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THE SPLITTING PROBLEM FOR SUBSPACES OF TENSOR PRODUCTS OF OPERATOR ALGEBRAS

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ABSTRACT. The main result of this paper is that if N is a von Neumann algebra that is a factor and has the weak* operator approximation property (the weak* OAP), and if R is a von Neumann algebra, then every σ-weakly closed subspace of $N\bar{\otimes}R$ that is an $N\bar{\otimes}\mathbb{C}1_R$-bimodule (under multiplication) splits, in the sense that there is a σ-weakly closed subspace T of R such that $S = N\bar{\otimes}T$. Note that if S is a von Neumann subalgebra of $N\bar{\otimes}R$, then S is an $N\bar{\otimes}\mathbb{C}1_R$-bimodule if and only if $N\bar{\otimes}\mathbb{C}1_R \subset S$. So this result is a generalization (in the case where N has the weak* OAP) of the result of Ge and Kadison that if N is a factor, then every von Neumann subalgebra M of $N\bar{\otimes}R$ that contains $N\bar{\otimes}\mathbb{C}1_R$ splits. We also obtain other results concerning the splitting of σ-weakly closed subspaces of tensor products of von Neumann algebras and the splitting of normed closed subspaces of C*-algebras that generalize results previously obtained for von Neumann subalgebras and C*-subalgebras.

In this article we are concerned with the following question: if N and R are von Neumann algebras, and if S is a σ-weakly closed $N\bar{\otimes}\mathbb{C}1_R$-bimodule of $N\bar{\otimes}R$, when do we have

$$S = N\bar{\otimes}T \tag{1}$$

for some σ-weakly closed subspace T of R? If (1) holds for some T, then we say that S splits. Note that if a σ-weakly closed subspace S of $N\bar{\otimes}R$ splits, then it is an $N\bar{\otimes}\mathbb{C}1_R$-bimodule. So this requirement is necessary for splitting. This problem has been previously studied in the case when $S = M$ is a von Neuman subalgebra of $N\bar{\otimes}R$ containing $N\bar{\otimes}\mathbb{C}1_R$ (which is obviously equivalent to M being an $N\bar{\otimes}\mathbb{C}1_R$-bimodule). Ge and Kadison showed in [4] that if N is a factor, then every von Neumann algebra M satisfying $N\bar{\otimes}\mathbb{C}1_R \subset M \subset N\bar{\otimes}R$ splits. In [10] Strătilă and Zsidó extended this result by showing that if N is a von Neumann algebra with center $Z(N)$, and if H is a Hilbert space, then a von Neumann algebra M such that $N\bar{\otimes}\mathbb{C}1_{B(H)} \subset M \subset N\bar{\otimes}B(H)$ is of the form $N\bar{\otimes}P$ for some von Neumann subalgebra P of $B(H)$ if and only if $M \cap (Z(N)\bar{\otimes}B(H)) = Z(N)\bar{\otimes}P$. By modifying the methods of [10], we are able to extend their result (and so the result of Ge and Kadison) to the case where S is a σ-weakly closed subspace of $N\bar{\otimes}R$ that is an $N\bar{\otimes}\mathbb{C}1_R$-bimodule. However, we have to add a condition to N, namely that N satisfies Property S_σ (introduced by the author in [6]) or, equivalently, the weak* operator approximation property. We also observe that the proof of the main result

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in [16] (and Theorem 3.3 in [15]) can be easily modified to give a splitting result for certain norm closed subspaces of the spatial (= minimal) tensor product of C*-algebras. We would like to thank the referee for helpful comments and suggestions.

Our first result is an extension of Theorem 3.5 in [10]. Since the proof of Theorem 1 is essentially the same as the proof of Theorem 3.5 in [10], it is omitted. (The only difference between Theorem 1 below and Theorem 3.5 in [10] is that in [10] it is shown that if $N \subset M \subset N \vee N_0$ is an intermediate von Neumann algebra, then $\Phi(M) = M \cap N_0 = N' \cap M$ rather than $\Phi(S) = S \cap N_0 = N' \cap S$ if S is a σ-weakly closed subspace of $N \vee N_0$ that is an N-bimodule. The only place in the proof of the equality $\Phi(M) = M \cap N_0$ in [10] where the fact that M is a von Neumann algebra is used is that $N \subset M$ implies that M is an N-bimodule, while the proof in [10] of the equality $M \cap N_0 = N' \cap M$ is valid for any subset M of $N \vee N_0$.)

Theorem 1. *Let $N, N_0 \subset B(H)$ be commuting von Neumann algebras with common center Z, and suppose N_0 is type I. Then*

(1) *$\Phi \to \Phi|_N$ establishes a one-to-one correspondence between normal conditional expectations $\Phi : N \vee N_0 \to N_0$ and normal conditional expectations from N to Z;*
(2) *the normal conditional expectations $N \vee N_0 \to N_0$ separate the points of $N \vee N_0$;*
(3) *for every normal conditional expectation $\Phi : N \vee N_0 \to N_0$ and every σ-weakly closed subspace S of $N \vee N_0$ that is an N-bimodule, we have*

$$\Phi(S) = S \cap N_0 = N' \cap S.$$

The proof of our main result (Theorem 2) uses Fubini products. So before proving the result we recall some of the basic definitions and results that we need. (See [6] and [7] for more details.) If N and R are von Neumann algebras, then for each $\phi \in N_*$, the right slice map $R_\phi : N \bar{\otimes} R \to R$ is defined by $R_\phi = \phi \otimes id_R$. Note that R_ϕ is the unique σ-weakly continuous linear map from $N \bar{\otimes} R \to R$ satisfying

$$R_\phi(a \otimes b) = \phi(a)b, \qquad a \in N, b \in R.$$

Similarly, if $\psi \in R_*$, the left slice map $L_\psi : N \bar{\otimes} R \to N$ is defined by $L_\psi = id_N \otimes \psi$. If S and T are σ-weakly subspaces of N and R respectively, then the Fubini product of S and T is the σ-weakly closed subspace of $N \bar{\otimes} R$ defined by

$$F(S,T) = \{\, x \in N \bar{\otimes} R : R_\phi(x) \in T \text{ and } L_\psi \in S \text{ for all } \phi \in N_* \text{ and } \psi \in R_* \,\}.$$

It is shown in [6] that $F(S,T)$ only depends on S and T and not on the containing von Neumann algebras. So if $N \subset B(H)$ and $R \subset B(K)$, we can replace N by $B(H)$ and R by $B(K)$ in the definition of $F(S,T)$. It is easy to see that we always have $S \bar{\otimes} T \subset F(S,T)$. A σ-weakly closed subspace S of $B(H)$ is said to have Property S_σ if $F(S,T) = S \bar{\otimes} T$ for all σ-weakly subspaces $T \subset B(K)$ for any Hilbert space K (as shown in [6], it suffices to consider the case where K is separable and infinite dimensional). If $1_{B(H)} \in S \subset B(H)$ and $1_{B(K)} \in T \subset B(K)$ are σ-weakly closed subspaces, then $(S \bar{\otimes} T)' = F(S', T')$ (see [6]). In particular, if N and R are von Neumann algebras, then with $S = N'$ and $T = R'$, we get that

$$F(N,R) = (N' \bar{\otimes} R')' = N'' \bar{\otimes} R'' = N \bar{\otimes} R$$

by Tomita's Commutation Theorem, and, in fact, the above calculation shows that Tomita's Commutation Theorem is equivalent to the statement that $F(N,R) =$

$N\bar{\otimes}R$ whenever N and R are von Neumann algebras. However, not only are there subspaces S and T such that $F(S,T) \neq S\bar{\otimes}T$, but it is shown in [7] that for each of the types II_1, II_∞ and III_λ, $0 \leq \lambda \leq 1$, there is a separably acting factor N of that type and a unital σ-weakly closed subalgebra T of $B(K)$ (where K is a separable infinite-dimensional Hilbert space) such that $N\bar{\otimes}T \neq F(N,T)$. (We cannot choose N to be type I, because all type I von Neumann algebras have Property S_σ [6, Theorem 1.9].) It is also shown in [7] that a subspace $S \subset B(H)$ has Property S_σ if and only if it has the weak* OAP. (A subspace $S \subset B(H)$ has the weak* OAP if there is a net $\{\Phi_i\}$ of normal finite rank completely bounded maps from S to S such that for any Hilbert space K and for any $x \in S\bar{\otimes}B(K)$, we have that $(\Phi_i \otimes id_{B(K)})(x) \to x$ σ-weakly. See [3] for a detailed treatment of the weak* OAP and other approximation properties of operator spaces.)

Theorem 2. *Let N be a von Neumann algebra with center $Z(N)$, let R be a von Neumann algebra, and suppose S is a σ-weakly closed subspace of $N\bar{\otimes}R$ that is a $N\bar{\otimes}\mathbb{C}1_R$-bimodule such that*

$$S \cap (Z(N)\bar{\otimes}R) = Z(N)\bar{\otimes}T$$

for some σ-weakly closed subspace T of R. Then

$$N\bar{\otimes}T \subset S \subset F(N,T).$$

In particular, if N has the weak OAP, or if S is a von Neumann algebra, then*

$$S = N\bar{\otimes}T.$$

Proof. We can assume that $R \subset B(H)$ for some Hilbert space H. Then $S \subset N\bar{\otimes}R \subset N\bar{\otimes}B(H)$, and S is an $N\bar{\otimes}\mathbb{C}1_{B(H)}$-bimodule. Moreover, we have that $S \cap (Z(N)\bar{\otimes}B(H)) = (S \cap (N\bar{\otimes}R)) \cap (Z(N)\bar{\otimes}B(H)) = S \cap ((N\bar{\otimes}R) \cap (Z(N)\bar{\otimes}B(H))) = S \cap (Z(N)\bar{\otimes}R) = Z(N)\bar{\otimes}T$. Let $\phi \in N_*$ be a normal state, let $\psi = \phi|_{Z(N)}$, and let e be the support projection of ψ. Then there exists a positive normal $Z(N)$-module map Φ_1 from N into $Z(N)$ such that $\phi = \psi \circ \Phi_1$ and such that $\Phi_1(1) = e$ (see Theorem 1 in [5] or Proposition 1.4 in [9]). It follows from this result (applied to all normal states of N) and a Zorn's lemma argument that the normal conditional expectations from N onto $Z(N)$ separate the points of N. Now let Φ_2 be any normal conditional expectation from N onto $Z(N)$, and define Φ on N by $\Phi(x) = \Phi_1(x) + (1-e)\Phi_2(x)$. Then Φ is a normal conditional expectation from N onto $Z(N)$ such that $\phi = \psi \circ \Phi$. Since $\Phi \otimes id_{B(H)}$ is a normal conditional expectation from $(N\bar{\otimes}\mathbb{C}1_{B(H)}) \vee (Z(N)\bar{\otimes}B(H)) = N\bar{\otimes}B(H)$ to $Z(N)\bar{\otimes}B(H)$, it follows from Theorem 1 that

$$(\Phi \otimes id_{B(H)})(x) \in S \cap (Z(N)\bar{\otimes}B(H)) = Z(N)\bar{\otimes}T \text{ for all } x \in S,$$

and so $R_\psi((\Phi \otimes id_{B(H)})(x)) \in T$. But

$$(2)\quad R_\psi((\Phi\otimes id_{B(H)})(x)) = (\psi\otimes id_{B(H)})\circ(\Phi\otimes id_{B(H)})(x) = \phi\otimes id_{B(H)}(x) = R_\phi(x),$$

and so $R_\phi(x) \in T$ whenever $x \in S$ and ϕ is a normal state of N. Since the map $\phi \to R_\phi$ is linear, and since every $\phi \in N_*$ is a linear combination of normal states, we also have that $R_\phi(x) \in T$ for all $\phi \in N_*$. Moreover, since $S \subset N\bar{\otimes}R$, $L_\psi(x) \in N$ for all $\psi \in R_*$. Hence $S \subset F(N,T)$. By assumption, $Z(N)\bar{\otimes}T \subset S$ and S is an $N\bar{\otimes}\mathbb{C}1_R$-bimodule; so $(N\bar{\otimes}\mathbb{C}1_R)(Z(N)\bar{\otimes}T) = N\bar{\otimes}T \subset S$. Hence $N\bar{\otimes}T \subset S \subset F(N,T)$. As noted above, if N has the weak* OAP, then N has Property S_σ; so in this case we always have $N\bar{\otimes}T = F(N,T)$. Finally, if S is a von Neumann algebra, then $S \cap (Z(N)\bar{\otimes}R) = Z(N)\bar{\otimes}T$ is also a von Neumann algebra, and so T is a von Neumann algebra. Hence we again have $F(N,T) = N\bar{\otimes}T$. □

If N is a factor, then

$$S \cap (Z(N)\bar{\otimes}R) = S \cap (\mathbb{C}1_N\bar{\otimes}R) \subset \mathbb{C}1_N\bar{\otimes}R = \{\, 1_N \otimes b : b \in R\}.$$

So we always have $S \cap (Z(N)\bar{\otimes}R) = \mathbb{C}1_N\bar{\otimes}T = Z(N)\bar{\otimes}T$, where T is a σ-weakly closed subspace of R. (In fact, $T = \{R_\phi(x) : x \in S \cap (Z(N)\bar{\otimes}R) \text{ and } \phi \in N_*\}$.) Thus we get the following extension of the result of Ge and Kadison.

Theorem 3. *Let N be a factor, let R be a von Neumann algebra, and suppose S is a σ-weakly closed subspace of $N\bar{\otimes}R$ that is an $N\bar{\otimes}\mathbb{C}1_R$-bimodule. Then there is a σ-weakly closed subspace T of R such that*

$$N\bar{\otimes}T \subset S \subset F(N,T).$$

In particular, if N has the weak OAP, or if S is a von Neumann algebra, then*

$$S = N\bar{\otimes}T.$$

Our next result shows that the requirement in Theorem 3 that N have the weak* OAP is necessary for all σ-weakly closed $N\bar{\otimes}\mathbb{C}1_{B(H)}$-bimodules of $N\bar{\otimes}B(H)$ to split.

Theorem 4. *Suppose N is a factor without the weak* OAP, and that H is a separable infinite-dimensional Hilbert space. Then there is a unital (but not self-adjoint) σ-weakly closed subalgebra S of $N\bar{\otimes}B(H)$ that is an $N\bar{\otimes}\mathbb{C}1_{B(H)}$-bimodule, but which does not split.*

Proof. Since N is a von Neumann algebra without the weak* OAP, and so without Property S_σ, there is an abelian subalgebra A of $B(H)$ that is reflexive (and so σ-weakly closed and unital) such that $F(N,A) \neq N\bar{\otimes}A$. (See Remark 1.1 in [7], noting that von Neumann algebras are always reflexive.) Let $S = F(N,A)$. Then S is a σ-weakly closed unital subalgebra of $N\bar{\otimes}B(H)$. (See Remark 3.16 in [7].) For $\phi \in N_*$ and for $a \in N$, define the normal linear functionals $a\phi$ and ϕa on N by

$$a\phi(b) = \phi(ab) \text{ and } \phi a(b) = \phi(ba) \text{ for all } b \in N.$$

Then for all $\phi \in N_*$, $a \in N$, and $x \in N\bar{\otimes}B(H)$, we have

$$R_\phi((a \otimes 1_{B(H)})x) = R_{a\phi}(x) \text{ and } R_\phi(x(a \otimes 1_{B(H)})) = R_{\phi a}(x), \tag{3}$$

as can be easily seen (first check the equalities in (3) for x of the form $x = b \otimes c$). It follows immediately from (3) that if $x \in F(N,A)$, then $(a\otimes 1_{B(H)})x$ and $x(a\otimes 1_{B(H)})$ are also in $F(N,A)$ for all $a \in N$; so S is an $N\bar{\otimes}\mathbb{C}1_{B(H)}$-bimodule. Finally, note that if $S = N\bar{\otimes}T$ for some σ-weakly closed subspace T of $B(H)$, then $T = A$, which contradicts $F(N,A) \neq N\bar{\otimes}A$. Hence S does not split (and so is not selfadjoint). □

Combining Theorems 3 and 4, we get the following result.

Theorem 5. *A factor N has Property S_σ if and only if for every von Neumann algebra R (or just $R = B(H)$), every σ-weakly closed $N\bar{\otimes}\mathbb{C}1_R$-bimodule of $N\bar{\otimes}R$ splits.*

The von Neumann subalgebra version of Theorem 3 in [4] is used to show the following result ([4, Theorem H]): if N and R are factors, if N is injective, and if P is a maximal injective von Neumann subalgebra of R, then $N\bar{\otimes}P$ is a maximal injective von Neumann subalgebra of $N\bar{\otimes}R$. Injectivity makes sense for norm closed subspaces of $B(H)$, when these are viewed as operator spaces. By definition, an operator space V is injective if for any operator space $W_0 \subset W$, every complete contraction from W_0 to V has a completely contractive extension from W to V. Since $B(H)$ is injective, it is not hard to show that an operator space $V \subset B(H)$ is injective if and only if there is a completely contractive projection from $B(H)$ onto V. (See [3, Proposition 4.1.6].) Using ideas from the proof of Theorem H, we obtain the next result.

Theorem 6. *Let N be an injective factor, and let R be a von Neumann algebra. Then*

(1) *if T is a maximal σ-weakly closed injective subspace of R, then $N\bar{\otimes}T$ is a maximal σ-weakly closed injective $N\bar{\otimes}\mathbb{C}1_R$-bimodule of $N\bar{\otimes}R$;*
(2) *if A is a maximal σ-weakly closed injective unital (but not necessarily self-adjoint) subalgebra of R, then $N\bar{\otimes}A$ is a maximal σ-weakly closed injective unital subalgebra of $N\bar{\otimes}R$.*

Proof. (1) We first show that $N\bar{\otimes}T$ is injective. Since T is σ-weakly closed, it is dual as a Banach space. So it follows from Theorem 1.3 in [1] that there is an injective von Neumann algebra P and a projection e in P such that T is completely isometric and weak* homeomorphic to $eP(1_P - e)$, and hence $N\bar{\otimes}T$ is completely isometric and weak* homeomorphic to $N\bar{\otimes}eP(1_P - e)$. Let $f = 1_N \otimes e$, and let $M = N\bar{\otimes}P$. Then $1_M - f = 1_N \otimes (1_P - e)$, and so $N\bar{\otimes}eP(1_P - e) = fM(1_M - f)$. Since N and P are injective von Neumann algebras, so is M ([8, Proposition 10.24]). Thus another application of Theorem 1.3 in [1] shows that $N\bar{\otimes}T$ is injective. Now suppose that S is a σ-weakly closed injective $N\bar{\otimes}\mathbb{C}1_R$-bimodule of $N\bar{\otimes}R$, and that $N\bar{\otimes}T \subset S$. Since N is injective, it has Property S_σ. (It is shown in [6] that semidiscrete von Neumann algebras have Property S_σ, and, as shown in [13], semidiscreteness is equivalent to injectivity.) Hence by Theorem 3, there is a σ-weakly closed subspace T_0 of R such that $S = N\bar{\otimes}T_0$, and since $\mathbb{C}1_N\bar{\otimes}T \subset S = F(N, T_0)$, $T \subset T_0$. Suppose $N \subset B(K)$ for a Hilbert space K. Then $S \subset B(K)\bar{\otimes}B(H) = B(K \otimes H)$ is injective. So there is a completely contractive projection Ψ from $B(K \otimes H)$ onto S. Let ϕ be a normal state of N, and let Φ be the unique normal linear map from $N\bar{\otimes}R$ onto $\mathbb{C}1_N\bar{\otimes}R$ such that $\Phi(a \otimes b) = \phi(a)1_N \otimes b$ for all $a \in N$ and $b \in R$. Then it is easily checked that $\Phi \circ \Psi$ is a completely contractive projection from $B(K \otimes H)$ onto $\mathbb{C}1_N\bar{\otimes}T_0$. Hence $\mathbb{C}1_N\bar{\otimes}T_0$ is injective, from which it follows easily that T_0 is injective. By the maximality of T, $T = T_0$, and so $S = N\bar{\otimes}T$. Hence $N\bar{\otimes}T$ is a maximal σ-weakly closed injective $N\bar{\otimes}\mathbb{C}1_R$-bimodule of $N\bar{\otimes}R$.

(2) Let B be a σ-weakly closed injective unital subalgebra of $N\bar{\otimes}R$ such that $N\bar{\otimes}A \subset B$. Then since A is unital, $N\bar{\otimes}\mathbb{C}1_R \subset B$, and so, since B is an algebra, B is an $N\bar{\otimes}\mathbb{C}1_R$-bimodule. Thus, by the same argument as in (1), $B = N\bar{\otimes}A_0$, where $A \subset A_0$ and A_0 is an injective σ-weakly closed subspace of R. Moreover, if a and b are elements of A_0, then $1_N \otimes a$ and $1_N \otimes b$ are elements of the algebra B; so $1_N \otimes ab = (1_N \otimes a)(1_N \otimes b) \in B = F(N, A_0)$, and so $ab \in A_0$. Hence A_0 is also a unital subalgebra of R, and so by the maximality of A, $A = A_0$, and so $B = N\bar{\otimes}A$. Thus $N\bar{\otimes}A$ is a maximal σ-weakly closed injective unital subalgebra of $N\bar{\otimes}R$. □

We conclude this paper with a result that generalizes the main result in [16] (and Theorem 3.3(1) in [15]). First we need some terminology. If A and B are C*-algebras, we denote by $A \otimes B$ the minimal or spatial tensor product of A and

B. (If $A \subset B(H)$ and $B \subset B(K)$, then $A \otimes B$ is just the norm closed linear span in $B(H)\bar{\otimes}B(K)$ of $\{a \otimes b : a \in A, b \in B\}$.) Slice maps and Fubini products can be defined as in the von Neumann algebra case, with the σ-weak topology replaced by the norm topology. (So, for example, if $\phi \in A^*$, then the right slice map R_ϕ is the unique norm bounded linear map from $A \otimes B$ to B satisfying $R_\phi(a \otimes b) = \phi(a)b$ for all $a \in A$ and $b \in B$.) In contrast to the case of von Neumann algebras, the Fubuni product depends on A and B. (See section 5 of [7].) Since our first C*-algebra will be fixed, we use the following special notation: if A and B are C*-algebras, and if T is a norm closed subspace of B, then

$$F_B(T) = \{x \in A \otimes B : R_\phi(x) \in T \text{ for all } \phi \in A^*\}.$$

Note that since every bounded linear functional on A is a linear combination of states, we also have that $F_B(T)$ is the norm closed linear span of $\{x \in A \otimes B : R_\phi(x) \in T$ for all states ϕ of $A\}$. A C*-algebra A is said to have Property S for subspaces (see [7]) if for every pair (T, B), where B is a C*-algebra and T is a norm closed subspace of B, we have $F_B(T) = A \otimes T$. (A C*-algebra is said to have Property S if $F_B(C) = A \otimes C$ whenever C is a C*-subalgebra of a C*-algebra B. Property S was introduced by Wassermann in [11], and was the inspiration for both Property S_σ and Property S for subspaces. In [12], Wassermann showed that there are C*-algebras that do not have Property S.) It is shown in [7] that a C*-algebra has Property S for subspaces if and only if for any C*-algebra B there is a net $\{\Phi_i\}$ of finite rank completely bounded maps from A to A such that $(\Phi_i \otimes id_B)(x) \to x$ in the point norm topology of $A \otimes B$. A C*-algebra with this approximation property is said to have the strong operator approximation property, or strong OAP (see p. 185 of [2] or Chapter 11 of [3]). Wassermann showed in [14] that nuclear C*-algebras have Property S, and his proof of this result (Proposition 10 in [14]) shows that nuclear C*-algebras have Property S for subspaces (and also shows directly that nuclear C*-algebras have the strong OAP).

The main result (Theorem) in [16] is that if A, D and C are unital C*-algebras, if A is simple and nuclear, and if $A \otimes \mathbb{C}1_D \subset C \subset A \otimes D$, then $C = A \otimes B$ for some C*-subalgebra B of D. This result was obtained independently by Zacharias, using elementary maps (Theorem 3.3(1) of [15]), where it was explicitly observed that the only property of nuclearity that Zsido used was the fact that a nuclear C*-algebra has Property S. So the result remains valid if we just assume that A is simple and has Property S. Our last theorem follows from a straightforward modification of the proof of [16, Theorem], or the proof of [15, Theorem 3.3(1)]; so the proof is omitted.

Theorem 7. *Let A and B be unital C*-algebras, with A simple, and suppose S is a norm closed subspace of $A \otimes B$ that is also an $A \otimes \mathbb{C}1_B$-bimodule (under multiplication). Then*

(1) *there is a norm closed subspace T of B such that $A \otimes T \subset S \subset F_B(T)$;*
(2) *if A has Property S for subspaces, or, equivalently, if A has the strong OAP (in particular, if A is nuclear), then $S = A \otimes T$ for some norm closed subspace T of B.*

Remark. Theorem 3.3(2) in [15] can also be generalized to the subspace case, with the obvious modifications to the proof. The general result is: Suppose A and B are unital C*-algebras and that A contains a unital abelian C*-subalgebra D that has the pure state extension property (each pure state on D extends uniquely to a pure state on A). Suppose C is a norm closed subspace of $A \otimes B$ that is an $A \otimes \mathbb{C}1_B$-bimodule, and that $B_\omega = B_0$ for each pure state ω on A and some norm closed subspace B_0 of B. (See [15] for the definition of B_ω. Note that if we just assume C is an $A \otimes \mathbb{C}1_B$-bimodule, then the proofs of (i) and (iii) of Proposition 3.2 in [15] are still valid.) Then $A \otimes B_0 \subset C \subset F_B(B_0)$, and thus $C = A \otimes B_0$ if A has Property S for subspaces.

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