 = y_2^2, \quad \tilde{c}_1y_2 = y_1^2, \quad \tilde{c}_1^2 = y_1^2 + y_1y_2 + y_2^2
\]
identifying $\tilde{c}_1 = v_n\tilde{c}_2^{n-1} + y_1 + y_2$ for $D_8$ and $\tilde{c}_1 = v_n\tilde{c}_2^{n-1}$ for $Q_8$.

The generators $y_i$ can be identified with the Euler classes of the representations $\rho_i: G \to V \to \langle \tilde{a}_i \rangle \to \mathbb{C}^*$. Switching from $\tilde{c}_2$ to $c_2 = c_1(\rho)$, we may write $c_2 = \tilde{c}_2 \mod (y_1,y_2)^2$. Then $v_n\tilde{c}_2^{n-1} - v_n\tilde{c}_2^{n-1} \mod (y_1,y_2)^2$. We also have $c_1 = \tilde{c}_1 \mod (y_1,y_2)^2$, by considering restrictions to maximal abelian subgroups; see below. Hence relation (4.1) in the theorem holds modulo $(y_1,y_2)^3$ with $\tilde{c}_1$ replaced by $c_1$.

We want to compute the restrictions of $c_2$ to the maximal subgroups of $G$. Consider $G = D_8$ first. Let $C = \langle a^2 \rangle$ be the centre of $D_8$, and $A_i = \langle a_i \rangle$. The maximal subgroups are $A = \langle a \rangle \cong \mathbb{Z}/4$ and $C \times A_i \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Let $\rho_A: A \to \mathbb{C}^*$ be a faithful representation of $A$. Then $c_1(\rho_A)$ restricts to the generator $u$ of the centre, and identifying classes with their images under restriction, we may write
\[
K(n)^*(BA) \cong K(n)^*[u]/[4](u) \cong K(n)^*[u^4] ; \\
K(n)^*(BC \times A_i) \cong K(n)^*[u,y_i]/([2](u),[2](y_i)) \cong K(n)^*[u,y_i]/(u^{2n},y_i^{2n}).
\]
We have $\text{Res}_A(\rho) = \rho_A \otimes \rho_A$, and since $\rho = \text{Ind}_A^G(\rho_A)$, the double coset formula gives $\text{Res}_A(\rho) = \rho_A \otimes \rho_A^3$. The restrictions of the total Chern class are $\text{Res}_A(c(\rho)) = (1 + u)(1 + [-1]u)$ and $\text{Res}_{C \times A_i}(c(\rho)) = (1 + u)(1 + u + K(\eta) y_i)$. Thus we obtain the following restrictions:
\[
\text{Res}_A(c_2) = ([{-1}(u)]u)u = u^2 + v_nu^{2n+1} \mod (u^{2n+1});
\]
\[
\text{Res}_A(c_1) = \text{Res}_A(y_1) = [2](u) = v_nu^{2n}.
\]
Similarly, we get

\begin{align}
(4.4) \quad \text{Res}_{C \times A_1}(c_2) &= u(u + K(n) y_i) = u^2 + u y_i + v_n u^{2n-1} + 1 y_i^{2n-1}; \\
(4.5) \quad \text{Res}_{C \times A_1}(c_1) &= u + (u + K(n) y_i) = y_i + v_n u^{2n-1} y_i^{2n-1}.
\end{align}

Next consider the quaternion case. Here the maximal subgroups are \( B_1 = \langle a_1 \rangle \), \( B_2 = \langle a_2 \rangle \), and \( B_3 = \langle a_1 a_2 \rangle \), all isomorphic to \( \mathbb{Z}/4 \). If \( c_i: B_i \to \mathbb{C}^* \) is a faithful representation, we have \( \rho \cong \text{Ind}_{B_i}^{B}(c_i) \) for each \( B_i \), and, similar to the above, we can see that

\begin{equation}
(4.6) \quad \text{Res}_{B_i}(c_2) = u_i^2 + v_n u_i^{2n+1} \mod (u_i^{2n+1}) \text{ in } K(n)^*(B B_i) \cong K(n)^*[u]/(u_i^{4n}).
\end{equation}

To finish this section, we consider a compact group defined by \( Q_8 \) or \( D_8 \). When a group \( G \) has centre \( C \cong \mathbb{Z}/2 \), let us write \( \hat{G} \) for the central product \( G \times_C S^1 \), identifying \( C \) with \( \{1, -1\} \subset S^1 \). Then \( \hat{D}_8 \cong \hat{Q}_8 \). Using Lemma \( \ref{lem:quotient} \) we easily see:

**Theorem 4.2.** There is an additive isomorphism

\[ K(n)^*(B \hat{D}_8) \cong (K(n)^* \{ c_1 \} \oplus K(n)^*[y_1, y_2]/(\pi, y_1^{2n}, y_2^{2n}))|_{c_2}. \]

The multiplicative structure is given by \( (4.1) \mod (y_1, y_2)^3 \) in Theorem \( \ref{thm:main} \). \( \square \)

## 5. Extraspecial groups of order \( 2^5 \)

In this section we consider the central products \( G = D_8 \circ D_8 \) and \( G = D_8 \circ Q_8 \). In both cases, \( G \) is generated by elements \( a_1, \ldots, a_4 \) of order 2, and we have an extension

\begin{equation}
(5.1) \quad 1 \to G' \to G \to V \to 0 \quad \text{with} \quad G' \cong D_8, \quad V \cong \mathbb{Z}/2 \times \mathbb{Z}/2
\end{equation}

and trivial \( V \)-action on \( G' \). Set \( G_{ij} = \langle a_i, a_j \rangle \subset G \), numbering the generators \( a_i \) such that \( G' = G_{12} \) and \( A_i = \langle a_i \rangle \). Then \( G_{34} \cong D_8 \) or \( Q_8 \), and \( G_{34}/C = V \) for \( C \) = centre of \( G \). This allows us to keep the notation for \( K(n)^*(B \hat{D}_8) \) from the previous section. Furthermore, let \( H^*(BV; \mathbb{F}_2) = \mathbb{F}_2[x_3, x_4] \), and \( y_1, y_4, \alpha \in H^*(BV) \) correspond to \( x_3^2, x_4^2 \), and \( x_3 x_4 + x_3 x_4^2 \), respectively. We consider the spectral sequence

\begin{equation}
(5.2) \quad E_2^{*,*} = H^*(BV; \hat{K}(n)^*(B \hat{D}_8)) \implies \hat{K}(n)^*(B G). \quad \square
\end{equation}

**Lemma 5.1.** In the above spectral sequence, we have

\[ d_3 c_2 = c_1 \otimes \alpha \mod (y_1, y_2)^2. \]

**Proof.** For dimensional reasons, \( d_3 c_2 = (\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 c_1) \otimes \alpha \mod (y_1, y_2)^2 \) with \( \lambda_i \in \mathbb{F}_2 \). Consider the map of spectral sequences induced by

\[
\begin{array}{cccccc}
1 & \longrightarrow & A_1 \times C & \longrightarrow & A_1 \times G_{34} & \longrightarrow & V = G_{34}/C & \longrightarrow & 0 \\
& & \downarrow i & & \downarrow i & & & & \\
1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & V & \longrightarrow & 0.
\end{array}
\]

Since \( \text{Res}_{A_1 \times C}(c_2) = u^2 + u y_1 \mod (y_1^2) \) and \( d_3 u = 1 \otimes \alpha \), we get

\[ i^*(d_3 c_2) = d_3 (u^2 + u y_1) = y_1 \otimes \alpha \mod (y_1^2). \]
and hence $\lambda_1 + \lambda_3 = 1$. Similarly, replacing $A_1$ with $A_2$, we get $\lambda_2 + \lambda_3 = 1$. Finally, consider the inclusion of $A$ into $G_{12}$:

$$
1 \longrightarrow A \longrightarrow A \times_C G_{34} \longrightarrow V \longrightarrow 0
$$

$$
1 \longrightarrow G_{12} \longrightarrow G \longrightarrow V \longrightarrow 0.
$$

Now modulo $u^{2n+1}$, we have $\text{Res}_A(c_2) = u^2 + uv_n u^{2n+1}$ and thus $j^*(d_3 c_2) = v_n u^{2n} \otimes \alpha$. Since $\text{Res}_A(c_1) = v_n (\text{Res}_A(c_2))^{2n-1} = v_n u^{2n}$, we get $\lambda_1 + \lambda_2 + \lambda_3 = 1$, too. 

Therefore Theorem 5.1 gives

$$
d_3(y, c_2) = y_1 c_1 \otimes \alpha = y_1^2 \otimes \alpha \mod (y_1, y_2)^3,
$$

$$
d_3(c_1, c_2) = c_2^2 \otimes \alpha = y_1 y_2 \otimes \alpha \mod (y_1, y_2)^3.
$$

Using these formulae, it is easy to see that $\tilde{K} = \ker(d_3 |_{\tilde{K}((n)^*(BD_8))})$ is generated as a $K(n)^*$-algebra by

$$
y_1, y_2, c_2^2 \text{ (which gives } c_1), 2c_2,
$$

$$
b_1 = y_1^{2n-1} c_2, b_2 = y_2^{2n-1} c_2, y_1 b_2 = y_1 y_2^{2n-1} c_2.
$$

The last three terms are in $\tilde{K}$ since $v_n y_1^{2n} = 0$ in $K(n)^*(BD_8)$. More precisely, we have

**Lemma 5.2.** In the spectral sequence 5.2, the kernel $\tilde{K}$ and the homology $H$ with respect to $d_3 \otimes \alpha^{-1}$ are given additively as

$$
\tilde{K} \cong \left( \tilde{K}((n)^*[y_1, y_2]/(\tilde{\pi}, [2](y_1)) \oplus \tilde{K}((n)^*[c_1]}{1, 2c_2} \right) \left[ c_2^2 \right] / (c_2^{2n-1}),
$$

$$
H \cong K(n)^*[1, y_1, y_2, b_1, b_2, y_1 b_2] \left[ c_2^2 \right] / (c_2^{2n-1})
$$

where $\tilde{\pi} = y_1 y_2 (y_1 + \tilde{K}(n) y_2)$. Note that $2b_1 = 2c_2 y_1^{2n-1}$. 

A similar statement holds for the associated compact group:

**Lemma 5.3.** In the spectral sequence $1 \rightarrow \tilde{G}' \rightarrow \tilde{G} \rightarrow V \rightarrow 1$, the kernel $\tilde{K}$ and the homology $H$ are given additively as

$$
\tilde{K} \cong \left( \tilde{K}((n)^*[y_1, y_2]/(\tilde{\pi}, [2](y_1)) \oplus \tilde{K}((n)^*[c_1]}{1, 2c_2} \right) \left[ c_2^2 \right] / (c_2^{2n-1}),
$$

$$
H \cong K(n)^*[1, y_1, y_2] \left[ c_2^2 \right].
$$

We want to show that all elements in $\tilde{K}$ are permanent. Let $t = \text{Tr}_{G_{12} \times A_3}^G(c_2 \otimes 1)$. By the double coset formula, we get

$$
\text{Res}_{G_{12}}(t) = \sum_{g \in G_{12} \setminus G/G_{12} \times A_3} \text{Tr}_{G_{12} \setminus (G_{12} \times A_3)^g} \text{Res}_{G_{12} \setminus (G_{12} \times A_3)^g}^G(g^*(c_2 \otimes 1)).
$$

Here $G_{12} \setminus G / G_{12} \times A_3 \cong A_4$ and $G_{12} \setminus (G_{12} \times A_3)^g = G_{12}$ for all $g \in A_4$. Hence

$$
\text{Res}_{G_{12}}(t) = c_2 + a_4^* c_2 = 2c_2.
$$

Therefore, $t \in K((n)^*(BG)$ corresponds to the element $[2c_2] \in E_{0,1}$. Next we look for elements corresponding to the $b_i$ of Lemma 5.1. Let $A' = \langle a_3 a_4 \rangle$; this is cyclic of order 4. Let $\rho_{A'}$ be a faithful one-dimensional representation of $A'$.
Set \( \rho' = \text{Ind}_{A}^{G}(\rho_A) \) and \( c_2' = c_2(\rho') \). Define \( t_i = \text{Tr}_{G \times A_i}^{G}(c_2' \otimes 1) \) for \( i = 1, 2 \). We claim the following identities:

\[
\begin{align*}
  v_n b_1 &= \text{Res}_{G \times A_i}^{G}(t - t_2 + y_2^2 - y_1 y_2), \\
  v_n b_2 &= \text{Res}_{G \times A_i}^{G}(t - t_1 + y_1^2 - y_1 y_2).
\end{align*}
\]

It suffices to check them on the abelian subgroups of \( G_{12} \), by [3]. Thus we need to compute the restrictions to \( C \times A_1 \) and \( A \). Since \( \rho \) restricts to \( \eta + \eta \lambda_i \) on \( C \times A_i \) and to \( \rho_A + \rho_A^3 \) on \( A \), we have

\[
\begin{align*}
  \text{Res}_{C \times A_i}^{G}(t) &= 2u(u + K(n) y_i), \\
  \text{Res}_{C \times A_1}^{G}(b_1) &= u(u + K(n) y_1)y_1^{2n-1}, \\
  \text{Res}_{C \times A_2}^{G}(b_1) &= 0.
\end{align*}
\]

Here \( z = c_1(\rho_A) \) denotes the generator of \( K(n)^*(BA) \cong K(n)^*[z]/[4](z) \); clearly \( y_1, y_2 \) restrict to \([2](z)\).

Now \( C \times A_1 \setminus G/34 \times A_2 \cong 1 \) and \( (C \times A_1) \cap (G_{34} \times A_2) = C \); the double coset formula then says

\[
\begin{align*}
  \text{Res}_{C \times A_1}^{G}(t_2) &= \text{Tr}_{C \times A_1}^{C \times A_2} \text{Res}_{C \times A_2}^{C \times A_1}(c_2' \otimes 1) = \text{Tr}_{C}^{C}(u^2) \\
  &= u^2 \text{Tr}_{A_1}^{A_1}(1) = u^2(2 - v_n y_1^{2n-1})
\end{align*}
\]

where we used the fact (see, e.g., [3] or [4])

\[
\text{Tr}_{A_1}^{A_1}(1) = [2](y_1) / y_1 = 2 - v_n y_1^{2n-1}.
\]

Similarly, we have \( C \times A_2 \setminus G/34 \times A_2 \cong A_1 \) and \( (C \times A_2) \cap (G_{34} \times A_2) \cong C \times A_2 \), whence

\[
\begin{align*}
  \text{Res}_{C \times A_2}^{G}(t_2) &= \text{Res}_{C \times A_2}^{G \times A_2}(1 + a_1^*) (c_2' \otimes 1) \\
  &= c_2(2\eta) + c_2(\eta \otimes \lambda_2) = u^2 + (u + K(n) y_2)^2.
\end{align*}
\]

By the double coset formula again,

\[
\text{Res}_{A}^{G}(t_i) = \text{Tr}_{C}^{A}(u^2) = z^2 \text{Tr}_{C}^{A}(1) = z^2 [4](z) / [2](z).
\]

Thus

\[
\begin{align*}
  \text{Res}_{C \times A_1}^{G}(t - t_2 + y_2^2 - y_1 y_2) - \text{Res}_{C \times A_1}^{G}(v_n b_1) &= (u(u + K(n) y_1) - u^2)(2 - v_n y_1^{2n-1})
\end{align*}
\]

Let \( \chi \) be a generalized character of \( C \times A_1 \). If \( \chi(y_1) = 0 \), then

\[
\chi((u(u + K(n) y_1) - u^2)(2 - v_n y_1^{2n-1})) = 2\chi(u^2) - \chi(u^2) = 0,
\]

whereas if \( \chi(y_1) \neq 0 \), then \( \chi(2 - v_n y_1^{2n-1}) = [2](\chi(y_1)) / \chi(y_1) = 0 \). Secondly,

\[
\begin{align*}
  \text{Res}_{C \times A_2}^{G}(t - t_2 + y_2^2 - y_1 y_2) - \text{Res}_{C \times A_2}^{G}(v_n b_1) &= 2u(u + K(n) y_2) - (u + K(n) y_2)^2 - u^2 + y_2^2.
\end{align*}
\]

Any generalized character \( \chi \) with \( \chi(u) = 0 \) or \( \chi(y_2) = 0 \) clearly annihilates this expression. So assume, without loss of generality, that \( \chi(u) = \pi \), where \( \pi \) is a uniformizing element. Any other nonzero root of the 2-series is of the form \( \zeta \pi \) for a \((2^n - 1)\)-st root of unity \( \zeta \). Then \( [\zeta]\pi = \zeta \pi \) and \( \pi + K(n) \zeta \pi = \pi + K(n) [\zeta]\pi = [1 + \zeta]\pi = 1 + \zeta \pi \), since \((1 + \zeta)^{2n-1} \equiv 1 \mod 2 \). Thus

\[
\chi(2u(u + K(n) y_2) - (u + K(n) y_2)^2 - u^2 + y_2^2)) = 2\pi(1 + \zeta)\pi - (1 + \zeta)^2 \pi^2 - \pi^2 + \zeta^2 \pi^2 = 0.
\]
Finally,
\[
\text{Res}_A^G(t - t_2 + y_2^2 - y_1y_2) - \text{Res}_A^{G^2}(v_n b_1)
= 2z[3](z) - z^2 \frac{[4](z)}{[2](z)} - v_n z[3](z)([2](z))^{2^n - 1} = (z[3](z) - z^2 \frac{[4](z)}{[2](z)}
\]
where we used \(v_n([2](z))^{2^n - 1} = 2 - \frac{[4](z)}{[2](z)}\). Let \(\alpha\) denote the value of a character on \(z\). Then either \([4](\alpha)/[2](\alpha) = 0\), if \([2](\alpha) \neq 0\), or \([4](\alpha)/[2](\alpha) = 2\), if \([2](\alpha) = 0\), and in that case \(\alpha[3](\alpha) = \alpha^2 = \alpha(\alpha + \tilde{K}(n)[2](\alpha)) - \alpha^2 = \alpha^2 - \alpha^2 = 0\).
This finishes the proof of equation (5.4). The other equation follows by exchanging the indices 1 and 2. Thus the assumptions of Lemma (3.1) hold, yielding

**Theorem 5.4.** Let \(G\) be an extraspecial group of order 32. Then \(K(n)^*(BG)\) is concentrated in even degrees and generated by transfers of Euler classes. \(\square\)

In the compact case it suffices to show that \(c_1\) is a permanent cycle. Suppose that \(d_r c_1 = x \otimes f \alpha \neq 0\) for \(3 \leq r \leq 2^{n+1} - 1\). Note that \(x \otimes f \pi^2 = x \otimes f \pi \neq 0\) in \(E_r^{s*}\). But \(d_r(c_1 \otimes \alpha)\) must be zero in \(E_r^{s*}\), since it is so in \(E_1^{s*}\). This is a contradiction. The term \(E_{2n+1}^{s*}\) is generated by even-dimensional elements, and \(c_1\) is a permanent cycle.

From Lemma (5.3) and the formula in the proof of Lemma (3.1), we get
\[
\text{gr} \tilde{K}(n)^*(BG) \cong \tilde{K} \oplus K \otimes \mathbb{F}_2[y_3, y_4]^+/(\pi_{34}) \oplus H \otimes \mathbb{F}_2[y_3, y_4]/(q_{34})\{\pi_{34}\}
\]
(5.7)
\[
\cong (\tilde{K}(n)^*\{y_1, y_2]/(\pi_{12}, [2](y_i)) \oplus \tilde{K}(n)^*\{c_1\})\{1, 2c_2\}
\]
\[
\oplus ((K(n)^*\{y_1, y_2]/(\pi_{12}, [2](y_i)) \oplus K(n)^*\{c_1\}) \otimes \mathbb{F}_2[y_3, y_4]^+/(\pi_{34})
\]
\[
\oplus (K(n)^*\{1, y_1, y_2\} \otimes \mathbb{F}_2[y_3, y_4]/(q_{34})\{\pi_{34}\})[c_2^2].
\]

6. **Euler characteristics of extraspecial \(p\)-groups**

In this section we give the Euler characteristic of an extraspecial \(p\)-group. The result is not new; the same formula was obtained by Brunetti [2].

The Morava \(K\)-theory Euler characteristic \(\chi_{n,p}(G)\) of a finite group \(G\), i.e., the difference between the ranks of the even and odd degree parts of \(K(n)^*(BG)\), can be computed using the formula from [3]:
\[
\chi_{n,p}(G) = \sum_{A < G} \frac{|A|}{|G|} \mu_{A(G)}(A) \chi_{n,p}(A)
\]
(6.1)
where the sum is over all abelian subgroups \(A < G\) and \(\mu_{A(G)}\) is a Möbius function defined recursively by
\[
\sum_{A' < A} \mu_{A(G)}(A') = 1
\]
(6.2)
where the sum is over all abelian subgroups \(A' < G\) contained in \(A\). In particular, \(\mu_{A(G)}(A) = 1\) when \(A\) is maximal. It is easy to see that one only has to consider subgroups arising as intersections of maximal ones. Furthermore, one clearly has \(\chi_{n,p}(A) = |A(p)|^n\) where \(A(p)\) denotes the \(p\)-part of the abelian group \(A\).

The abelian subgroups of an extraspecial \(p\)-group \(D(m) = p^{1+2m}\) are in one-to-one correspondence with those subspaces \(W\) of the central quotient \(V \cong \mathbb{F}_p\) that are isotropic with respect to the bilinear form
\[
b(x, y) = x_1y_2 + x_2y_1 + \cdots + x_{2m-1}y_{2m} + x_{2m}y_{2m-1}.
\]
Let \( \alpha_i^{(m)} \) denote the number of such subspaces of dimension \( i \). Note that the maximal dimension of a \( b \)-isotropic subspace is \( m \).

The following lemma is an easy exercise in counting:

**Lemma 6.1.** \( \alpha_i^{(m)} = \prod_{j=1}^{i} \frac{p^{2(m-j+1)} - 1}{p^j - 1}. \)

The Möbius function on abelian subgroups can be computed via a Möbius function on \( b \)-isotropic subspaces defined as in (6.2). Let \( \gamma_k^{(m)} \) denote its value on a subspace of dimension \( k \): by symmetry, it is constant on subspaces of the same rank. Furthermore, it only depends on the codimension of a \( b \)-isotropic subspace in a maximal one, independent of \( m \); this follows by considering \( W^\perp / W \). The following formula can be proved inductively, see [2].

**Lemma 6.2.** \( \gamma_k^{(m)} = (-p)^{(m-k)^2}. \)

Since a \( b \)-isotropic subspace \( W \) of dimension \( i \) gives rise to an abelian subgroup of index \( 2m - i \), we arrive at

**Proposition 6.3 ([2]).** The Morava \( K \)-theory Euler characteristic of \( G = \mathbb{F}_p^{1+2m} \) is given by

\[
\chi_{n, p}(G) = \sum_{i=0}^{m} \alpha_i^{(m)} \gamma_i^{(m)} p^{(i+1)n} = \sum_{i=0}^{m} (-1)^{m-i} \alpha_i^{(m)} p^{(m-i-1)^2 + (n-1)(i+1)}
\]

with \( \alpha \) and \( \gamma \) as in the two lemmas above.

For example, for \( D_8 \) and \( D(2) = \mathbb{F}_2^{1+4} \) we obtain

\[
\chi_{n, 2}(D_8) = \frac{3}{2} 4^n - \frac{1}{2} 2^n, \quad \text{and} \quad \chi_{n, 2}(D(2)) = \frac{15}{4} (8^n - 4^n) + 2^n.
\]

This agrees with the Euler characteristics that we can compute using Corollary 3.2 as we shall now see. Let \( Y_{1,2} = K(n)^* \{[y_1, y_2]/\{\pi, y_1^{2n}, y_2^{2n}\} \}, \) and denote by \( \chi(-) \) the dimension of a \( K(n)^* \)-vector space. Then one easily computes \( \chi(Y_{1,2}) = 3 \cdot (2^n - 1). \)

We have \( K(n)^*(BD_8) \cong (Y_{1,2} \oplus K(n)^* \{1, c_1\}) \otimes \mathbb{F}_2/c_2 \otimes \mathbb{Z}/2/c_2^{2n-1} \) is a \( \mathbb{F}_2 \)-module.

Hence

\[
\chi(K(n)^*(BD_8)) = (3 \cdot (2^n - 1) + 2) 2^{n-1} - 3 \cdot 2^{n-1} - 2^{n-1}.
\]

Next consider \( K(n)^*(BD(2)) \). First note

\[
\chi(K/2) = \chi(Y_{1,2} + K(n)^* \{1, c_1\}) \otimes \mathbb{Z}/2/c_2^{2n-1} = (3 \cdot (2^n - 1) + 2) \cdot 2^{n-2} = (6 \cdot 2^n - 2) \cdot 2^{n-2},
\]

where we used the fact that we can take either \( y_1^{2}c_2 \) or \( 2y_2^{2}c_2 \) as a basis element and may neglect the summand \( K(n)^* \{b_1, b_2, y_2b_1\} \). Then

\[
\chi(K \otimes Y_{3,4}) = \chi((Y_{1,2} + K(n)^* \{1, c_1, b_1, b_2, y_2b_1\}) \otimes Y_{3,4}) \cdot 2^{n-2}
\]

\[
= (3(2^n - 1) + 5) \cdot 3(2^n - 1) \cdot 2^{n-2} = (9 \cdot 2^{2n} - 3 \cdot 2^n - 6) \cdot 2^{n-2};
\]

\[
\chi(H \otimes \mathbb{Z}/2[y_3, y_4]/(q_{34}, y_3^{2n-1}, y_4^{2n-1}) \{\pi\}) = (6 \cdot (2^n - 1)(2^n - 2)) \cdot 2^{n-2}
\]

\[
= (6 \cdot 2^{2n} - 18 \cdot 2^n + 12) \cdot 2^n.
\]

Therefore, we have \( \chi(K(n)^*(BD(2)) = (15 \cdot 2^{2n} - 15 \cdot 2^n + 4) \cdot 2^{n-2} = \chi_{n, 2}(D(2)) \).
References


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