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CONTROL OF RADII OF CONVERGENCE AND EXTENSION OF SUBANALYTIC FUNCTIONS

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ABSTRACT. Let g: $U \to \mathbb{R}$ denote a real analytic function on an open subset U of $\mathbb{R}^n$, and let $\Sigma \subset \partial U$ denote the points where g does not admit a local analytic extension. We show that if g is semialgebraic (respectively, globally subanalytic), then Σ is semialgebraic (respectively, subanalytic) and g extends to a semialgebraic (respectively, subanalytic) neighbourhood of $\overline{U}\backslash\Sigma$. (In the general subanalytic case, Σ is not necessarily subanalytic.) Our proof depends on controlling the radii of convergence of power series G centred at points b in the image of an analytic mapping φ, in terms of the radii of convergence of $G \circ \widehat{\varphi}_a$ at points $a \in \varphi^{-1}(b)$, where $\widehat{\varphi}_a$ denotes the Taylor expansion of φ at a.

1. INTRODUCTION

Let g: $U \to \mathbb{R}$ denote a real-analytic function defined on an open subset U of $\mathbb{R}^n$, such that g is *semialgebraic* (respectively, *subanalytic*); i.e., the graph of g is semialgebraic (respectively, subanalytic) as a subset of $\mathbb{R}^n \times \mathbb{R}$. Chris Miller raised the following questions about extension of the domain of g:

(1.1) Let Σ denote the subset of the boundary ∂U of U consisting of points where g does not admit an analytic extension (to some neighbourhood). Is Σ a closed semialgebraic (respectively, subanalytic) subset of ∂U?

If the answer to (1.1) is "yes", then:

(1.2) Can g be extended to an analytic function defined on a semialgebraic (respectively, subanalytic) neighbourhood of $\overline{U}\backslash\Sigma$?

(1.3) Can g be extended to a neighbourhood of $\overline{U}\backslash\Sigma$ as an analytic function that is semialgebraic (respectively, subanalytic)?

Note that if g: $U \to \mathbb{R}$ is semialgebraic (respectively, subanalytic and bounded), then U is semialgebraic (respectively, subanalytic). If g is semialgebraic, then a local analytic extension at a point of ∂U is semialgebraic (on a suitable neighbourhood). An analytic function that is semialgebraic is called *algebraic* or *Nash*.

We will show that the answer to (1.1) is "yes" if g is a semialgebraic function or a bounded (or global) subanalytic function (Theorem 2.3 below); the proof is a direct application of the "graphic point" argument used in [BM, §7] to show that

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the set of smooth points of a subanalytic set is subanalytic. The answer to (1.1) is "no" in the general subanalytic case (Example 2.1).

The answer to (1.2) is "yes" under the hypotheses above (Theorem 2.4). It follows that the answer to (1.3) is "yes" in the semialgebraic (or global semianalytic) case (Corollary 2.5 and Remark 2.6(2)), but (1.3) remains open in the general global subanalytic case. Our proof of Theorem 2.4 depends on controlling the radii of convergence of g at points of $\partial U \backslash \Sigma$. This essentially means controlling the radii of convergence of power series G centred at points b in the image of an analytic mapping φ, in terms of the radii of convergence of $G \circ \widehat{\varphi}_a$ at points $a \in \varphi^{-1}(b)$. ($G \circ \widehat{\varphi}_a$ denotes the formal composition of G with the Taylor expansion $\widehat{\varphi}_a$ of φ at a.) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let V be an open neighbourhood of 0 in $\mathbb{K}^n$, and let φ: $V \to \mathbb{K}^n$ denote an analytic mapping whose Jacobian determinant

$$\Delta := \det \frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(x_1, \ldots, x_n)}$$

does not vanish identically in a neighbourhood of 0. We write $\mathcal{O}_b$ to denote the ring of germs of analytic functions at a point b of $\mathbb{K}^n$, and $\widehat{\mathcal{O}}_b$ to denote the formal completion of $\mathcal{O}_b$; i.e., $\widehat{\mathcal{O}}_b$ is the ring of formal power series centred at b. Tougeron [T, 5.10] and Chaumat and Chollet [CC, §17] have shown that, for a given point $a \in V$, there exist constants $\lambda \in \mathbb{N}$ and $c > 0$ such that, if $G \in \widehat{\mathcal{O}}_{\varphi(a)}$ and $F = G \circ \widehat{\varphi}_a$ converges in a ball $|x - a| < r$, where $r \leq 1$, then G has radius of convergence $r_G \geq c r^{\lambda}$. The power λ can be chosen to be independent of a, but c cannot be chosen uniformly:

Example 1.1. Let φ: $\mathbb{C} \to \mathbb{C}$ denote the mapping $\varphi(x) = x^2$. If $G \in \widehat{\mathcal{O}}_0$ and $F = G \circ \widehat{\varphi}_0$ converges in a disk $|x| < r$, then G converges in $|y| < r^2$. On the other hand, if $f(x) = x$ and $a \neq 0$, then we can solve $f = g \circ \varphi$ for $g \in \mathcal{O}_{\varphi(a)}$, but g converges only in $|y - \varphi(a)| < |a|^2$.

We will show that, in general, the lack of uniformity in the constant c is of the same nature as in the preceding elementary example:

Theorem 1.2. *Let φ: $V \to \mathbb{K}^n$ denote an algebraic (respectively, analytic) mapping from an open neighbourhood V of 0 in $\mathbb{K}^n$. Assume that the Jacobian determinant Δ of φ does not vanish identically in a neighbourhood of 0. Then we can find a neighbourhood W of 0 in V, constants $\delta > 0$ and $\lambda \in \mathbb{N}$, and a finite filtration of W by closed algebraic (respectively, analytic) subsets,*

$$W = X_0 \supset X_1 \supset \cdots \supset X_{k+1} = \emptyset,$$

such that, for all $j = 0, \ldots, k$, there exists $\alpha = \alpha(j) \in \mathbb{N}^n$ with the properties that, for every $a \in X_j \backslash X_{j+1}$:

(1) *$D^\alpha \Delta(a) \neq 0$, where D^α denotes the partial derivative $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.*

(2) *If $G \in \widehat{\mathcal{O}}_{\varphi(a)}$ and $F = G \circ \widehat{\varphi}_a$ converges in a ball $|x - a| < r$, where $r \leq 1$, then G has radius of convergence*

$$r_G \geq \delta |D^\alpha \Delta(a)|^2 r^\lambda.$$

Corollary 1.3. *Let U be an open subset of $\mathbb{R}^n$ and let g: $U \to \mathbb{R}$ denote an analytic function that is semialgebraic (respectively, globally subanalytic). Let Σ be a closed semialgebraic (respectively, subanalytic) subset of ∂U, and assume that g*

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admits a local analytic extension at each point of $\partial U \backslash \Sigma$. *Then there is a partition* $\mathcal{P} = \{S_i\}$ *of* $\partial U \backslash \Sigma$ *such that* $\mathcal{P}$ *is finite (respectively, locally finite in* ∂U*), each* S_i *is a semialgebraic (respectively, bounded subanalytic) subset of* $\mathbb{R}^n$*, and, for each* i*, there is a continuous semialgebraic (respectively, subanalytic) function* $\rho_i \colon S_i \to \mathbb{R}$ *such that, for all* $b \in S_i$*,* g *extends to an analytic function that is semialgebraic (respectively, subanalytic) on the ball of radius* $\rho_i(b)$ *centred at* b.

We sketch a proof of Theorem 1.2 in Section 3 below; the theorem follows immediately from estimates used by Augustin Mouze in [M] to prove a more precise version of the theorem of Chaumat and Chollet [CC]. In Section 2, we deduce Corollary 1.3 from Theorem 1.2 and we reply to Miller's questions. I am happy to acknowledge my discussions with Anne-Marie Chollet, Augustin Mouze and Vincent Thilliez concerning [CC] and [M], and to thank Artur Piękosz for pointing out some errors in earlier manuscripts. Piękosz has extended Theorem 2.3 to "k-subanalytic functions" (composites of global subanalytic functions and power functions).

2. Extension of semialgebraic and subanalytic functions

The following example shows that the answer to question (1.1) above is "no" in general for a subanalytic function on a subanalytic domain.

Example 2.1. Let $U = \{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$ and let

$$g(x,y) = \frac{1}{x^2\left(\sin^2 \dfrac{1}{x} + y^2\right)}.$$

The denominator of $g(x,y)$ has zero-set

$$Z = \{x = 0\} \cup \left\{ y = 0,\ \sin \frac{1}{x} = 0 \right\},$$

and is analytic except at points where $x = 0$. Therefore, $g(x,y)$ is subanalytic. (In fact, it is semianalytic.) But $\Sigma = Z \cap \{x \geq 0,\ y \geq 0\}$ is not subanalytic.

Let $g \colon U \to \mathbb{R}$ denote an analytic function on an open subset U of $\mathbb{R}^n$ such that g is semialgebraic (respectively, subanalytic). We compactify $\mathbb{R}^n$ to real projective space $\mathbb{P}^n$, and we compactify $\mathbb{R}$ to $\mathbb{P}^1 = S^1$; write ∞ for the point at infinity of the latter. The following conditions are obviously equivalent:

(2.2)

(i) g is semialgebraic.

(ii) The graph of g is semialgebraic as a subset of $\mathbb{R}^n \times S^1$.

(iii) The graph of g is semialgebraic as a subset of $\mathbb{P}^n \times S^1$.

For the subanalytic analogues of these conditions, we have (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). We say that g is a *global subanalytic* function if it satisfies the analogue of (ii); i.e., the graph of g is subanalytic as a subset of $\mathbb{R}^n \times S^1$. Clearly, a bounded subanalytic function is globally subanalytic.

Theorem 2.3. *Let* U *be an open subset of* $\mathbb{R}^n$ *and let* $g \colon U \to \mathbb{R}$ *denote an analytic function. Let* $\Sigma \subset \partial U$ *denote the set of points at which* g *does not admit an analytic extension. If* g *is semialgebraic (respectively, globally subanalytic), then* Σ *is a closed semialgebraic (respectively, subanalytic) set.*

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Proof. Let X denote the closure of the graph of g in $\mathbb{P}^n \times S^1$. Since the question is local, if g is globally subanalytic, then we can reduce to the case that the graph of g is subanalytic in $\mathbb{P}^n \times S^1$. So we assume that X is semialgebraic (respectively, subanalytic).

By the uniformization theorem for semialgebraic or subanalytic sets [BM, Theorem 0.1], there is a compact real algebraic (respectively, analytic) manifold M of dimension n, and an algebraic (respectively, analytic) mapping $\Phi = (\varphi, f)$: $M \to \mathbb{P}^n \times S^1$ such that $\Phi(M) = X$. We can assume that Φ (and therefore φ) has maximum rank n on each component of M. There is a bound s on the number of connected components of the fibres of φ [BM, Theorem 3.14]. Let M^s_φ denote the *s-fold fibre-product* of M with respect to φ; i.e.,

$$M^s_\varphi := \{\underline{x} = (x^1, \dots, x^s) \in M^s : \ \varphi(x^1) = \cdots = \varphi(x^s)\},$$

and let $\underline{\varphi}$: $M^s_\varphi \to \mathbb{P}^n$ denote the mapping induced by φ.

We say that $\underline{a} \in M^s_\varphi$ is an *s-fold graphic point* of Φ if there exists $g^{\underline{a}} \in \mathcal{O}_{\underline{\varphi}(\underline{a})}$ such that $f_{a^i} = g^{\underline{a}} \circ \varphi_{a^i}$, $i = 1, \dots, s$, where $\underline{a} = (a^1, \dots, a^s)$ (and f_{a^i}, φ_{a^i} denote the germs of f, φ at a^i). Let $\underline{E} \subset M^s_\varphi$ denote the subset of points that are *not* s-fold graphic points of Φ. Then $\underline{E}$ is a closed algebraic (respectively, analytic) subset of M^s_φ [BM, Corollary 7.13]. But g extends to a neighbourhood of a point $b \in \partial U$ precisely when $b \notin \underline{\varphi}(\underline{E})$ and $(b, \infty) \notin X$. □

Proof of Corollary 1.3. Let g: $U \to \mathbb{R}$ be an analytic function that is semialgebraic (respectively, globally subanalytic). Let $Y \subset \mathbb{P}^n \times S^1$ (respectively, $Y \subset \mathbb{R}^n \times S^1$) denote the closure of the graph of g. By the uniformization theorem [BM, Theorem 0.1], there is an algebraic (respectively, analytic) manifold M and a proper algebraic (respectively, analytic) mapping $\Phi = (\varphi, f)$: $M \to \mathbb{P}^n \times S^1$ (respectively, $\Phi = (\varphi, f)$: $M \to \mathbb{R}^n \times S^1$) such that $\Phi(M) = Y$. Again, we can assume that Φ (and hence φ) has maximum rank n on each component of M. Suppose $a \in M$ and $b = \varphi(a) \in \partial U$. If g extends to an analytic function in a neighbourhood of b, then $f_a = g_b \circ \varphi_a$, where g_b denotes the germ of an extension at b. Corollary 1.3 is now a simple consequence of Theorem 1.2. □

Theorem 2.4. *Let U be an open subset of $\mathbb{R}^n$ and let g: $U \to \mathbb{R}$ be an analytic function that is semialgebraic (respectively, globally subanalytic). Let Σ be a closed semialgebraic (respectively, subanalytic) subset of ∂U. Assume that g admits a local analytic extension at each point of $\partial U \backslash \Sigma$. Then g extends to an analytic function defined in a semialgebraic (respectively, subanalytic) neighbourhood of $\overline{U} \backslash \Sigma$.*

Proof. Consider a partition $\mathcal{P} = \{S_i\}$ of $\partial U \backslash \Sigma$ and continuous semialgebraic (respectively, subanalytic) functions ρ_i: $S_i \to \mathbb{R}$ satisfying the conclusion of Corollary 1.3. For each i, set

$$U_i = \bigcup_{b \in S_i} \{y : \ |y - b| < \rho_i(b)\}.$$

Then U_i is an open semialgebraic (respectively, subanalytic) neighbourhood of S_i. Let $V' = \bigcup_i U_i$. Then V' is an open semialgebraic (respectively, subanalytic) neighbourhood of $\partial U \backslash \Sigma$. By Corollary 1.3, g extends to $U \cup V'$ but, in general, only as a multivalued function.

Let $\mathcal{T}$ denote a finite semialgebraic triangulation of $\mathbb{P}^n$ (respectively, a locally finite subanalytic triangularization of $\mathbb{R}^n$) that is compatible with U, Σ and each U_i [H], and let $\mathcal{B}(\mathcal{T})$ denote the barycentric subdivision of $\mathcal{T}$. Let V denote the union

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of all open simplices of $\mathcal{B}(\mathcal{T})$ that are adherent to $\partial U \backslash \Sigma$. (An *open simplex* means a simplex minus its boundary.) Then V is an open semialgebraic (respectively, subanalytic) neighbourhood of $\partial U \backslash \Sigma$, and g extends to an analytic function on $U \cup V$. □

Corollary 2.5. *Under the hypotheses of Theorem 2.4 in the semialgebraic case, g extends to a neighbourhood of $\overline{U} \backslash \Sigma$ as an analytic function that is semialgebraic.*

Proof. Since g is semialgebraic, there is a proper real algebraic subset Z of $\mathbb{R}^{n+1}$ containing the graph of g, and thus containing the graph of an analytic extension h of g to a semialgebraic neighbourhood of $\overline{U} \backslash \Sigma$. It follows that h is semialgebraic. □

Remarks 2.6. (1) The assertions of Theorems 2.3 and 2.4 (and of Corollary 2.5) under the weaker hypothesis that g is semialgebraic (respectively, globally subanalytic) but not necessarily analytic throughout U follow immediately from the theorems as stated.

(2) There are analogues of Theorems 2.3, 2.4 and Corollary 2.5 for a bounded (or global) semianalytic function $g \colon U \to \mathbb{R}$.

3. Control of radii of convergence

In this section, we show that Theorem 1.2 follows from the estimates used by Mouze [M] to prove his generalization of the theorem of Tougeron and Chaumat-Chollet. Let $\varphi \colon V \to \mathbb{K}^n$ denote an algebraic (respectively, analytic) mapping, where V is a neighbourhood of 0 in $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let

$$\frac{\partial \varphi}{\partial x} = \frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(x_1, \ldots, x_n)},$$

so that

$$\frac{\partial \varphi}{\partial x} \cdot \left(\frac{\partial \varphi}{\partial x}\right)^{\#} = \Delta \cdot I,$$

where $(\cdot)^{\#}$ denotes the transposed matrix of cofactors, $\Delta = \det(\partial\varphi/\partial x)$ and I is the identity matrix. Write $T_{(i)}^{(j)}$ for the ji cofactor of $\partial\varphi/\partial x$. Shrinking V if necessary, we can choose constants $c_1 \geq 1$ and $c_2 \geq 1$ such that, for all $\alpha \in \mathbb{N}^n$ and $a \in V$, $|D^\alpha \Delta(a)| \leq \alpha! c_1 c_2^{|\alpha|}$ and each $|D^\alpha T_{(i)}^{(j)}(a)| \leq \alpha! c_1 c_2^{|\alpha|}$.

We can assume that $\varphi(0) = 0$. Let $\mu(a)$ (respectively, $\nu(a)$) denote the order of vanishing of Δ (respectively, of $(\partial\varphi/\partial x)^{\#}$) at a point $a \in V$.

Consider $f = g \circ \varphi$, where g is an analytic function in a neighbourhood of $\varphi(a)$, $a \in V$. By the chain rule,

$$\sum_{j=1}^{n} \left(\frac{\partial g}{\partial y_j} \circ \varphi\right) \frac{\partial \varphi_j}{\partial x_i} = \frac{\partial f}{\partial x_i}, \qquad i = 1, \ldots, n.$$

By Cramer's rule,

$$\Delta \cdot \left(\frac{\partial g}{\partial y_j} \circ \varphi\right) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} T_{(i)}^{(j)}, \qquad j = 1, \ldots, n.$$

From the formulas obtained by repeated differentiation, we get analytic functions T_α^β on V, for all $\alpha, \beta \in \mathbb{N}^n$, such that

$$\Delta^{2|\beta|-1} \cdot \left((D^\beta g) \circ \varphi\right) = \sum_{|\alpha| \leq |\beta|} T_\alpha^\beta D^\alpha f,$$

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where, at each $a \in V$, T_α^β has order $\geq |\alpha| - \mu(a) + |\beta|\big(\mu(a) + \nu(a) - 1\big)$ (cf. [M, Lemme 3]). Moreover, there are constants $c_3 \geq 1$, $c_4 \geq 1$ depending only on c_1, c_2 and n, such that, for all $\alpha, \beta, \gamma \in \mathbb{N}^n$ and all $a \in V$,

$$\left|\frac{D^\gamma T_\alpha^\beta(a)}{\gamma!}\right| \leq c_3^{|\beta|} c_4^{|\beta|+|\gamma|-|\alpha|} \frac{(|\beta| + |\gamma| - |\alpha|)!}{|\gamma|!}$$

[M, Lemme 4].

Let $a \in V$ and let $b = \varphi(a)$. If $F \in \widehat{\mathcal{O}}_a$ and $\alpha \in \mathbb{N}^n$, let $D^\alpha F$ denote the formal derivative of F of order α; thus, if $f \in \mathcal{O}_a$ and $\widehat{f}_a$ denotes the Taylor series of f at a, then $D^\alpha \widehat{f}_a = (D^a f)\widehat{\ }_a$. Consider $F = G \circ \widehat{\varphi}_a$, where G is a formal power series centred at b. Write $G = \sum_{\beta \in \mathbb{N}^n} G_\beta (y-b)^\beta$, $F = \sum_{\alpha \in \mathbb{N}^n} F_\alpha (x-a)^\alpha$. Assume that F converges in a ball $|x - a| \leq r$, where $r \leq 1$; thus, if $c = 1/r$, then there is a constant c' such that $|F_\alpha| \leq c' c^{|\alpha|}$, for all $\alpha \in \mathbb{N}^n$. As above, for all $\beta \in \mathbb{N}^n$,

$$\widehat{\Delta}_a^{2|\beta|-1} \cdot \big((D^\beta G) \circ \widehat{\varphi}_a\big) = \sum_{|\alpha| \leq |\beta|} (T_\alpha^\beta)\widehat{\ }_a \cdot D^\alpha F. \tag{3.1}$$

Let H^β denote the right-hand side of (3.1); write $H^\beta = \sum_{\gamma \in \mathbb{N}^n} H_\gamma^\beta (x-a)^\gamma$. It is not difficult to estimate

$$|H_\gamma^\beta| \leq 2^{|\beta|+|\gamma|} (2n)^{|\beta|} |\beta|! c_3^{|\beta|} c_4^{|\beta|+|\gamma|-1} c_5 c' c^{|\gamma|+\mu(a)-|\beta|(\mu(a)+\nu(a)-1)},$$

where c_5 depends only on c_3, c_4 and n [M, (5.9)].

We now compare terms of degree $(2|\beta| - 1)\mu(a)$ in (3.1). Order multi-indices $\gamma \in \mathbb{N}^n$ according to the lexicographic order of $(|\gamma|, \gamma_1, \ldots, \gamma_n)$. Let α denote the smallest γ such that $D^\gamma \Delta(a) \neq 0$. Then $(D^{(2|\beta|-1)\alpha} \Delta^{2|\beta|-1})(a) = \big(D^\alpha \Delta(a)\big)^{2|\beta|-1}$ and, from (3.1),

$$\big(D^\alpha \Delta(a)\big)^{2|\beta|-1} \beta! G_\beta = H_{(2|\beta|-1)\alpha}^\beta.$$

We obtain

$$|G_\beta| \leq c_6 c_7^{|\beta|} c' c^{|\beta|(\mu(a)-\nu(a)+1)} \cdot \frac{1}{(D^\alpha \Delta(a))^{2|\beta|-1}},$$

where c_6, c_7 depend only on n, $\mu(a)$, c_3, c_4, c_5. In other words, G converges in an open ball centred at b, of radius at least

$$\frac{1}{c_7} |D^\alpha \Delta(a)|^2 r^{\mu(a)-\nu(a)+1}.$$

Now, shrinking V if necessary, we can assume that $\mu(a) \leq \mu(0)$ and $\nu(a) \leq \nu(0)$, for all $a \in V$. Therefore, V has a finite partition into sets

$$\Sigma_{\mu,\nu} := \big\{a \in V : \ \mu(a) = \mu, \ \nu(a) = \nu\big\};$$

each $\Sigma_{\mu,\nu}$ is a difference of closed algebraic (respectively, analytic) subsets of V.

For each $a \in V$, let $\alpha(a)$ denote the smallest multi-index γ such that $D^\gamma \Delta(a) \neq 0$ (in particular, $|\alpha(a)| = \mu(a)$). Clearly, there is a finite filtration of V by closed algebraic (respectively, analytic) subsets $V = X_0 \supset X_1 \supset \cdots$ such that, for each $j = 0, 1, \ldots$, there exist μ, ν and $\alpha \in \mathbb{N}^n$ such that $X_j \backslash X_{j+1} \subset \Sigma_{\mu,\nu}$ and $\alpha(a) = \alpha$ for all $a \in X_j \backslash X_{j+1}$. Theorem 1.2 follows.

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