

## MORAVA $K$ -THEORY OF EXTRASPECIAL 2-GROUPS

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ABSTRACT. We compute the Morava  $K$ -theory of some extraspecial 2-groups and associated compact groups.

### 1. INTRODUCTION

Let  $G$  be a finite group and  $BG$  denote its classifying space. Not that many computations for the Morava  $K$ -theory of  $BG$  have been carried out, the most notable exception being Kriz's article [5] and its successor [6], where he calculates just enough about the 3-primary second Morava  $K$ -theory of the 3-Sylow subgroup of  $GL_4(\mathbb{F}_3)$  to conclude that it cannot be concentrated in even degrees, the first such example known. Other computations can be found in [1], [3], [4], [8], [9], [10], and [11].

In this paper we present a few more calculations concerning extraspecial 2-groups. We mainly work with integral Morava  $K$ -theory for the prime 2, which shall be denoted  $\tilde{K}(n)$ . This is a complex oriented cohomology theory with coefficients  $\tilde{K}(n)^* \cong W\mathbb{F}_{2^n}[v_n, v_n^{-1}]$ , the ring of Laurent polynomials over the Witt ring  $W\mathbb{F}_{2^n}$ , with  $v_n$  of degree  $-2(2^n - 1)$ . It has a complex orientation  $x$  such that the 2-series of the associated formal group law takes the form  $[2](x) = 2x - v_n x^{2^n}$ . Sometimes we switch to the mod 2 reduction  $K(n)$ .

In Section 2 we describe the groups we want to study and recall Quillen's computation of their mod 2 cohomology. As a corollary we consider a slight modification serving as motivation for our calculational approach. Section 3 contains the main technical result, Lemma 3.1, which under favourable circumstances computes the spectral sequence of an extension of  $\mathbb{Z}/2 \times \mathbb{Z}/2$  by a "good" group in the sense of Hopkins-Kuhn-Ravenel, i.e., whose Morava  $K$ -theory is generated by transfers of Euler classes. The next two sections contain applications to extraspecials of order 8 and 32. Section 4 is a rehash of the already known computations for  $D_8$  and  $Q_8$  and serves mainly to set up notation for the next section, where we deal with the central products  $D_8 \circ D_8$  and  $D_8 \circ Q_8$ . We need some of the multiplicative structure for  $D_8$ , and make repeated use of generalized characters à la Hopkins-Kuhn-Ravenel [3]. We also consider the associated compact groups that arise by replacing the centre  $\mathbb{Z}/2$  by the circle group  $S^1$ . The last section contains calculations of the Euler

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characteristics of extraspecial groups (for any prime), due also to Brunetti [2]. We omit proofs, since they are now available in [2].

## 2. EXTRASPECIAL 2-GROUPS

There are three types of (almost) extraspecial 2-groups, the so-called real, complex and quaternion types. These may be described as central products. Let  $D_8$  and  $Q_8$  denote the dihedral, respectively quaternion, group of order 8. The extraspecials of real type have order  $2^{2m+1}$  for some  $m > 0$  and correspond to  $m$ -fold central products of  $D_8$  (for the quaternion type replace one of copy  $D_8$  with a  $Q_8$ ) whereas the complex type is obtained as the central product of a real extraspecial with a cyclic group of order four.

In this section we try to motivate our subsequent computations, and thus concentrate on the real case only. So let  $D(m) := D_8 \circ \cdots \circ D_8$  ( $m$  copies); in Hall-Senior notation this group is known as  $2_+^{1+2m}$ . Its mod 2 cohomology was computed by Quillen [7]: one has a central extension

$$(2.1) \quad 1 \rightarrow \mathbb{Z}/2 \longrightarrow D(m) \longrightarrow E \rightarrow 1$$

where  $E \cong (\mathbb{Z}/2)^{2m}$  is a  $2m$ -dimensional vector space over  $\mathbb{F}_2$ . The Serre spectral sequence associated to this extension takes the form

$$(2.2) \quad E_2 = H^*(BE; H^*(B\mathbb{Z}/2)) \cong \mathbb{F}_2[u] \otimes \mathbb{F}_2[x_1, \dots, x_{2m}]$$

with  $u$  and  $x_i$  in degree one; the extension class is  $q := x_1x_2 + \cdots + x_{2m-1}x_{2m}$ . Quillen's computation can be summarised as follows:

**Theorem 2.1** (Quillen [7]). *The only differentials in the spectral sequence (2.2) are  $d_2u = q$ ,  $d_{2^k+1}u^{2^k} = Q_{k-1}q$  for  $1 \leq k < m$ , where  $Q_i$  stands for Milnor's primitive operation in the Steenrod algebra. The sequence  $(q, Q_0q, \dots, Q_{m-2}q)$  is regular, and  $u^{2^m}$  is a permanent cycle since it represents the Euler class  $w_{2m}$  of the spin representation  $\Delta$ . Thus*

$$H^*(D(m); \mathbb{F}_2) \cong \mathbb{F}_2[w_{2m}] \otimes \mathbb{F}_2[x_1, \dots, x_{2m}] / (q, Q_0q, \dots, Q_{m-2}q).$$

The nontrivial Stiefel-Whitney classes of  $\Delta$  are  $w_{2m}$  and  $w_{2m-2i}$ ,  $0 \leq i \leq m$ .  $\square$

Knowing the result, one can slightly rearrange the computation.  $D(m+1)$  contains  $D(m)$  as a normal subgroup with quotient  $\mathbb{Z}/2 \times \mathbb{Z}/2$ , i.e., one has an extension

$$(2.3) \quad 1 \rightarrow D(m) \longrightarrow D(m+1) \longrightarrow V \rightarrow 1$$

with  $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$  acting trivially on the kernel. The Serre spectral sequence corresponding to (2.3) has  $E_2$ -term given by

$$(2.4) \quad E_2 = H^*(BV; H^*(BD(m))) \cong \mathbb{F}_2[x_{2m+1}, x_{2m+2}] \otimes H^*(BD(m)).$$

**Corollary 2.2.** *The spectral sequence (2.4) collapses on the  $E_3$ -page. The only nontrivial differential is  $d_2w_{2m} = x_{2m+1}x_{2m+2} \otimes w_{2m-1}$ .*

*Proof.* Since the cohomology of extraspecial 2-groups of real type is detected on maximal elementary abelian subgroups, the action of  $d_2$  can be worked out by looking at the restrictions to those subgroups. Each maximal elementary abelian  $W$  is of the form  $C \times U$  where  $C$  is the centre and  $U$  a maximal isotropic subspace

of the central quotient  $E$ . (Recall from [7] that  $q$  may be regarded as a quadratic form on  $E$ .) The corresponding extension is of the form

$$1 \rightarrow C \times U \rightarrow D_8 \times U \rightarrow V \rightarrow 1,$$

and the only differential is  $d_2u = x_{2m+1}x_{2m+2}$ . Quillen tells us that  $\Delta$  restricts to  $W$  as  $\chi \otimes \text{reg}(U)$ , where  $\chi$  is the nontrivial character of  $C$  and  $\text{reg}(U)$  the regular representation of  $U$ . Applying the formula expressing  $w_*(\chi \otimes \text{reg}(U))$  in terms of  $w_*(\chi)$  and  $w_*(\text{reg}(U))$  we obtain

$$w_i(\chi \otimes \text{reg}(U)) = \sum_{j=0}^i \binom{2^m - i + j}{j} w_1(\chi)^j w_{i-j}(\text{reg}(U)).$$

So  $w_{2^m}$  restricts to  $\sum_{k=0}^m u^{2^k} w_{2^m-2^k}(\text{reg}(U))$ , since other Stiefel-Whitney classes of  $\text{reg}(U)$  are zero, and  $w_{2^m-1}$  restricts to  $w_{2^m-1}(\text{reg}(U))$ . Thus  $d_2$  is as claimed; the rest follows from a Poincaré series calculation.  $\square$

Note that  $w_{2^m}^2$  represents the Euler class of the spin representation of  $D(m+1)$ . Furthermore, there are extension problems in the  $E_\infty$ -term. Let  $q_m = x_1x_2 + \dots + x_{2m-1}x_{2m}$  denote the extension class of  $D(m)$ . Then  $q_m$  drops in filtration to  $x_{2m+1}x_{2m+2}$  (so we get the relation  $q_{m+1} = 0$ ), and the other relations follow as solutions to extension problems related to  $Q_iq_m = 0$  and  $x_{2m+1}x_{2m+2}w_{2^m-1} = 0$ .

The (additive) simplicity of the spectral sequence of this extension is what lets us believe it to be possible to emulate this computation in Morava  $K$ -theory. In the subsequent sections we shall try to prove that the Atiyah-Hirzebruch-Serre spectral sequence of (2.3) behaves analogously, meaning it has only two differentials (the second being  $v_n \otimes Q_n$ , see below).

### 3. SPECTRAL SEQUENCE CALCULATIONS

In this section we consider the Atiyah-Hirzebruch-Serre spectral sequence associated to extensions

$$1 \rightarrow G' \rightarrow G \rightarrow V \rightarrow 0$$

with  $V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ , acting trivially on  $G'$ . The spectral sequence has  $E_2$ -term given by

$$(3.1) \quad E_2^{*,*} = H^*(\mathbb{Z}/2 \otimes \mathbb{Z}/2; \tilde{K}(n)^*(BG')) \implies \tilde{K}(n)^*(BG).$$

**Lemma 3.1.** *Let  $G$  be as above. Suppose  $K(n)^{\text{odd}}(BG') = 0$  for all  $n \geq 1$ , and moreover that all elements in  $E_4^{0,*}$  are permanent cycles. Then  $\tilde{K}(n)^{\text{odd}}(BG) = 0$  and  $\tilde{K}(n)^*(BG)$  has no  $p$ -torsion, and  $K(n)^{\text{odd}}(BG) = 0$ .*

*Proof.*  $K(n)^{\text{odd}}(BG') = 0$  implies  $\tilde{K}(n)^{\text{odd}}(BG') = 0$  and  $\tilde{K}(n)^*(BG')$  is  $p$ -torsion free. One has  $H^*(BV; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2]$ ; setting  $y_i = x_i^2$  and  $\alpha = x_1^2x_2 + x_1x_2^2$ , the  $E_2$ -page of the spectral sequence is

$$E_2^{*,*'} \cong \begin{cases} \tilde{K}(n)^*(BG') & \text{for } * = 0, \\ \tilde{K}(n)^*(BG') \otimes \mathbb{F}_2[y_1, y_2, \alpha]/(\alpha^2 = y_1^2y_2 + y_1y_2^2) & \text{for } * > 0. \end{cases}$$

We shall write  $\pi$  for the element  $y_1^2y_2 + y_1y_2^2$ . The first potentially nontrivial differential is  $d_3$ . Any even (respectively odd) degree element in  $E_2^{*,*}$  is of the form  $x \otimes f$  ( $x \otimes f\alpha$ ) for some  $x \in \tilde{K}(n)^*(BG')$  and  $f \in \mathbb{F}_2[y_1, y_2]$ . We shall first consider the case  $n \geq 2$ , the argument for  $n = 1$  being similar (see the remark at

the end). Note that  $d_3$  is zero on any element of  $\mathbb{F}_2[y_1, y_2, \alpha]/(\alpha^2 = y_1^2 y_2 + y_1 y_2^2)$  by comparison to the Atiyah-Hirzebruch spectral sequence for  $V$  and  $n \geq 2$ . Hence  $d_3(x \otimes f) = x' \otimes f\alpha$  and  $d_3(x \otimes f\alpha) = x' \otimes f\pi$  for some  $x' \in K(n)^*(BG')$ . Thus we obtain additive isomorphisms

$$\begin{cases} E_4^{0,*} \cong \tilde{K}, \\ E_4^{>0,*} \cong K \otimes \mathbb{F}_2[y_1, y_2]/(\pi) \oplus H \otimes \mathbb{F}_2[y_1, y_2]\{\alpha, \pi\} \end{cases}$$

where  $\tilde{K} = \text{Ker}(d_3|_{\tilde{K}(n)^*(BG')})$ ,  $K = \text{Ker}(d_3|_{K(n)^*(BG')}) = \tilde{K}/(\tilde{K} \cap 2E_2^{0,*})$ , and  $H = H(K(n)^*(BG'); d_3 \otimes \alpha^{-1})$ . As a  $\tilde{K}(n)^*$ -algebra, the  $E_4$ -page is generated by  $\alpha, y_i$ , and the generators in  $\tilde{K}$ . By hypothesis, all but  $\alpha$  are permanent cycles. So the next nonzero differential is

$$d_{2n+1-1}(\alpha) = v_n \otimes Q_n \alpha = v_n \otimes (y_1^{2^n} y_2 + y_1 y_2^{2^n}) = v_n \otimes q\pi$$

where  $q = (y_1^{2^n} y_2 + y_1 y_2^{2^n})/\pi = (y_1^{2^n-2} + y_1^{2^n-3} y_2 + \dots + y_2^{2^n-2})$ . Thus we get

$$\begin{cases} E_{2n+1}^{0,*} \cong \tilde{K}, \\ E_{2n+1}^{>0,*} \cong K \otimes \mathbb{F}_2[y_1, y_2]/(\pi) \oplus H \otimes \mathbb{F}_2[y_1, y_2]/(q)\{\pi\}. \end{cases}$$

This is concentrated in even degrees, whence  $E_{2n+1} \cong E_\infty$  and  $\tilde{K}(n)^{\text{odd}}(BG) = 0$ . It remains to prove that  $\tilde{K}(n)^*(BG)$  has no 2-torsion. Let  $0 \neq x \in \tilde{K}(n)^*(BG)$ . Represent  $x$  by  $x' \in E_\infty$ . If  $x' \in E_\infty^{0,*}$  then it cannot be 2-torsion, since  $\tilde{K}(n)^*(BG')$  is 2-torsion free. If  $x'$  is in  $K \otimes \mathbb{F}_2[y_1, y_2]/(\pi)$ , we may write  $x' = \sum \bar{x} \otimes f$  with  $\bar{x} \in K, f \in \mathbb{F}_2[y_1, y_2]/(\pi)$ . Rewrite  $f$  as  $y_1 f_1 + \lambda y_2^s, \lambda \in \mathbb{F}_2$ . Since  $2y_i = v_n y_i^{2^n}$  in  $\tilde{K}(n)^*(BG)$  (this is immediate from the calculation for cyclic groups),  $2x$  can be represented by

$$(2x)' = \sum v_n \bar{x} \otimes (y_1^{2^n} f_1 + \lambda y_2^{2^n+s-1}).$$

We claim that the right-hand side of this expression is nonzero: if  $\lambda \neq 0$ , it does not lie in the ideal  $(y_1 y_2) \supset (\pi)$ , and if  $\lambda = 0$ , then  $y_1^{2^n} f \in (\pi)$  implies  $y_1 f \in (\pi)$ . Lastly suppose  $x' \in H \otimes \mathbb{F}_2[y_1, y_2]/(y_1^{2^n-2} + \dots + y_2^{2^n-2})\{\pi\} \subset H \otimes \mathbb{F}_2[y_1, y_2]/(Q_n \alpha)$ . Write  $x' = \sum \bar{x} \otimes f\pi$  and  $f\pi = y_1 f_1$ . Then  $(2x)' \neq 0$  if  $v_n \otimes f_1 y_1^{2^n} \neq 0$ . But  $f_1 y_1^{2^n} \in (Q_n \alpha)$  implies  $f_1 y_1 \in (Q_n \alpha)$ : tensoring up with the finite field of  $2^n$  elements  $\mathbb{F}_{2^n}$  yields

$$Q_n \alpha = y_1^{2^n} y_2 + y_1 y_2^{2^n} = \prod_{\mu \in \mathbb{F}_{2^n}} (y_1 + \mu y_2).$$

Finally, for  $n = 1$  the differential  $d_3$  is given by  $v_1 \pi$ ; the claim follows by filtering  $E_2^{*,*}$  by powers of  $\pi$  and setting  $q = 1$ . □

Since  $\tilde{K}$  is 2-torsion free and the map defined by

$$ay_1^i \mapsto ay_1^{i+2^n-1} \quad \text{and} \quad y_2^i \mapsto y_2^{i+2^n-1}$$

on  $E_\infty^{>0,*}$  is injective, one easily sees

**Corollary 3.2.** *Suppose  $G$  is as in Lemma 3.1. Then there is an additive isomorphism*

$$\begin{aligned} K(n)^*(BG) &\cong E_\infty^{0,*}/2 \oplus E_\infty^{>0,*}/(y_1^{2^n}, y_2^{2^n}) \\ &\cong \tilde{K}/2 \oplus K \otimes \mathbb{Z}/2[y_1, y_2]^+ / (y_1^{2^n}, y_2^{2^n}, \pi) \\ &\quad \oplus H \otimes \mathbb{Z}/2[y_1, y_2]/(y_1^{2^n-1}, y_2^{2^n-1}, q)\{\pi\}. \quad \square \end{aligned}$$

4. THE CASES  $D_8$  AND  $Q_8$

The groups  $D_8$  and  $Q_8$  have presentations

$$\begin{aligned} D_8 &= \langle a_1, a \mid a_1^2 = a^4 = 1, [a_1, a] = a^2 \rangle, \\ Q_8 &= \langle a_1, a_2 \mid a_1^4 = a_2^4 = 1, [a_1, a_2] = a_1^2 = a_2^2 \rangle, \end{aligned}$$

respectively. Thus there are central extensions of the form  $\mathbb{Z}/2 \rightarrow G \rightarrow V$  for  $G$  either  $D_8$  or  $Q_8$ , i.e., we have  $G' = \mathbb{Z}/2$  in the setup of Section 3. Setting  $a_2 = aa_1$  in the case of  $D_8$ , the quotient  $V$  is generated by the cosets  $\bar{a}_i$  for either group; let  $x_i \in H^*(BV; \mathbb{F}_2)$  be dual to  $\bar{a}_i$ . Recall that  $\tilde{K}(n)^*(B\mathbb{Z}/2) \cong \tilde{K}(n)^*[u]/(2u - v_n u^{2^n})$  where  $u$  is the Euler class of the nontrivial linear character  $\eta$  of  $\mathbb{Z}/2$ . In the spectral sequence (3.1), we get  $d_3 u = \alpha$ . Hence  $H = \text{Ker}(d_3)/\text{Im}(d_3 \otimes \alpha^{-1}) = 0$ , and  $u^2$  is a permanent cycle, since it is the restriction of the Euler class of the irreducible two-dimensional complex representation  $\rho$  of  $G$  to the fibre. Thus

$$\begin{aligned} E_\infty^{0,*} &\cong \tilde{K}(n)^*[u^2]/((2u - v_n u^{2^n}) \cap \tilde{K}(n)^*[u^2]) \cong \tilde{K}(n)^*[u^2]\{1, 2u\} \\ E_\infty^{>0,*} &\cong \tilde{K}(n)^*[u^2]/(v_n u^{2^n}) \otimes \mathbb{F}_2[y_1, y_2]/(\pi). \end{aligned}$$

It follows that  $\tilde{K}(n)^*(BG)$  is concentrated in even degrees and has no 2-torsion, whence  $K(n)^*(BG) \cong \tilde{K}(n)^*(BG)/(2)$ . Choosing an element  $\bar{c}_2 \in \tilde{K}(n)^*(BG)$  represented by  $u^2$ , one obtains

**Theorem 4.1** ([9], [8]). *Let  $G$  be either  $D_8$  or  $Q_8$ . Then there is an additive isomorphism*

$$K(n)^*(BG) \cong (K(n)^*\{\bar{c}_1\} \oplus K(n)^*[y_1, y_2]/(\pi, y_1^{2^n}, y_2^{2^n}))[\bar{c}_2]/(\bar{c}_2^{2^{n-1}}).$$

The multiplicative structure is given by

$$(4.1) \quad \bar{c}_1 y_1 = y_1^2, \quad \bar{c}_1 y_2 = y_2^2, \quad \bar{c}_1^2 = y_1^2 + y_1 y_2 + y_2^2$$

identifying  $\bar{c}_1 = v_n \bar{c}_2^{2^{n-1}} + y_1 + y_2$  for  $D_8$  and  $\bar{c}_1 = v_n \bar{c}_2^{2^{n-1}}$  for  $Q_8$ . □

The generators  $y_i$  can be identified with the Euler classes of the representations  $\rho_i: G \rightarrow V \rightarrow \langle \bar{a}_i \rangle \xrightarrow{\eta} \mathbb{C}^*$ . Switching from  $\bar{c}_i$  to  $c_i = c_i(\rho)$ , we may write  $c_2 = \bar{c}_2 \text{ mod } (y_1, y_2)^2$ . Then  $v_n \bar{c}_2^{2^{n-1}} = v_n c_2^{2^{n-1}} \text{ mod } (y_1, y_2)^{2^n}$ . We also have  $c_1 = \bar{c}_1 \text{ mod } (y_1, y_2)^2$ , by considering restrictions to maximal abelian subgroups; see below. Hence relation (4.1) in the theorem holds modulo  $(y_1, y_2)^3$  with  $\bar{c}_i$  replaced by  $c_i$ .

We want to compute the restrictions of  $c_2$  to the maximal subgroups of  $G$ . Consider  $G = D_8$  first. Let  $C = \langle a^2 \rangle$  be the centre of  $D_8$ , and  $A_i = \langle a_i \rangle$ . The maximal subgroups are  $A = \langle a \rangle \cong \mathbb{Z}/4$  and  $C \times A_i \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ . Let  $\rho_A: A \rightarrow \mathbb{C}^*$  be a faithful representation of  $A$ . Then  $c_1(\rho_A)$  restricts to the generator  $u$  of the centre, and identifying classes with their images under restriction, we may write

$$K(n)^*(BA) \cong K(n)^*[u]/[4](u) \cong K(n)^*[u]/(u^{4^n});$$

$$K(n)^*(BC \times A_i) \cong K(n)^*[u, y_i]/([2](u), [2](y_i)) \cong K(n)^*[u, y_i]/(u^{2^n}, y_i^{2^n}).$$

We have  $\text{Res}_A(\rho_i) = \rho_A \otimes \rho_A$ , and since  $\rho = \text{Ind}_A^G(\rho_A)$ , the double coset formula gives  $\text{Res}_A(\rho) = \rho_A \oplus \rho_A^{-1}$ . The restrictions of the total Chern class are  $\text{Res}_A(c(\rho)) = (1 + u)(1 + [-1]u)$  and  $\text{Res}_{C \times A_i}(c(\rho)) = (1 + u)(1 + u + {}_{K(n)}y_i)$ . Thus we obtain the following restrictions:

$$(4.2) \quad \text{Res}_A(c_2) = ([-1](u)u)u = u^2 + v_n u^{2^n+1} \text{ mod } (u^{2^{n+1}});$$

$$(4.3) \quad \text{Res}_A(c_1) = \text{Res}_A(y_i) = [2](u) = v_n u^{2^n}.$$

Similarly, we get

$$(4.4) \quad \text{Res}_{C \times A_i}(c_2) = u(u +_{K(n)} y_i) = u^2 + uy_i + v_n u^{2^{n-1}+1} y_i^{2^{n-1}};$$

$$(4.5) \quad \text{Res}_{C \times A_i}(c_1) = u + (u +_{K(n)} y_i) = y_i + v_n u^{2^{n-1}} y_i^{2^{n-1}}.$$

Next consider the quaternion case. Here the maximal subgroups are  $B_1 = \langle a_1 \rangle$ ,  $B_2 = \langle a_2 \rangle$ , and  $B_3 = \langle a_1 a_2 \rangle$ , all isomorphic to  $\mathbb{Z}/4$ . If  $e_i: B_i \rightarrow \mathbb{C}^*$  is a faithful representation, we have  $\rho \cong \text{Ind}_{B_i}^{Q_8}(e_i)$  for each  $B_i$ , and, similar to the above, we can see that

$$(4.6) \quad \text{Res}_{B_i}(c_2) = u_i^2 + v_n u_i^{2^n+1} \pmod{(u^{2^{n+1}})} \text{ in } K(n)^*(BB_i) \cong K(n)^*[u_i]/(u_i^{4^n}).$$

To finish this section, we consider a compact group defined by  $Q_8$  or  $D_8$ . When a group  $G$  has centre  $C \cong \mathbb{Z}/2$ , let us write  $\tilde{G}$  for the central product  $G \times_C S^1$ , identifying  $C$  with  $\{1, -1\} \subset S^1$ . Then  $\tilde{D}_8 \cong \tilde{Q}_8$ . Using Lemma 3.1, we easily see:

**Theorem 4.2.** *There is an additive isomorphism*

$$K(n)^*(B\tilde{D}_8) \cong (K(n)^*\{\bar{c}_1\} \oplus K(n)^*[y_1, y_2]/(\pi, y_1^{2^n}, y_2^{2^n}))[\bar{c}_2].$$

The multiplicative structure is given by (4.1) mod  $(y_1, y_2)^3$  in Theorem 4.1. □

### 5. EXTRASPECIAL GROUPS OF ORDER $2^5$

In this section we consider the central products  $G = D_8 \circ D_8$  and  $G = D_8 \circ Q_8$ . In both cases,  $G$  is generated by elements  $a_1, \dots, a_4$  of order 2, and we have an extension

$$(5.1) \quad 1 \rightarrow G' \rightarrow G \rightarrow V \rightarrow 0 \quad \text{with } G' \cong D_8, V \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

and trivial  $V$ -action on  $G'$ . Set  $G_{ij} = \langle a_i, a_j \rangle \subset G$ , numbering the generators  $a_i$  such that  $G' = G_{12}$  and  $A_i = \langle a_i \rangle$ . Then  $G_{34} \cong D_8$  or  $Q_8$ , and  $G_{34}/C = V$  for  $C =$  centre of  $G$ . This allows us to keep the notation for  $K(n)^*(BD_8)$  from the previous section. Furthermore, let  $H^*(BV; \mathbb{F}_2) = \mathbb{F}_2[x_3, x_4]$ , and  $y_3, y_4, \alpha \in H^*(BV)$  correspond to  $x_3^2, x_4^2$ , and  $x_3^2 x_4 + x_3 x_4^2$ , respectively. We consider the spectral sequence

$$(5.2) \quad E_2^{*,*} = H^*(BV; \tilde{K}(n)^*(BD_8)) \implies \tilde{K}(n)^*(BG).$$

**Lemma 5.1.** *In the above spectral sequence, we have*

$$d_3 c_2 = c_1 \otimes \alpha \pmod{(y_1, y_2)^2}.$$

*Proof.* For dimensional reasons,  $d_3 c_2 = (\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 c_1) \otimes \alpha \pmod{(y_1, y_2)^2}$  with  $\lambda_i \in \mathbb{F}_2$ . Consider the map of spectral sequences induced by

$$\begin{array}{ccccccc} 1 & \longrightarrow & A_1 \times C & \longrightarrow & A_1 \times G_{34} & \longrightarrow & V = G_{34}/C \longrightarrow 0 \\ & & \downarrow i & & \downarrow i & & \parallel \\ 1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & V \longrightarrow 0. \end{array}$$

Since  $\text{Res}_{A_1 \times C}(c_2) = u^2 + uy_1 \pmod{(y_1^2)}$  and  $d_3 u = 1 \otimes \alpha$ , we get

$$i^*(d_3 c_2) = d_3(u^2 + uy_1) = y_1 \otimes \alpha \pmod{(y_1^2)}$$

and hence  $\lambda_1 + \lambda_3 = 1$ . Similarly, replacing  $A_1$  with  $A_2$ , we get  $\lambda_2 + \lambda_3 = 1$ . Finally, consider the inclusion of  $A$  into  $G_{12}$ :

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \longrightarrow & A \times_C G_{34} & \longrightarrow & V \longrightarrow 0 \\ & & \downarrow j & & \downarrow j & & \parallel \\ 1 & \longrightarrow & G_{12} & \longrightarrow & G & \longrightarrow & V \longrightarrow 0. \end{array}$$

Now modulo  $u^{2^{n+1}}$ , we have  $\text{Res}_A(c_2) = u^2 + v_n u^{2^n+1}$  and thus  $j^*(d_3 c_2) = v_n u^{2^n} \otimes \alpha$ . Since  $\text{Res}_A(c_1) = v_n (\text{Res}_A(c_2))^{2^{n-1}} = v_n u^{2^n}$ , we get  $\lambda_1 + \lambda_2 + \lambda_3 = 1$ , too.  $\square$

Therefore Theorem 4.1 gives

$$\begin{aligned} d_3(y_i c_2) &= y_i c_1 \otimes \alpha = y_1^2 \otimes \alpha \pmod{(y_1, y_2)^3}, \\ d_3(c_1 c_2) &= c_1^2 \otimes \alpha = y_1 y_2 \otimes \alpha \pmod{(y_1, y_2)^3}. \end{aligned}$$

Using these formulae, it is easy to see that  $\tilde{K} = \text{Ker}(d_3|_{\tilde{K}(n)^*(BD_8)})$  is generated as a  $K(n)^*$ -algebra by

$$(5.3) \quad \begin{aligned} &y_1, y_2, c_2^2 \text{ (which gives } c_1), 2c_2, \\ &b_1 = y_1^{2^n-1} c_2, b_2 = y_2^{2^n-1} c_2, y_1 b_2 = y_1 y_2^{2^n-1} c_2. \end{aligned}$$

The last three terms are in  $\tilde{K}$  since  $v_n y_i^{2^n} = 0$  in  $K(n)^*(BD_8)$ . More precisely, we have

**Lemma 5.2.** *In the spectral sequence (5.2), the kernel  $\tilde{K}$  and the homology  $H$  with respect to  $d_3 \otimes \alpha^{-1}$  are given additively by*

$$\begin{aligned} \tilde{K} &\cong \left( \begin{aligned} &(\tilde{K}(n)^*[y_1, y_2]/(\tilde{\pi}, [2](y_i)) \oplus \tilde{K}(n)^*\{c_1\}\{1, 2c_2\}) \\ &+ \tilde{K}(n)^*\{b_1, b_2, y_1 b_2\} \end{aligned} \right) [c_2^2]/(c_2^{2^{n-1}}), \\ H &\cong K(n)^*\{1, y_1, y_2, b_1, b_2, y_1 b_2\}[c_2^2]/(c_2^{2^{n-1}}) \end{aligned}$$

where  $\tilde{\pi} = y_1 y_2 (y_1 +_{\tilde{K}(n)} y_2)$ . Note that  $2b_i = 2c_2 y_i^{2^n-1}$ .  $\square$

A similar statement holds for the associated compact group:

**Lemma 5.3.** *In the spectral sequence  $1 \rightarrow \tilde{G}' \rightarrow \tilde{G} \rightarrow V \rightarrow 1$ , the kernel  $\tilde{K}$  and the homology  $H$  are given additively by*

$$\begin{aligned} \tilde{K} &\cong (\tilde{K}(n)^*[y_1, y_2]/(\tilde{\pi}, [2](y_i)) \oplus \tilde{K}(n)^*\{c_1\}\{1, 2c_2\}[c_2^2]), \\ H &\cong K(n)^*\{1, y_1, y_2\}[c_2^2]. \quad \square \end{aligned}$$

We want to show that all elements in  $\tilde{K}$  are permanent. Let  $t = \text{Tr}_{G_{12} \times A_3}^G(c_2 \otimes 1)$ . By the double coset formula, we get

$$\text{Res}_{G_{12}}^G(t) = \sum_{g \in G_{12} \backslash G/G_{12} \times A_3} \text{Tr}_{G_{12} \cap (G_{12} \times A_3)^g}^{G_{12}} \text{Res}_{G_{12} \cap (G_{12} \times A_3)^g}^{(G_{12} \times A_3)^g} g^*(c_2 \otimes 1).$$

Here  $G_{12} \backslash G/G_{12} \times A_3 \cong A_4$  and  $G_{12} \cap (G_{12} \times A_3)^g = G_{12}$  for all  $g \in A_4$ . Hence

$$\text{Res}_{G_{12}}^G(t) = c_2 + a_4^* c_2 = 2c_2.$$

Therefore,  $t \in \tilde{K}(n)^*(BG)$  corresponds to the element  $[2c_2] \in E_\infty^{0,*}$ .

Next we look for elements corresponding to the  $b_i$  of Lemma 5.2. Let  $A' = \langle a_3 a_4 \rangle$ ; this is cyclic of order 4. Let  $\rho'_{A'}$  be a faithful one-dimensional representation of  $A'$ .

Set  $\rho' = \text{Ind}_{A'}^{G_{34}}(\rho_{A'})$  and  $c'_2 = c_2(\rho')$ . Define  $t_i = \text{Tr}_{G_{34} \times A_i}^G(c'_2 \otimes 1)$  for  $i = 1, 2$ . We claim the following identities:

$$(5.4) \quad v_n b_1 = \text{Res}_{G_{12}}^G(t - t_2 + y_2^2 - y_1 y_2),$$

$$(5.5) \quad v_n b_2 = \text{Res}_{G_{12}}^G(t - t_1 + y_1^2 - y_1 y_2).$$

It suffices to check them on the abelian subgroups of  $G_{12}$ , by [3]. Thus we need to compute the restrictions to  $C \times A_i$  and  $A$ . Since  $\rho$  restricts to  $\eta + \eta\lambda_i$  on  $C \times A_i$  and to  $\rho_A + \rho_A^3$  on  $A$ , we have

$$\begin{aligned} \text{Res}_{C \times A_i}^G(t) &= 2u(u + \tilde{K}(n) y_i), & \text{Res}_A^G(t) &= 2z[3](z), \\ \text{Res}_{C \times A_1}^{G_{12}}(b_1) &= u(u + \tilde{K}(n) y_1)y_1^{2^n-1}, & \text{Res}_A^{G_{12}}(b_i) &= z[3](z)([2]z)^{2^n-1}, \\ \text{Res}_{C \times A_2}^{G_{12}}(b_1) &= 0. \end{aligned}$$

Here  $z = c_1(\rho_A)$  denotes the generator of  $\tilde{K}(n)^*(BA) \cong \tilde{K}(n)^*[[z]]/[4](z)$ ; clearly  $y_1, y_2$  restrict to  $[2](z)$ .

Now  $C \times A_1 \backslash G/G_{34} \times A_2 \cong 1$  and  $(C \times A_1) \cap (G_{34} \times A_2) = C$ ; the double coset formula then says

$$\begin{aligned} \text{Res}_{C \times A_1}^G(t_2) &= \text{Tr}_C^{C \times A_1} \text{Res}_C^{G_{34} \times A_2}(c'_2 \otimes 1) = \text{Tr}_C^{C \times A_1}(u^2) \\ &= u^2 \text{Tr}_{\{1\}}^{A_1}(1) = u^2(2 - v_n y_1^{2^n-1}) \end{aligned}$$

where we used the fact (see, e.g., [3] or [5])

$$(5.6) \quad \text{Tr}_{\{1\}}^{A_1}(1) = \frac{[2](y_1)}{y_1} = 2 - v_n y_1^{2^n-1}.$$

Similarly, we have  $C \times A_2 \backslash G/G_{34} \times A_2 \cong A_1$  and  $(C \times A_2) \cap (G_{34} \times A_2) = C \times A_2$ , whence

$$\begin{aligned} \text{Res}_{C \times A_2}^G(t_2) &= \text{Res}_{C \times A_2}^{G_{34} \times A_2}(1 + a_1^*)(c'_2 \otimes 1) \\ &= c_2(2\eta) + c_2(\eta \otimes \lambda_2) = u^2 + (u + \tilde{K}(n) y_2)^2. \end{aligned}$$

By the double coset formula again,

$$\text{Res}_A^G(t_i) = \text{Tr}_C^A(u^2) = z^2 \text{Tr}_C^A(1) = z^2 \frac{[4](z)}{[2](z)}.$$

Thus

$$\text{Res}_{C \times A_1}^G(t - t_2 + y_2^2 - y_1 y_2) - \text{Res}_{C \times A_1}^{G_{12}}(v_n b_1) = (u(u + \tilde{K}(n) y_1) - u^2)(2 - v_n y_1^{2^n-1}).$$

Let  $\chi$  be a generalized character of  $C \times A_1$ . If  $\chi(y_1) = 0$ , then

$$\chi((u(u + \tilde{K}(n) y_1) - u^2)(2 - v_n y_1^{2^n-1})) = 2(\chi(u)^2 - \chi(u)^2) = 0,$$

whereas if  $\chi(y_1) \neq 0$ , then  $\chi(2 - v_n y_1^{2^n-1}) = [2](\chi(y_1))/\chi(y_1) = 0$ . Secondly,

$$\text{Res}_{C \times A_2}^G(t - t_2 + y_2^2 - y_1 y_2) - \text{Res}_{C \times A_2}^{G_{12}}(v_n b_1) = 2u(u + \tilde{K}(n) y_2) - (u + \tilde{K}(n) y_2)^2 - u^2 + y_2^2.$$

Any generalized character  $\chi$  with  $\chi(u) = 0$  or  $\chi(y_2) = 0$  clearly annihilates this expression. So assume, without loss of generality, that  $\chi(u) = \pi$ , where  $\pi$  is a uniformizing element. Any other nonzero root of the 2-series is of the form  $\zeta\pi$  for a  $(2^n - 1)$ -st root of unity  $\zeta$ . Then  $[\zeta](\pi) = \zeta\pi$  and  $\pi + \tilde{K}(n) \zeta\pi = \pi + \tilde{K}(n) [\zeta](\pi) = [1 + \zeta](\pi) = (1 + \zeta)\pi$ , since  $(1 + \zeta)^{2^n-1} \equiv 1 \pmod{2}$ . Thus

$$\chi(2u(u + \tilde{K}(n) y_2) - (u + \tilde{K}(n) y_2)^2 - u^2 + y_2^2) = 2\pi(1 + \zeta)\pi - (1 + \zeta)^2 \pi^2 - \pi^2 + \zeta^2 \pi^2 = 0.$$

Finally,

$$\begin{aligned} & \text{Res}_A^G(t - t_2 + y_2^2 - y_1y_2) - \text{Res}_A^{G^{12}}(v_nb_1) \\ &= 2z[3](z) - z^2 \frac{[4](z)}{[2](z)} - v_n z[3](z)([2](z))^{2^n-1} = (z[3](z) - z^2) \frac{[4](z)}{[2](z)} \end{aligned}$$

where we used  $v_n([2](z))^{2^n-1} = 2 - [4](z)/[2](z)$ . Let  $\alpha$  denote the value of a character on  $z$ . Then either  $[4](\alpha)/[2](\alpha) = 0$ , if  $[2](\alpha) \neq 0$ , or  $[4](\alpha)/[2](\alpha) = 2$ , if  $[2](\alpha) = 0$ , and in that case  $\alpha[3](\alpha) - \alpha^2 = \alpha(\alpha +_{\tilde{K}(n)} [2](\alpha)) - \alpha^2 = \alpha^2 - \alpha^2 = 0$ . This finishes the proof of equation (5.4). The other equation follows by exchanging the indices 1 and 2. Thus the assumptions of Lemma (3.1) hold, yielding

**Theorem 5.4.** *Let  $G$  be an extraspecial group of order 32. Then  $K(n)^*(BG)$  is concentrated in even degrees and generated by transfers of Euler classes.*  $\square$

In the compact case it suffices to show that  $c_1$  is a permanent cycle. Suppose that  $d_r c_1 = x \otimes f\alpha \neq 0$  for  $3 \leq r \leq 2^{n+1} - 1$ . Note that  $x \otimes f\alpha^2 = x \otimes f\pi \neq 0$  in  $E_r^{*,*}$ . But  $d_r(c_1 \otimes \alpha)$  must be zero in  $E_r^{*,*}$ , since it is so in  $E_4^{*,*}$ . This is a contradiction. The term  $E_{2^{n+1}}^{*,*}$  is generated by even-dimensional elements, and  $c_1$  is a permanent cycle.

From Lemma (5.3) and the formula in the proof of Lemma (3.1), we get

$$\begin{aligned} (5.7) \quad & \text{gr } \tilde{K}(n)^*(B\tilde{G}) \cong \tilde{K} \oplus K \otimes \mathbb{F}_2[y_3, y_4]^+ / (\pi_{34}) \oplus H \otimes \mathbb{F}_2[y_3, y_4] / (q_{34}) \{ \pi_{34} \} \\ & \cong (\tilde{K}(n)^*[y_1, y_2] / (\tilde{\pi}_{12}, [2](y_i)) \oplus \tilde{K}(n)^*\{c_1\}) \{1, 2c_2\} \\ & \oplus ((K(n)^*[y_1, y_2] / (\pi_{12}, [2](y_i)) \oplus K(n)^*\{c_1\}) \otimes \mathbb{F}_2[y_3, y_4]^+ / (\pi_{34})) \\ & \oplus (K(n)^*\{1, y_1, y_2\} \otimes \mathbb{F}_2[y_3, y_4] / (q_{34}) \{ \pi_{34} \}) [c_2^2]. \end{aligned}$$

6. EULER CHARACTERISTICS OF EXTRASPECIAL  $p$ -GROUPS

In this section we give the Euler characteristic of an extraspecial  $p$ -group. The result is not new; the same formula was obtained by Brunetti [2].

The Morava  $K$ -theory Euler characteristic  $\chi_{n,p}(G)$  of a finite group  $G$ , i.e., the difference between the ranks of the even and odd degree parts of  $K(n)^*(BG)$ , can be computed using the formula from [3]:

$$(6.1) \quad \chi_{n,p}(G) = \sum_{A < G} \frac{|A|}{|G|} \mu_{\mathcal{A}(G)}(A) \chi_{n,p}(A)$$

where the sum is over all abelian subgroups  $A < G$  and  $\mu_{\mathcal{A}(G)}$  is a Möbius function defined recursively by

$$(6.2) \quad \sum_{A' < A} \mu_{\mathcal{A}(G)}(A') = 1$$

where the sum is over all abelian subgroups  $A' < G$  contained in  $A$ . In particular,  $\mu_{\mathcal{A}(G)}(A) = 1$  when  $A$  is maximal. It is easy to see that one only has to consider subgroups arising as intersections of maximal ones. Furthermore, one clearly has  $\chi_{n,p}(A) = |A_{(p)}|^n$  where  $A_{(p)}$  denotes the  $p$ -part of the abelian group  $A$ .

The abelian subgroups of an extraspecial  $p$ -group  $D(m) = p_+^{1+2m}$  are in one-to-one correspondence with those subspaces  $W$  of the central quotient  $V \cong \mathbb{F}_p^{2m}$  that are isotropic with respect to the bilinear form

$$b(x, y) = x_1y_2 + x_2y_1 + \dots + x_{2m-1}y_{2m} + x_{2m}y_{2m-1}.$$

Let  $\alpha_i^{(m)}$  denote the number of such subspaces of dimension  $i$ . Note that the maximal dimension of a  $b$ -isotropic subspace is  $m$ .

The following lemma is an easy exercise in counting:

**Lemma 6.1.**  $\alpha_i^{(m)} = \prod_{j=1}^i \frac{p^{2(m-j+1)} - 1}{p^j - 1}$ . □

The Möbius function on abelian subgroups can be computed via a Möbius function on  $b$ -isotropic subspaces defined as in (6.2). Let  $\gamma_k^{(m)}$  denote its value on a subspace of dimension  $k$ : by symmetry, it is constant on subspaces of the same rank. Furthermore, it only depends on the *codimension* of a  $b$ -isotropic subspace in a maximal one, independent of  $m$ ; this follows by considering  $W^\perp/W$ . The following formula can be proved inductively, see [2].

**Lemma 6.2.**  $\gamma_k^{(m)} = (-p)^{(m-k)^2}$ . □

Since a  $b$ -isotropic subspace  $W$  of dimension  $i$  gives rise to an abelian subgroup of index  $2m - i$ , we arrive at

**Proposition 6.3** ([2]). *The Morava  $K$ -theory Euler characteristic of  $G = p_+^{1+2m}$  is given by*

$$\chi_{n,p}(G) = \sum_{i=0}^m \frac{\alpha_i^{(m)} \gamma_i^{(m)}}{p^{2m-i}} p^{(i+1)n} = \sum_{i=0}^m (-1)^{m-i} \alpha_i^{(m)} p^{(m-i-1)^2 + (n-1)(i+1)}$$

with  $\alpha$  and  $\gamma$  as in the two lemmas above.

For example, for  $D_8$  and  $D(2) = 2_+^{1+4}$  we obtain

$$\begin{aligned} \chi_{n,2}(D_8) &= \frac{3}{2}4^n - \frac{1}{2}2^n, \quad \text{and} \\ \chi_{n,2}(D(2)) &= \frac{15}{4}(8^n - 4^n) + 2^n. \end{aligned}$$

This agrees with the Euler characteristics that we can compute using Corollary 3.2, as we shall now see. Let  $Y_{i,j} = K(n)^*[y_i, y_j]^+ / (\pi, y_i^{2^n}, y_j^{2^n})$ , and denote by  $\chi(-)$  the dimension of a  $K(n)^*$ -vector space. Then one easily computes  $\chi(Y_{i,j}) = 3 \cdot (2^n - 1)$ . We have  $K(n)^*(BD_8) \cong (Y_{1,2} \oplus K(n)^*\{1, c_1\}) \otimes \mathbb{Z}/2[c_2] / (c_2^{2^n-1})$ . Hence

$$\chi(K(n)^*(BD_8)) = (3 \cdot (2^n - 1) + 2)2^{n-1} = 3 \cdot 2^{2n-1} - 2^{n-1}.$$

Next consider  $K(n)^*(BD(2))$ . First note

$$\begin{aligned} \chi(\tilde{K}/2) &= \chi((Y_{1,2} + K(n)^*\{1, c_1\})\{1, 2c_2\} \otimes \mathbb{Z}/2[c_2^2] / (c_2^{2^n-1})) \\ &= (3 \cdot (2^n - 1) + 2) \cdot 2 \cdot 2^{n-2} = (6 \cdot 2^n - 2) \cdot 2^{n-2}, \end{aligned}$$

where we used the fact that we can take either  $y_i^j c_2$  or  $2y_i^j c_2$  as a basis element and may neglect the summand  $K(n)^*\{b_1, b_2, y_2 b_1\}$ . Then

$$\begin{aligned} \chi(K \otimes Y_{3,4}) &= \chi((Y_{1,2} + K(n)^*\{1, c_1, b_1, b_2, y_2 b_1\}) \otimes Y_{34}) \cdot 2^{n-2} \\ &= (3(2^n - 1) + 5) \cdot 3(2^n - 1) \cdot 2^{n-2} = (9 \cdot 2^{2n} - 3 \cdot 2^n - 6) \cdot 2^{n-2}; \\ \chi(H \otimes \mathbb{Z}/2[y_3, y_4] / (q_{34}, y_3^{2^n-1}, y_4^{2^n-1})\{\pi\}) &= (6 \cdot (2^n - 1)(2^n - 2)) \cdot 2^{n-2} \\ &= (6 \cdot 2^{2n} - 18 \cdot 2^n + 12) \cdot 2^{n-2}. \end{aligned}$$

Therefore, we have  $\chi(K(n)^*(BD(2))) = (15 \cdot 2^{2n} - 15 \cdot 2^n + 4) \cdot 2^{n-2} = \chi_{n,2}(D(2))$ .

## REFERENCES

1. M. Brunetti. Morava  $K$ -theories and dihedral groups. *Ricerche Mat.* **44** (1995), 31–40. MR **98f**:55004
2. M. Brunetti. The  $K(n)$ -Euler characteristic of extraspecial  $p$ -groups. *J. Pure Appl. Algebra* **155** (2001), 105–113. MR **2002d**:55002
3. M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.* **13** (2000), 553–594. MR **2001k**:55015
4. J. R. Hunton. The Morava  $K$ -theories of wreath products. *Math. Proc. Cambridge Philos. Soc.* **107** (1990), 309–318. MR **91a**:55004
5. I. Kriz. Morava  $K$ -theory of classifying spaces: Some calculations. *Topology* **36** (1997), 1247–1273. MR **99a**:55016
6. I. Kriz and K. Lee. Odd-degree elements in the Morava  $K(n)$  cohomology of finite groups. *Topology Appl.* **103** (2000), 229–241. MR **2001f**:55008
7. D. G. Quillen. The mod 2 cohomology rings of extra-special 2-groups and the spinor groups. *Math. Ann.* **194** (1971), 197–212. MR **44**:7582
8. B. Schuster. On the Morava  $K$ -theory of some finite 2-groups. *Math. Proc. Cambridge Philos. Soc.* **121** (1997), 7–13. MR **97i**:55008
9. M. Tezuka and N. Yagita. Cohomology of finite groups and Brown-Peterson cohomology. *Algebraic Topology (Arcata, CA, 1986)*, 396–408, Lecture Notes in Math. **1370**, Springer-Verlag, Berlin, 1989. MR **90i**:55011
10. M. Tezuka and N. Yagita. Cohomology of finite groups and Brown-Peterson cohomology II. *Homotopy theory and related topics (Kinosaki, 1988)*, 57–69. Lecture Notes in Math. **1418**, Springer-Verlag, Berlin, 1990. MR **92b**:55003
11. N. Yagita. Cohomology for groups of rank $_p(G) = 2$  and Brown-Peterson cohomology. *J. Math. Soc. Japan* **45** (1993), no. 4, 627–644. MR **94k**:55008

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