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CONVERGENCE OF SEQUENCES OF PAIRWISE INDEPENDENT RANDOM VARIABLES

N. ETEMADI AND A. LENZHEN

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ABSTRACT. In spite of the fact that the tail σ-algebra of a sequence of pairwise independent random variables may not be trivial, we have discovered that if such a sequence converges in probability or almost everywhere, then the limit has to be a constant. This enables us to provide necessary and sufficient conditions for its convergence, in terms of its marginal distribution functions.

Let $\{X_n\colon n \geq 1\}$ be a sequence of pairwise independent random variables. It is known (see Robertson and Womack (1985)), that the elements of the tail σ-algebra induced by such a sequence may not satisfy the 0-1 law. Therefore, it came to us as a pleasant surprise when we discovered that the limit, in probability or almost everywhere, of such a sequence is a constant in all cases, and it is even more interesting that the almost everywhere convergence of such sequences depends only on the distribution function of individual X_n's. These can all be described in the following theorem.

Theorem. *Let $\{X_n : n \geq 1\}$ be a sequence of pairwise independent random variables and let m_{X_n} be a median of X_n. Then*

(a) *X_n converges in probability iff for all $\epsilon > 0$, $\mathbf{P}\{|X_n - m_{X_n}| > \epsilon\} \to 0$ and m_{X_n} converges,*

(b) *X_n converges almost everywhere iff for all $\epsilon > 0$, $\sum_{n=1}^{\infty} \mathbf{P}\{|X_n - m_{X_n}| > \epsilon\} < \infty$ and m_{X_n} converges.*

Proof. Only the necessity parts in (a) and (b) need justification. Assume that X_n converges in probability. Construct a sequence of *independent* random variables $\{Y_n : n \geq 1\}$ such that Y_n has the same distribution as X_n for each $n \geq 1$. Since convergence in probability involves the joint distribution of two random variables at a time and (X_i, X_j) has the same distribution as (Y_i, Y_j) for all $i, j \geq 1$, Y_n also converges in probability. Thus it converges almost everywhere to the same limit on a subsequence and the Kolmogorov 0-1 law tells us that the limit is a constant, say c. But given $\epsilon > 0$, $\mathbf{P}\{|X_n - c| > \epsilon\} = \mathbf{P}\{|Y_n - c| > \epsilon\}$. Therefore, X_n converges in probability (almost everywhere) iff $\mathbf{P}\{|X_n - c| > \epsilon\} \to 0$ ($\sum_{n=1}^{\infty} \mathbf{P}\{|X_n - c| > \epsilon\} < \infty$), where we used the fact that the Borel-Cantelli lemma for independent random variable remains intact for pairwise independent random variables; see Theorem 4.2.5 in Chung (1974). Also $X_n \xrightarrow{\mathbf{P}} c$ implies that $m_{X_n} \to c$. As a result we can

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replace the constant c by m_{X_n}, if we wish, at the expense of ϵ, and since $\epsilon > 0$ is arbitrary, we are through. □

References

[1] Chung, K. L., *A course in probability theory.* 2nd edition, Academic Press, 1974. MR **49:**11579

[2] Robertson, J. B. and Womack, J. M. (1985), *A pairwise independent stationary stochastic process.* Statist. Probab. Lett. **3**, 195-199. MR **87d:**60040

Department of Mathematics, Statistics, and Computer Science (M/C 249), 322 Science and Engineering Offices, 851 South Morgan Street, Chicago, Illinois 60607-7045

E-mail address: etemadi@uic.edu

Department of Mathematics, Statistics, and Computer Science (M/C 249), 322 Science and Engineering Offices, 851 South Morgan Street, Chicago, Illinois 60607-7045

E-mail address: lenzhen@math.uic.edu