ON GROUP OPERATIONS ON HOMOGENEOUS SPACES

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Abstract. It is proved that every countably infinite homogeneous regular space admits a structure of any countably infinite group with continuous left shifts.

It is well known that not all infinite groups admit a non-discrete group topology (see, for example, [1, §9]). However, every infinite group admits a non-discrete zero-dimensional topology with continuous left shifts [3], [4], and every countably infinite group admits a non-discrete zero-dimensional topology with continuous shifts and inversion [6]. It is well known also that not all homogeneous spaces admit a structure of a topological group and even a structure of a group with continuous left shifts (see, for example, [1, §10]). The aim of this note is to prove that every countably infinite homogeneous regular space admits a structure of any countably infinite group with continuous left shifts.

We begin with the Boolean version of this result.

Theorem 1. Let X be a countably infinite homogeneous regular space. Then there is a Boolean group operation + on X with continuous shifts.

To prove Theorem 1 we need the following lemma.

Lemma. Let X be a countably infinite homogeneous regular space and let x, y ∈ X, x ≠ y. Then there are a clopen neighborhood U of x and a homeomorphism h : X → X such that h(x) = y, h(U) = X \ U and h² = id_X.

Proof.Enumerate X as {x_n : n < ω}. Since X is homogeneous, there is a homeomorphism g_0 : X → X with g_0(x) = y. Since X is countable and regular, therefore zero-dimensional, one may choose a clopen neighborhood U_0 of x with U_0 \ g_0(U_0) = ∅. Put X_0 = U_0 ∪ g_0(U_0) and define h_0 : X_0 → X_0 by

h_0(x) = \begin{cases} g_0(x) & \text{if } x \in U_0, \\ g_0^{-1}(x) & \text{if } x \in g_0(U_0). \end{cases}

If X_0 = X, put U = U_0 and h = h_0. Otherwise, choose the first element x_{n_1} in the sequence {x_n : n < ω} with x_{n_1} \notin X_0 and pick any y_{n_1} ∈ X \ (X_0 ∪ {x_{n_1}}). Let g_1 : X → X be a homeomorphism with g_1(x_{n_1}) = y_{n_1} and let U_1 be a clopen
neighborhood of \( x_{n_1} \) with \( U_1 \cap g_1(U_1) = \emptyset \) and \( U_1 \cup g_1(U_1) \subseteq X \setminus X_0 \). Put \( X_1 = X_0 \cup U_1 \cup g_1(U_1) \) and extend \( h_0 \) to \( h_1 : X_1 \to X_1 \) by

\[
h_1(x) = \begin{cases} 
h_0(x) & \text{if } x \in X_0, 
g_1(x) & \text{if } x \in U_1, 
g_1^{-1}(x) & \text{if } x \in g_1(U_1). 
\end{cases}
\]

If \( X_1 = X \), put \( U = U_0 \cup U_1 \) and \( h = h_1 \). Otherwise, choose the first element \( x_{n_2} \) in the sequence \( \{x_n : n < \omega \} \) with \( x_{n_2} \notin X_1 \), pick any \( y_{n_2} \in X \setminus (X_1 \cup \{x_{n_2}\}) \), and so forth.

If \( X_n = X \) for some \( n \), put \( U = \bigcup_{i \leq n} U_i \) and \( h = h_n \). Otherwise, put \( U = \bigcup_{n < \omega} U_n \) and \( h = \bigcup_{n < \omega} h_n \). \( \square \)

**Proof of Theorem 1.** We can assume that \( X \) is not discrete. Pick any element \( u \in X \) and enumerate the set \( X \setminus \{u\} \) as \( \{x_n : n < \omega\} \). By the Lemma, we can construct a decreasing sequence \( \{U_n : n < \omega\} \) of clopen neighborhoods of \( u \) with \( U_0 = X \) and a sequence of homeomorphisms \( h_n : U_n \to U_n \) such that \( h_n(U_{n+1}) = U_n \setminus U_{n+1} \) and \( h_n^2 = id_{U_n} \). By induction on \( n \), it is easy to verify that for each \( n < \omega \), the subsets \( h_0^{x_0} \cdot \cdots \cdot h_n^{x_n}(U_{n+1}) \), where \( n_i \in \{0, 1\} \), form a partition of \( X \) and every \( x \in X \) can be uniquely written in the form \( x = h_0^{x_0} \cdot \cdots \cdot h_n^{x_n}(y) \), where \( y \in U_{n+1} \). We construct sequences \( \{U_n : n < \omega\} \) and \( \{h_n : n < \omega\} \) satisfying, in addition, the condition \( x_n \in B_n = \{h_0^{x_0} \cdot \cdots \cdot h_n^{x_n}(u) : n_i \in \{0, 1\}, i \leq n\} \). To see that this can be done, assume that \( x_n \notin B_{n-1} \). Then \( x_n \in h_0^{x_0} \cdot \cdots \cdot h_{n-1}^{x_{n-1}}(y_n) \) for some \( n_i \in \{0, 1\} \) and \( y_n \in U_n \). Choose \( h_n \) such that \( h_n(u) = y_n \).

Now let \( x, y \in X \). By the construction, there exists large enough \( n \) such that \( x \) and \( y \) can be uniquely written in the form \( x = h_0^{x_0} \cdot \cdots \cdot h_n^{x_n}(u) \), \( y = h_0^{y_0} \cdot \cdots \cdot h_n^{y_n}(u) \). Put \( x + y = h_0^{x_0+y_0} \cdot \cdots \cdot h_n^{x_n+y_n}(u) \). It is clear that the operation is well defined and that \( (X, +) \) is a Boolean group with zero \( u \). We note that for every \( z \in U_{n+1} \), \( x + z = h_0^{x_0} \cdot \cdots \cdot h_n^{x_n}(z) \). To check that the left shifts in \( (X, +) \) are continuous, let \( U \) be a neighborhood of \( x \). Choose a neighborhood \( V \) of \( u \) such that \( U \subseteq U_{n+1} \) and \( h_0^{x_0} \cdot \cdots \cdot h_n^{x_n}(V) \subseteq U \). Then \( W = h_0^{x_0} \cdot \cdots \cdot h_n^{x_n}(V) \) is a neighborhood of \( y \) and for every \( z \in V \), \( x + h_0^{x_0} \cdot \cdots \cdot h_n^{x_n}(z) = x + (y + z) = (x + y) + z = h_0^{x_0+y_0} \cdot \cdots \cdot h_n^{x_n+y_n}(z) \in U \); so \( x + W \subseteq U \). \( \square \)

Next we need the Local Isomorphism Theorem. It is close to [5] Theorem 2] (see also [2] Lemma 7.4).

A space with a group operation is called a **left topological group** if all left shifts are continuous. A space \( X \) with a partial binary operation \( \cdot \) and a distinguished element 1 is called a **local left topological group**, if for each element \( x \in X \) there is an open neighborhood \( U_x \) of 1 such that

1. for any \( y \in U_x \), \( x \cdot y \) is defined, \( x \cdot 1 = x \), \( x \cdot U_x \) is an open neighborhood of \( x \), and a mapping \( U_x \ni y \mapsto x \cdot y \in x \cdot U_x \) is a homeomorphism;
2. \((x \cdot y) \cdot z = x \cdot (y \cdot z)\) if \( y \in U_x \), \( z \in U_{x \cdot y} \cap U_y \), \( y \cdot z \in U_x \).

For a local left topological group, from this point on, when we write \( x \cdot y \) we mean \( y \in U_x \) and when we write \( x \cdot U \), where \( U \) is a neighborhood of 1, we mean \( U \subseteq U_x \).

A basic example of a local left topological group is an open neighborhood of the identity of a left topological group.

Let \( X \) and \( Y \) be local left topological groups. A map \( f : X \to Y \) is called a **local homomorphism** if \( f(1_X) = 1_Y \) and for every \( x \in X \) there exists a neighborhood \( U_x \)
of \( I_X \) such that \( f(xz) = f(x)f(z) \) for all \( z \in U_x \). A bijective local homomorphism \( f \) is called a local isomorphism if \( f^{-1} \) is also a local homomorphism. We observe that every open bijective local homomorphism is a local isomorphism.

**Theorem 2.** All countably infinite non-discrete regular left topological groups are local isomorphic.

**Proof.** Let \( X \) be the countably infinite Boolean group \( \bigoplus_\omega \mathbb{Z}_2 \) endowed with the direct sum topology, and let \( Y \) be an arbitrary countably infinite non-discrete regular left topological group. We shall define a local isomorphism \( f : X \to Y \).

Let \( F \) be the semigroup of words on the letters \( 0 \) and \( 1 \) with empty word \( \emptyset \), and let \( F' \) be the subsemigroup of \( F \) of words including \( \emptyset \), in which the last letter is \( 1 \). We define a bijection \( X \ni x \mapsto w(x) \in F' \) as follows. If \( 0 \ne x = (\varepsilon_n)_{n<\omega} \) and \( m = \max\{n < \omega : \varepsilon_n = 1\} \), we put \( w(x) = \varepsilon_0 \cdots \varepsilon_m \). If \( x = 0 \), we put \( w(x) = \emptyset \).

For every \( w \in F \), \( |w| \) will denote the length of \( w \). For every \( n < \omega \), put \( W_n = \{ w \in F : |w| = n \} \). Each nonempty \( w \in F \) has a unique representation in the form \( w = w_1w_2\cdots w_k \), where \( w_l = 0^il^{j_l} \), \( 1 \leq l \leq k \), \( i_1,j_k \in \omega \), \( j_1,i_2,j_2,\ldots ,i_k \in \mathbb{N} \) (if \( k = 1 \), the requirement is \( i_1,j_1 \in \omega \) and \( i_1 + j_1 \in \mathbb{N} \)). This representation will be called canonical. Words of the form \( 0^i1^j \), where \( i,j \in \omega \) and \( i + j \in \mathbb{N} \), will be called basic. If a word \( w \) is basic or \( w = \emptyset \), we put \( w' = \emptyset \) and \( w^* = w \). Otherwise, if \( w = w_1w_2\cdots w_k \) is the canonical representation, we put \( w' = w_1w_2\cdots w_{k-1} \) and \( w^* = 0w_1\cdots w_{k-1}.w_k \).

Enumerate \( Y \setminus \{ 1_Y \} \) as \( \{y_n : 0 < n < \omega \} \). We shall construct a clopen \( Y(w) \subseteq Y \) and \( y(w) \in Y(w) \) for every \( w \in F \) such that \( Y(\emptyset) = Y, y(\emptyset) = 1_Y \) and the following conditions hold for all \( n \in \mathbb{N} \):

1. \( Y(w0) \cup Y(w1) = Y(w) \) and \( Y(w0) \cap Y(w1) = \emptyset \) for all \( w \in W_{n-1} \);
2. \( y(\emptyset) = y(w) \) for all \( w \in W_{n-1} \);
3. \( Y(w) = y(w')Y(w^*) \) and \( y(w) = y(w')y(w^*) \) for all \( w \in W_n \);
4. \( y_n \in \{y(w) : w \in W_n\} \).

We take as \( Y(\emptyset) \) a clopen neighborhood of \( 1_Y \) such that \( y_1 \notin Y(\emptyset) \). Put \( Y(1) = Y \setminus Y(\emptyset), y(0) = 1_Y \) and \( y(1) = y_1 \).

Suppose that \( Y(w) \) and \( y(w) \) have already been constructed for all \( w \in W_n \) such that conditions (1)–(4) hold.

It is obvious that the subsets \( Y(w) \), where \( w \in W_n \), form a partition of \( Y \). So, one of them, say \( Y(u) \), contains \( y_{n+1} \). For some \( z_{n+1} \in Y(u^+) \), \( y_{n+1} = y(u^+)z_{n+1} \). If \( z_{n+1} = y(u^+) \), we take as \( Y(0^{n+1}) \) any clopen neighborhood of \( 1_Y \) such that \( Y(w) \setminus y(w)Y(0^{n+1}) = \emptyset \) for all basic \( w \in W_n \). Then for every basic \( w \in W_n \), put \( y(w0) = y(w), Y(w0) = y(w)Y(0^{n+1}) \) and \( Y(w1) = Y(w) \setminus Y(w0) \), and take as \( Y(w1) \) any element of \( Y(w1) \). If \( z_{n+1} \neq y(u^+) \), we take as \( Y(0^{n+1}) \) in addition such that \( z_{n+1} \notin Y(u^+)Y(0^{n+1}) \) and \( y(u^+) = z_{n+1} \). For all non-basic \( v \in W_{n+1} \), we define \( Y(v) \) and \( y(v) \) by condition (3). Then \( y(v) = y(v')y(v^*) \in y(v')Y(v^*) = Y(v) \) and \( y_{n+1} = y(u^+)y(u^+) = y((u1)^+)y((u1)^+) = y(u1) \). To check conditions (2) and (1), let \( w \in W_n \). Then

\[
y(w0) = y((w0)^+)y((w0)^+) = y(w)y(0^{n+1}) = y(w),
\]

\[
Y(w0) = y(w)Y(0^{n+1}) = y(w')y(w^*)Y(0^{n+1}) = y(w')Y(w^*),
\]

\[
Y(w1) = y((w1)^+)Y((w1)^+) = y(w')Y(w^*),
\]
Therefore, group operation $+_X$ on $X$.

We may suppose that $F'' \ni w \mapsto y(w) \in Y$. It follows from (4), (2) and (1) that it is a bijection. Define the bijection $f : X \to Y$ by $f(x) = y(w(x))$. To verify that $f$ is a local homomorphism, let $x \in X$, $w(x) = u$. Take any $z \in U_{|u|+1}$, where $U_n = \{(\varepsilon_i)_{i<\omega} \in X : \varepsilon_i = 0 \text{ for all } i < n\}$. Then $w(z) = 0^{|u|+1}v$ and $w(x + z) = u0v$. It is easy to show by induction on the length of the canonical representations using (3) that $y(u0v) = y(u)y(0^{|u|+1}v)$. Therefore, $f(x + z) = y(u0v) = y(u)y(0^{|u|+1}v) = f(x)f(z)$. To see that $f$ is a local isomorphism, we note that $f(U_{n+1}) = Y(0^n)$; so $f$ is open.

Now we can prove our main result.

**Theorem 3.** Let $X$ be a countably infinite homogeneous regular space, and let $G$ be a countably infinite group. Then there is a group operation $*$ on $X$ with continuous left shifts such that $(X, *)$ is isomorphic to $G$.

**Proof.** We may suppose that $X$ is not discrete. By Theorem 1, there is a Boolean group operation $+$ on $X$ with continuous shifts. We endow $G$ with any non-discrete regular topology with continuous left shifts. By Theorem 2, there is a local isomorphism $f : (X, +) \to G$ (it even suffices that $f$ be a bijective local homomorphism). For any $x, y \in X$, we define $x \ast y = f^{-1}(f(x)f(y))$. Obviously, $(X, \ast)$ is a group isomorphic to $G$. Next, given any $x \in X$, we can choose a neighborhood $U$ of the identity such that $f(x + z) = f(x)f(z)$ for all $z \in U$, and then $x \ast z = f^{-1}(f(x)f(z)) = f^{-1}(f(x + z)) = x + z$. It follows from this that the left shifts of $(X, \ast)$ are continuous and open at the identity. Consequently, the left shifts of $(X, \ast)$ are continuous.

**References**


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