

SEBESTYÉN MOMENT PROBLEM: THE MULTI-DIMENSIONAL CASE

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To the memory of Gyula Farkas

ABSTRACT. Given a family $\{h_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}_+^{\Omega}}$ of vectors in a Hilbert space \mathcal{H} we characterize the existence of a family of commuting contractions $\mathbf{T} = \{T_{\omega}\}_{\omega \in \Omega}$ on \mathcal{H} having regular dilation and such that

$$h_{\mathbf{n}} = \mathbf{T}^{\mathbf{n}} h_{\mathbf{0}}, \quad \mathbf{n} \in \mathbb{Z}_+^{\Omega}.$$

The theorem is a multi-dimensional analogue for some well-known operator moment problems due to Sebestyén in case $|\Omega| = 1$ or, recently, to Găvruta and Păunescu in case $|\Omega| = 2$.

Let \mathcal{H} be a complex Hilbert space. Denote by $\mathcal{L}(\mathcal{H})$ the \mathbf{C}^* -algebra of bounded linear operators on \mathcal{H} .

Z. Sebestyén proposed in [5] the following operator moment problem:

Problem A. *Given a sequence $\{h_n\}_{n \geq 0}$ of vectors in \mathcal{H} , under what condition does there exist a contraction T on \mathcal{H} such that*

$$h_n = T^n h_0, \quad n \geq 0 ?$$

The two-dimensional analogue of this problem was proposed by Găvruta and Păunescu [2]:

Problem B. *Given a doubly indexed sequence $\{h_m^n\}_{m, n \geq 0}$ of vectors in \mathcal{H} , under what condition does there exist a commuting pair of contractions (S, T) on \mathcal{H} such that*

$$h_m^n = S^m T^n h_0^0, \quad m, n \geq 0 ?$$

The solution given by Z. Sebestyén to the first problem was the following:

Theorem A ([5]). *Let $\{h_n\}_{n \geq 0} \subset \mathcal{H}$. Problem A has a solution if and only if*

$$\left\| \sum_{m, m'} c_{m, m'} h_{m+m'} \right\|^2 \leq \sum_{m, m', n, n'} c_{m, m'} \overline{c_{n, n'}} \langle h_{(m-n)^+ + m'}, h_{(m-n)^- + n'} \rangle,$$

for every finite double sequence $\{c_{n, n'}\}_{n, n' \geq 0}$ of complex numbers (we used, for any integer m , the notation $m^+ = \max\{m, 0\}$ and $m^- = -\min\{m, 0\}$).

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The solution to the second problem uses the theory of regular dilations:

Theorem B ([2]). *Let $\{h_m^n\}_{m,n \geq 0}$ be a sequence of vectors that spans \mathcal{H} . Problem B has a solution (S, T) having regular dilation if and only if*

$$\left\| \sum_{m,m',n,n'} c_{m,m'}^{n,n'} h_{m+m'}^{n+n'} \right\|^2 \leq \sum_{\substack{m,n,p,q \\ m',n',p',q'}} c_{m,m'}^{n,n'} \overline{c_{p,p'}^{q,q'}} \langle h_{(m-p)^++m'}^{(n-q)^++n'}, h_{(m-p)^-+p'}^{(n-q)^-+q'} \rangle,$$

for every finite double sequence $\{c_{m,m'}^{n,n'}\}_{m,m',n,n' \geq 0}$ of complex numbers and the following regularity condition holds:

$$\|h_{m+1}^n\|^2 + \|h_m^{n+1}\|^2 \leq \|h_m^n\|^2 + \|h_{m+1}^{n+1}\|^2, \quad m, n \geq 0.$$

Before we formulate our multi-dimensional analogue of the previous moment problems we need to introduce some notions.

Let Ω be a nonempty set. Introduce \mathbb{Z}^Ω (respectively \mathbb{Z}_+^Ω) as the set of \mathbb{Z} -valued (respectively \mathbb{Z}_+ -valued) functions on Ω having finite support. With pointwise defined operation, \mathbb{Z}^Ω becomes an additive abelian group with unit $\mathbf{0}$ (the null function) having \mathbb{Z}_+^Ω as a subsemigroup. We identify some particular elements of \mathbb{Z}_+^Ω : for $\mathbf{m} = \{m_\omega\}_{\omega \in \Omega}$, let $\mathbf{m}^+ = \{m_\omega^+\}_{\omega \in \Omega}$ and $\mathbf{m}^- = \{m_\omega^-\}_{\omega \in \Omega}$; for any finite subset $v \subset \Omega$, let $e(v) = \{\varphi_v(\omega)\}_{\omega \in \Omega}$, φ_v being the characteristic function of v on Ω .

In what follows the term *multi-contraction* will denote a family $\mathbf{T} = \{T_\omega\}_{\omega \in \Omega}$ of commuting contractions on \mathcal{H} . If $\mathbf{T} = \{T_\omega\}_{\omega \in \Omega}$ is a multi-contraction and $\mathbf{m} = \{m_\omega\}_{\omega \in \Omega} \in \mathbb{Z}_+^\Omega$ we define, as usual, $\mathbf{T}^{\mathbf{m}} := \prod_{\omega \in \Omega} T_\omega^{m_\omega}$. A multi-unitary operator \mathbf{U} (multi-contraction consisting of unitary operators) acting on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ is said to be a *unitary dilation* of a multi-contraction \mathbf{T} if

$$\mathbf{T}^{\mathbf{m}} h = P_{\mathcal{H}} \mathbf{U}^{\mathbf{m}} h, \quad \mathbf{m} \in \mathbb{Z}_+^\Omega, \quad h \in \mathcal{H},$$

$P_{\mathcal{H}}$ being the orthogonal projection of \mathcal{K} onto \mathcal{H} . \mathbf{U} is called *regular* if, moreover,

$$(\mathbf{T}^{\mathbf{m}^-})^* \mathbf{T}^{\mathbf{m}^+} h = P_{\mathcal{H}} \mathbf{U}^{\mathbf{m}} h, \quad \mathbf{m} \in \mathbb{Z}^\Omega, \quad h \in \mathcal{H}$$

holds. Here \mathbf{U} is multi-unitary; so the definition of $\mathbf{U}^{\mathbf{m}}$ can be obviously extended to all of $\mathbf{m} \in \mathbb{Z}^\Omega$.

Remark. Regular dilations have been introduced and studied by S. Brehmer [1]. We should also mention here the paper of B. Sz.-Nagy [6], which brought more light on the subject.

By the classical theorem of M. A. Neumark [4] on positive definite functions on a group (see also [7]) we can state that a multi-contraction \mathbf{T} has regular dilation if and only if the operator-valued function

$$\mathbb{Z}^\Omega \ni \mathbf{m} \mapsto T(\mathbf{m}) := (\mathbf{T}^{\mathbf{m}^-})^* \mathbf{T}^{\mathbf{m}^+} \in \mathcal{L}(\mathcal{H})$$

is positive definite on \mathbb{Z}^Ω , i.e.,

$$(1) \quad \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^\Omega} \langle T(\mathbf{n} - \mathbf{m}) h_{\mathbf{n}}, h_{\mathbf{m}} \rangle \geq 0,$$

for any finite family $\{h_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}_+^\Omega}$ of vectors in \mathcal{H} .

Equivalently ([1]; cf. also [3], [7]), \mathbf{T} has regular dilation if and only if

$$(2) \quad \sum_{v \subset u} (-1)^{|v|} (T^{e(v)})^* T^{e(v)} \geq 0,$$

for any finite subset u of Ω .

Our terminology and approach in the following will be close to the one used in the monograph [7].

We are now ready to formulate the multi-dimensional analogue of the Sebestyén operator moment problem (Problem A):

Problem C. *Given a family $\{h_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}_+^\Omega}$ of vectors in \mathcal{H} , under what condition does there exist a multi-contraction \mathbf{T} on \mathcal{H} such that*

$$h_{\mathbf{n}} = \mathbf{T}^{\mathbf{n}} h_{\mathbf{0}}, \quad \mathbf{n} \in \mathbb{Z}_+^\Omega ?$$

Our solution applies to the theory of regular dilations (more precisely the results stated in the Remark) and also naturally extends the methods in [5].

Lemma. *Let $\{c_{\mathbf{n}, \mathbf{n}'}\}_{\mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^\Omega}$ and $\{d_{\mathbf{n}, \mathbf{n}'}\}_{\mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^\Omega}$ be given families of complex numbers. The following conditions are equivalent:*

(i)

$$\sum_{\mathbf{n}'} c_{\mathbf{n}, \mathbf{n}'} h_{\mathbf{n}'+\mathbf{p}} = \sum_{\substack{\mathbf{n}' \in \mathbb{Z}_+^\Omega \\ v \subset \Omega \text{ finite}}} (-1)^{|v|} d_{\mathbf{n}+e(v), \mathbf{n}'} h_{\mathbf{n}'+e(v)+\mathbf{p}}, \quad \mathbf{n}, \mathbf{p} \in \mathbb{Z}_+^\Omega,$$

for any family $\{h_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}_+^\Omega} \subset \mathcal{H}$ with finite support;

(ii)

$$\sum_{\mathbf{m} \geq \mathbf{n}; \mathbf{m}'} c_{\mathbf{m}, \mathbf{m}'} h_{\mathbf{m}+\mathbf{m}'-\mathbf{n}+\mathbf{p}} = \sum_{\mathbf{n}'} d_{\mathbf{n}, \mathbf{n}'} h_{\mathbf{n}'+\mathbf{p}}, \quad \mathbf{n}, \mathbf{p} \in \mathbb{Z}_+^\Omega,$$

for any family $\{h_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}_+^\Omega} \subset \mathcal{H}$ with finite support;

(iii)

$$c_{\mathbf{n}, \mathbf{n}'} = \sum_{v: e(v) \leq \mathbf{n}'} (-1)^{|v|} d_{\mathbf{n}+e(v), \mathbf{n}'-e(v)}, \quad \mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^\Omega;$$

(iv)

$$d_{\mathbf{n}, \mathbf{n}'} = \sum_{\mathbf{q} \leq \mathbf{n}'} c_{\mathbf{n}+\mathbf{q}, \mathbf{n}'-\mathbf{q}}, \quad \mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^\Omega.$$

Moreover, $\{c_{\mathbf{n}, \mathbf{n}'}\}$ has finite support if and only if $\{d_{\mathbf{n}, \mathbf{n}'}\}$ has finite support.

Proof. (i) \Leftrightarrow (iii) and (ii) \Leftrightarrow (iv) are easy observations (just compute the coefficient of each $h_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}_+^\Omega$).

For (iii) \Leftrightarrow (iv) observe firstly that

$$\begin{aligned} \sum_{\mathbf{q} \leq \mathbf{n}'} \sum_{v: e(v) \leq \mathbf{n}'-\mathbf{q}} (-1)^{|v|} d_{\mathbf{n}+\mathbf{q}+e(v), \mathbf{n}'-\mathbf{q}-e(v)} \\ \stackrel{\mathbf{q}+e(v)=\mathbf{p}}{=} \sum_{\mathbf{p}: \mathbf{p} \leq \mathbf{n}'} \sum_{v: e(v) \leq \mathbf{p}} (-1)^{|v|} d_{\mathbf{n}+\mathbf{p}, \mathbf{n}'-\mathbf{p}} = d_{\mathbf{n}, \mathbf{n}'}, \quad \mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^\Omega, \end{aligned}$$

since $\sum_{v:e(v)\leq \mathbf{p}} (-1)^{|v|} = 0$ for $\mathbf{p} = \mathbf{0}$ or 1 for $\mathbf{p} \neq \mathbf{0}$. Conversely,

$$\begin{aligned} \sum_{v:e(v)\leq \mathbf{n}'} (-1)^{|v|} \sum_{\mathbf{q}\leq \mathbf{n}'-e(v)} c_{\mathbf{n}+e(v)+\mathbf{q}, \mathbf{n}'-e(v)-\mathbf{q}} \\ \stackrel{\mathbf{q}+e(v)=\mathbf{p}}{=} \sum_{v:e(v)\leq \mathbf{n}'} (-1)^{|v|} \sum_{\mathbf{p}:e(v)\leq \mathbf{p}\leq \mathbf{n}'} c_{\mathbf{n}+\mathbf{p}, \mathbf{n}'-\mathbf{p}} \\ = \sum_{\mathbf{p}\leq \mathbf{n}'} c_{\mathbf{n}+\mathbf{p}, \mathbf{n}'-\mathbf{p}} \sum_{v:e(v)\leq \mathbf{p}} (-1)^{|v|} = c_{\mathbf{n}, \mathbf{n}'}, \quad \mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^\Omega, \end{aligned}$$

by the same argument as above. □

The solution for Problem C now follows:

Theorem C. *Let $\{h_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ be a given multi-indexed family of vectors in \mathcal{H} . The following conditions are equivalent:*

- (a) *Problem C has a solution $\mathbf{T} = \{T_\omega\}_{\omega\in\Omega}$ having regular dilation;*
- (b)

$$\| \sum_{\mathbf{m}, \mathbf{m}'} c_{\mathbf{m}, \mathbf{m}'} h_{\mathbf{m}+\mathbf{m}'} \|^2 \leq \sum_{\mathbf{m}, \mathbf{m}', \mathbf{n}, \mathbf{n}'} c_{\mathbf{m}, \mathbf{m}'} \overline{c_{\mathbf{n}, \mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n})^+ + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n})^- + \mathbf{n}'} \rangle,$$

for every finite family $\{c_{\mathbf{n}, \mathbf{n}'}\}_{\mathbf{n}, \mathbf{n}'\in\mathbb{Z}_+^\Omega}$ of complex numbers;

- (c)

$$\sum_{\mathbf{m}, \mathbf{m}', \mathbf{n}, \mathbf{n}'} c_{\mathbf{m}, \mathbf{m}'} \overline{c_{\mathbf{n}, \mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n})^+ + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n})^- + \mathbf{n}'} \rangle \geq 0,$$

for every finite family $\{c_{\mathbf{n}, \mathbf{n}'}\}_{\mathbf{n}, \mathbf{n}'\in\mathbb{Z}_+^\Omega}$ of complex numbers;

- (d)

$$\sum_{v\subset u} (-1)^{|v|} \left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}+e(v)} \right\|^2 \geq 0,$$

for every finite subset $u \subset \Omega$ and finite family $\{c_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}_+^\Omega}$ of complex numbers.

Proof. (b) \Rightarrow (c) is obvious. Replace $\{c_{\mathbf{n}, \mathbf{n}'}\}$ in (c) by the finite family $\{d_{\mathbf{n}, \mathbf{n}'}\}$ given by the previous Lemma (iii). The inequality in (c) becomes

$$\begin{aligned} \sum_{\mathbf{m}, \mathbf{n}} \langle \sum_{v, \mathbf{m}'} (-1)^{|v|} d_{\mathbf{m}+e(v), \mathbf{m}'} h_{\mathbf{m}'+e(v)+(\mathbf{m}-\mathbf{n})^+}, \\ \sum_{w, \mathbf{n}'} (-1)^{|w|} d_{\mathbf{n}+e(w), \mathbf{n}'} h_{\mathbf{n}'+e(w)+(\mathbf{m}-\mathbf{n})^-} \rangle \geq 0, \end{aligned}$$

by the same Lemma (i). Change variables $\mathbf{m} \leftrightarrow \mathbf{m} + e(v)$ and $\mathbf{n} \leftrightarrow \mathbf{n} + e(w)$ to obtain

$$\begin{aligned} (3) \quad \sum_{\mathbf{m}, \mathbf{m}', \mathbf{n}, \mathbf{n}'} \sum_{\substack{v:e(v)\leq \mathbf{m} \\ w:e(w)\leq \mathbf{n}}} (-1)^{|v|+|w|} d_{\mathbf{m}, \mathbf{m}'} \overline{d_{\mathbf{n}, \mathbf{n}'}} \langle h_{\mathbf{m}'+e(v)+(\mathbf{m}-e(v)-\mathbf{n}+e(w))^+}, \\ h_{\mathbf{n}'+e(w)+(\mathbf{m}-e(v)-\mathbf{n}+e(w))^-} \rangle \geq 0. \end{aligned}$$

For $\mathbf{p} = \{p_\omega\}_{\omega\in\Omega} \in \mathbb{Z}^\Omega$, define $\pi(\mathbf{p}) = \{\omega \in \Omega \mid p_\omega > 0\}$. Any finite set $v \subset \Omega$ with $e(v) \leq \mathbf{m}$ (or equivalently $v \subset \pi(\mathbf{m})$) can be partitioned by $v = v' \cup v''$ with $v' \subset \pi(\mathbf{m}) \setminus \pi(\mathbf{m} - \mathbf{n} + e(w))$ and $v'' \subset \pi(\mathbf{m}) \cap \pi(\mathbf{m} - \mathbf{n} + e(w))$.

Observe that, for $\mathbf{m} \neq \mathbf{n}$ (suppose $(\mathbf{m} - \mathbf{n})^+ \neq \mathbf{0}$, i.e., $\pi(\mathbf{m} - \mathbf{n}) \neq \emptyset$) and fixed $\mathbf{m}', \mathbf{n}', w$:

$$\begin{aligned} & \sum_{v \subset \pi(\mathbf{m})} (-1)^{|v|} \langle h_{\mathbf{m}'+e(v)+(\mathbf{m}-e(v)-\mathbf{n}+e(w))^+}, h_{\mathbf{n}'+e(w)+(\mathbf{m}-e(v)-\mathbf{n}+e(w))^-} \rangle \\ &= \sum_{v' \subset \pi(\mathbf{m}) \setminus \pi(\mathbf{m}-\mathbf{n}+e(w))} (-1)^{|v'|} \sum_{v'' \subset \pi(\mathbf{m}) \cap \pi(\mathbf{m}-\mathbf{n}+e(w))} (-1)^{|v''|} \\ & \quad \times \langle h_{\mathbf{m}'+e(v')+e(v'')+(\mathbf{m}-e(v')-e(v'')-\mathbf{n}+e(w))^+}, h_{\mathbf{n}'+e(w)+(\mathbf{m}-e(v')-e(v'')-\mathbf{n}+e(w))^-} \rangle \\ &= \sum_{v' \subset \pi(\mathbf{m}) \setminus \pi(\mathbf{m}-\mathbf{n}+e(w))} (-1)^{|v'|} \left[\sum_{v'' \subset \pi(\mathbf{m}) \cap \pi(\mathbf{m}-\mathbf{n}+e(w))} (-1)^{|v''|} \right] \\ & \quad \times \langle h_{\mathbf{m}'+e(v')+(\mathbf{m}-e(v')-\mathbf{n}+e(w))^+}, h_{\mathbf{n}'+e(w)+(\mathbf{m}-e(v')-\mathbf{n}+e(w))^-} \rangle = 0, \end{aligned}$$

the term in brackets being null ($\pi(\mathbf{m}) \cap \pi(\mathbf{m} - \mathbf{n} + e(w))$ contains $\pi(\mathbf{m} - \mathbf{n})$, which is non-void).

By this observation the positivity condition (3) becomes

$$\sum_{\mathbf{m}, \mathbf{m}', \mathbf{n}'} \sum_{v, w \subset \pi(\mathbf{m})} (-1)^{|v|+|w|} d_{\mathbf{m}, \mathbf{m}'} \overline{d_{\mathbf{m}, \mathbf{n}'}} \langle h_{\mathbf{m}'+e(v \cup w)}, h_{\mathbf{n}'+e(v \cup w)} \rangle \geq 0$$

or, equivalently,

$$\sum_{\mathbf{m}} \left[\sum_{u \subset \pi(\mathbf{m})} (-1)^{|u|} \left\| \sum_{\mathbf{m}'} d_{\mathbf{m}, \mathbf{m}'} h_{\mathbf{m}'+e(u)} \right\|^2 \right] \geq 0$$

since $\sum_{v, w: v \cup w = u} (-1)^{|v|+|w|} = (-1)^{|u|}$.

The family $\{d_{\mathbf{m}, \mathbf{m}'}\}$ with finite support being arbitrary, every sum in the brackets above must be positive or zero. The implication (c) \Rightarrow (d) is proved.

(d) \Rightarrow (b) Rewrite (d) in the form:

$$\sum_{\mathbf{m} \geq \mathbf{0}} \left[\sum_{u \subset \pi(\mathbf{m})} (-1)^{|u|} \left\| \sum_{\mathbf{m}'} d_{\mathbf{m}, \mathbf{m}'} h_{\mathbf{m}'+e(u)} \right\|^2 \right] \geq \left\| \sum_{\mathbf{m}'} d_{\mathbf{0}, \mathbf{m}'} h_{\mathbf{m}'} \right\|^2,$$

for any finite subset $u \subset \Omega$ and finite family $\{d_{\mathbf{m}, \mathbf{m}'}\}$ of complex numbers. Replace $\{d_{\mathbf{m}, \mathbf{m}'}\}$ by the finite family $\{c_{\mathbf{m}, \mathbf{m}'}\}$ given by Lemma (iv). Similar calculations as in the previous implication and further use of the Lemma lead us to condition (b).

(a) \Rightarrow (d) By (a) the solution to Problem C is a multi-contraction \mathbf{T} having regular dilation. Use the equalities $h_{\mathbf{n}} = \mathbf{T}^{\mathbf{n}} h_{\mathbf{0}}$, $\mathbf{n} \in \mathbb{Z}_+^{\Omega}$ in (2) to finally obtain (d).

(b) \Rightarrow (a) Let \mathcal{F}_0 be the linear space of all finite families $\{c_{\mathbf{n}, \mathbf{n}'}\}_{\mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^{\Omega}}$ of complex numbers. Endow \mathcal{F}_0 with the semi-inner product (in view of (b)):

$$\langle \{c_{\mathbf{n}, \mathbf{n}'}\}, \{d_{\mathbf{n}, \mathbf{n}'}\} \rangle_{\mathcal{F}_0} := \sum_{\mathbf{m}, \mathbf{m}', \mathbf{n}, \mathbf{n}'} c_{\mathbf{m}, \mathbf{m}'} \overline{d_{\mathbf{n}, \mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n})^+ + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n})^- + \mathbf{n}'} \rangle.$$

Factorize \mathcal{F}_0 with respect to the null space of $\langle \cdot, \cdot \rangle_{\mathcal{F}_0}$. Denote by \mathcal{F} its Hilbert space completion and by \mathcal{H}' the span of $\{h_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}_+^{\Omega}}$.

Define $V : \mathcal{F} \rightarrow \mathcal{H}'$,

$$V\{c_{\mathbf{n}, \mathbf{n}'}\} := \sum_{\mathbf{n}, \mathbf{n}'} c_{\mathbf{n}, \mathbf{n}'} h_{\mathbf{n} + \mathbf{n}'}, \quad \{c_{\mathbf{n}, \mathbf{n}'}\} \in \mathcal{F}_0.$$

By (a), V can be extended to a contraction on \mathcal{F} into \mathcal{H}' . Moreover, by easy computations,

$$V^*h_{\mathbf{k}} = \{\delta_{(\mathbf{n},\mathbf{n}')}^{(\mathbf{0},\mathbf{k})}\}_{\mathbf{n},\mathbf{n}' \in \mathbb{Z}_+^\Omega}, \quad \mathbf{k} \in \mathbb{Z}_+^\Omega$$

(δ denotes here the Kronecker symbol).

For fixed $\omega \in \Omega$, define U_ω on \mathcal{F}_0 by

$$U_\omega\{c_{\mathbf{n},\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'} := \{c_{\mathbf{n}+e(\{\omega\}),\mathbf{n}'}\}_{\mathbf{n},\mathbf{n}'}$$

U_ω can be extended on \mathcal{F} , and it is not hard to see that its adjoint U_ω^* is isometric. Let

$$T'_\omega = VU_\omega^*V^* \in \mathcal{L}(\mathcal{H}')$$

Each T_ω is a contraction on \mathcal{H}' . Moreover,

$$T'_\omega h_{\mathbf{k}} = VU_\omega^*V^*h_{\mathbf{k}} = VU_\omega^*\{\delta_{(\mathbf{n},\mathbf{n}')}^{(\mathbf{0},\mathbf{k})}\}_{\mathbf{n},\mathbf{n}'} = V\{\delta_{(\mathbf{n},\mathbf{n}')}^{(e(\{\omega\}),\mathbf{k})}\}_{\mathbf{n},\mathbf{n}'} = h_{\mathbf{k}+e(\{\omega\})}, \mathbf{k} \in \mathbb{Z}_+^\Omega$$

and, consequently,

$$T'_\omega T'_{\omega'} h_{\mathbf{n}} = h_{\mathbf{n}+e(\{\omega,\omega'\})} = T'_{\omega'} T'_\omega h_{\mathbf{n}}, \quad \{\omega,\omega'\} \subset \Omega, \mathbf{n} \in \mathbb{Z}_+^\Omega.$$

$\mathbf{T}' = \{T'_\omega\}_{\omega \in \Omega}$ is a multi-contraction on \mathcal{H}' and, since

$$\mathbf{T}'^{\mathbf{n}} h_{\mathbf{0}} = h_{\sum_{\omega \in \Omega} n_\omega e(\{\omega\})} = h_{\mathbf{n}}, \quad \mathbf{n} \in \mathbb{Z}_+^\Omega,$$

\mathbf{T}' is a solution for Problem C. In addition, by (c), for any $x_{\mathbf{m}} = \sum_{\mathbf{m}'} c_{\mathbf{m},\mathbf{m}'} h_{\mathbf{m}'}$ $\in \mathcal{H}'$ ($\mathbf{m} \in \mathbb{Z}_+^\Omega$),

$$\begin{aligned} & \sum_{\mathbf{m},\mathbf{n} \in \mathbb{Z}_+^\Omega} \langle [\mathbf{T}'^{(\mathbf{m}-\mathbf{n})^-}]^* \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^+} x_{\mathbf{m}}, x_{\mathbf{n}} \rangle \\ &= \sum_{\mathbf{m},\mathbf{n}} \langle \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^+} x_{\mathbf{m}}, \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^-} x_{\mathbf{n}} \rangle \\ &= \sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'} c_{\mathbf{m},\mathbf{m}'} \overline{c_{\mathbf{n},\mathbf{n}'}} \langle \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^+} h_{\mathbf{m}'}, \mathbf{T}'^{(\mathbf{m}-\mathbf{n})^-} h_{\mathbf{n}'} \rangle \\ &= \sum_{\mathbf{m},\mathbf{m}',\mathbf{n},\mathbf{n}'} c_{\mathbf{m},\mathbf{m}'} \overline{c_{\mathbf{n},\mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n})^+ + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n})^- + \mathbf{n}'} \rangle \geq 0. \end{aligned}$$

The positivity condition (1) is then verified and \mathbf{T}' has a regular dilation. Extend \mathbf{T}' on \mathcal{H} by $T_\omega = T'_\omega \oplus 0_{\mathcal{H}''}$ ($\omega \in \Omega$) corresponding to the orthogonal decomposition $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$. $\mathbf{T} = \{T_\omega\}_{\omega \in \Omega}$ is a multi-contraction on \mathcal{H} that extends \mathbf{T}' . \mathbf{T} is still a solution of Problem C and, since

$$\begin{aligned} & \langle \sum_{v \subset u} (-1)^{|v|} (\mathbf{T}^{e(v)})^* \mathbf{T}^{e(v)} (h' + h''), h' + h'' \rangle \\ &= \sum_{v \subset u} (-1)^{|v|} \|\mathbf{T}'^{e(v)} h'\|^2 + \|h'\|^2 + \|h''\|^2 \geq 0, \quad h' \in \mathcal{H}', h'' \in \mathcal{H}'', u \subset \Omega \text{ finite,} \end{aligned}$$

\mathbf{T} has regular dilation (following condition (2) for regularity). □

In particular, for Ω with $|\Omega| = 2$, we can improve the solution to Problem B. The condition that $\{h_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}_+^\Omega}$ spans \mathcal{H} is no longer necessary, and the regularity condition is actually a consequence of the positivity condition. More precisely, we have the following theorem.

Theorem B+. Let $\{h_n\}_{n \in \mathbb{Z}_+^2}$ be a given sequence of vectors in \mathcal{H} . The following conditions are equivalent:

(a) Problem B has a solution $\mathbf{T} = (T_1, T_2)$ having regular dilation;

(b)

$$\left\| \sum_{\mathbf{m}, \mathbf{m}'} c_{\mathbf{m}, \mathbf{m}'} h_{\mathbf{m} + \mathbf{m}'} \right\|^2 \leq \sum_{\mathbf{m}, \mathbf{m}', \mathbf{n}, \mathbf{n}'} c_{\mathbf{m}, \mathbf{m}'} \overline{c_{\mathbf{n}, \mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n}) + + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n}) - + \mathbf{n}'} \rangle,$$

for every finite family $\{c_{\mathbf{n}, \mathbf{n}'}\}_{\mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^2}$ of complex numbers;

(c)

$$\sum_{\mathbf{m}, \mathbf{m}', \mathbf{n}, \mathbf{n}'} c_{\mathbf{m}, \mathbf{m}'} \overline{c_{\mathbf{n}, \mathbf{n}'}} \langle h_{(\mathbf{m}-\mathbf{n}) + + \mathbf{m}'}, h_{(\mathbf{m}-\mathbf{n}) - + \mathbf{n}'} \rangle \geq 0,$$

for every finite family $\{c_{\mathbf{n}, \mathbf{n}'}\}_{\mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^2}$ of complex numbers;

(d)

$$\left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n} + e_1} \right\|^2 + \left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n} + e_2} \right\|^2 \leq \left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}} \right\|^2 + \left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n} + e_1 + e_2} \right\|^2,$$

$$\left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n} + e_1} \right\| \leq \left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}} \right\| \quad \text{and} \quad \left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n} + e_2} \right\| \leq \left\| \sum_{\mathbf{n}} c_{\mathbf{n}} h_{\mathbf{n}} \right\|,$$

for every finite family $\{c_{\mathbf{n}, \mathbf{n}'}\}_{\mathbf{n}, \mathbf{n}' \in \mathbb{Z}_+^2}$ of complex numbers (we use here $e_1 = (1, 0)$ and $e_2 = (0, 1)$).

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