

## ON GROUP OPERATIONS ON HOMOGENEOUS SPACES

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**ABSTRACT.** It is proved that every countably infinite homogeneous regular space admits a structure of any countably infinite group with continuous left shifts.

It is well known that not all infinite groups admit a non-discrete group topology (see, for example, [1, §9]). However, every infinite group admits a non-discrete zero-dimensional topology with continuous left shifts [3], [4], and every countably infinite group admits a non-discrete zero-dimensional topology with continuous shifts and inversion [6]. It is well known also that not all homogeneous spaces admit a structure of a topological group and even a structure of a group with continuous left shifts (see, for example, [1, §10]). The aim of this note is to prove that every countably infinite homogeneous regular space admits a structure of any countably infinite group with continuous left shifts.

We begin with the Boolean version of this result.

**Theorem 1.** *Let  $X$  be a countably infinite homogeneous regular space. Then there is a Boolean group operation  $+$  on  $X$  with continuous shifts.*

To prove Theorem 1 we need the following lemma.

**Lemma.** *Let  $X$  be a countably infinite homogeneous regular space and let  $x, y \in X$ ,  $x \neq y$ . Then there are a clopen neighborhood  $U$  of  $x$  and a homeomorphism  $h : X \rightarrow X$  such that  $h(x) = y$ ,  $h(U) = X \setminus U$  and  $h^2 = id_X$ .*

*Proof.* Enumerate  $X$  as  $\{x_n : n < \omega\}$ . Since  $X$  is homogeneous, there is a homeomorphism  $g_0 : X \rightarrow X$  with  $g_0(x) = y$ . Since  $X$  is countable and regular, therefore zero-dimensional, one may choose a clopen neighborhood  $U_0$  of  $x$  with  $U_0 \cap g_0(U_0) = \emptyset$ . Put  $X_0 = U_0 \cup g_0(U_0)$  and define  $h_0 : X_0 \rightarrow X_0$  by

$$h_0(x) = \begin{cases} g_0(x) & \text{if } x \in U_0, \\ g_0^{-1}(x) & \text{if } x \in g_0(U_0). \end{cases}$$

If  $X_0 = X$ , put  $U = U_0$  and  $h = h_0$ . Otherwise, choose the first element  $x_{n_1}$  in the sequence  $\{x_n : n < \omega\}$  with  $x_{n_1} \notin X_0$  and pick any  $y_{n_1} \in X \setminus (X_0 \cup \{x_{n_1}\})$ . Let  $g_1 : X \rightarrow X$  be a homeomorphism with  $g_1(x_{n_1}) = y_{n_1}$  and let  $U_1$  be a clopen

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neighborhood of  $x_{n_1}$  with  $U_1 \cap g_1(U_1) = \emptyset$  and  $U_1 \cup g_1(U_1) \subseteq X \setminus X_0$ . Put  $X_1 = X_0 \cup U_1 \cup g_1(U_1)$  and extend  $h_0$  to  $h_1 : X_1 \rightarrow X_1$  by

$$h_1(x) = \begin{cases} h_0(x) & \text{if } x \in X_0, \\ g_1(x) & \text{if } x \in U_1, \\ g_1^{-1}(x) & \text{if } x \in g_1(U_1). \end{cases}$$

If  $X_1 = X$ , put  $U = U_0 \cup U_1$  and  $h = h_1$ . Otherwise, choose the first element  $x_{n_2}$  in the sequence  $\{x_n : n < \omega\}$  with  $x_{n_2} \notin X_1$ , pick any  $y_{n_2} \in X \setminus (X_1 \cup \{x_{n_2}\})$ , and so forth.

If  $X_n = X$  for some  $n$ , put  $U = \bigcup_{i \leq n} U_i$  and  $h = h_n$ . Otherwise, put  $U = \bigcup_{n < \omega} U_n$  and  $h = \bigcup_{n < \omega} h_n$ . □

*Proof of Theorem 1.* We can assume that  $X$  is not discrete. Pick any element  $u \in X$  and enumerate the set  $X \setminus \{u\}$  as  $\{x_n : n < \omega\}$ . By the Lemma, we can construct a decreasing sequence  $\{U_n : n < \omega\}$  of clopen neighborhoods of  $u$  with  $U_0 = X$  and a sequence of homeomorphisms  $h_n : U_n \rightarrow U_n$  such that  $h_n(U_{n+1}) = U_n \setminus U_{n+1}$  and  $h_n^2 = id_{U_n}$ . By induction on  $n$ , it is easy to verify that for each  $n < \omega$  the subsets  $h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(U_{n+1})$ , where  $\varepsilon_i \in \{0, 1\}$ , form a partition of  $X$  and every  $x \in X$  can be uniquely written in the form  $x = h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(y)$ , where  $y \in U_{n+1}$ . We construct sequences  $\{U_n : n < \omega\}$  and  $\{h_n : n < \omega\}$  satisfying, in addition, the condition  $x_n \in B_n = \{h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(u) : \varepsilon_i \in \{0, 1\}, i \leq n\}$ . To see that this can be done, assume that  $x_n \notin B_{n-1}$ . Then  $x_n \in h_0^{\varepsilon_0} \cdots h_{n-1}^{\varepsilon_{n-1}}(y_n)$  for some  $\varepsilon_i \in \{0, 1\}$  and  $y_n \in U_n$ . Choose  $h_n$  with  $h_n(u) = y_n$ .

Now let  $x, y \in X$ . By the construction, there exists large enough  $n$  such that  $x$  and  $y$  can be uniquely written in the form  $x = h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(u)$ ,  $y = h_0^{\delta_0} \cdots h_n^{\delta_n}(u)$ . Put  $x + y = h_0^{\varepsilon_0 + \delta_0} \cdots h_n^{\varepsilon_n + \delta_n}(u)$ . It is clear that the operation is well defined and that  $(X, +)$  is a Boolean group with zero  $u$ . We note that for every  $z \in U_{n+1}$ ,  $x + z = h_0^{\varepsilon_0} \cdots h_n^{\varepsilon_n}(z)$ . To check that the left shifts in  $(X, +)$  are continuous, let  $U$  be a neighborhood of  $x + y$ . Choose a neighborhood  $V$  of  $u$  such that  $V \subseteq U_{n+1}$  and  $h_0^{\varepsilon_0 + \delta_0} \cdots h_n^{\varepsilon_n + \delta_n}(V) \subseteq U$ . Then  $W = h_0^{\delta_0} \cdots h_n^{\delta_n}(V)$  is a neighborhood of  $y$  and for every  $z \in V$ ,  $x + h_0^{\delta_0} \cdots h_n^{\delta_n}(z) = x + (y + z) = (x + y) + z = h_0^{\varepsilon_0 + \delta_0} \cdots h_n^{\varepsilon_n + \delta_n}(z) \in U$ ; so  $x + W \subseteq U$ . □

Next we need the Local Isomorphism Theorem. It is close to [5, Theorem 2] (see also [2, Lemma 7.4]).

A space with a group operation is called a *left topological group* if all left shifts are continuous. A space  $X$  with a partial binary operation  $\cdot$  and a distinguished element 1 is called a *local left topological group*, if for each element  $x \in X$  there is an open neighborhood  $U_x$  of 1 such that

- (1) for any  $y \in U_x$ ,  $x \cdot y$  is defined,  $x \cdot 1 = x$ ,  $x \cdot U_x$  is an open neighborhood of  $x$ , and a mapping  $U_x \ni y \mapsto x \cdot y \in x \cdot U_x$  is a homeomorphism;
- (2)  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$  if  $y \in U_x$ ,  $z \in U_{x \cdot y} \cap U_y$ ,  $y \cdot z \in U_x$ .

For a local left topological group, from this point on, when we write  $x \cdot y$  we mean  $y \in U_x$  and when we write  $x \cdot U$ , where  $U$  is a neighborhood of 1, we mean  $U \subseteq U_x$ . A basic example of a local left topological group is an open neighborhood of the identity of a left topological group.

Let  $X$  and  $Y$  be local left topological groups. A map  $f : X \rightarrow Y$  is called a *local homomorphism* if  $f(1_X) = 1_Y$  and for every  $x \in X$  there exists a neighborhood  $U_x$

of  $1_X$  such that  $f(xz) = f(x)f(z)$  for all  $z \in U_x$ . A bijective local homomorphism  $f$  is called a *local isomorphism* if  $f^{-1}$  is also a local homomorphism. We observe that every open bijective local homomorphism is a local isomorphism.

**Theorem 2.** *All countably infinite non-discrete regular left topological groups are local isomorphic.*

*Proof.* Let  $X$  be the countably infinite Boolean group  $\bigoplus_{\omega} \mathbb{Z}_2$  endowed with the direct sum topology, and let  $Y$  be an arbitrary countably infinite non-discrete regular left topological group. We shall define a local isomorphism  $f : X \rightarrow Y$ .

Let  $F$  be the semigroup of words on the letters 0 and 1 with empty word  $\emptyset$ , and let  $F'$  be the subsemigroup of  $F$  of words including  $\emptyset$ , in which the last letter is 1. We define a bijection  $X \ni x \mapsto w(x) \in F'$  as follows. If  $0 \neq x = (\varepsilon_n)_{n < \omega}$  and  $m = \max\{n < \omega : \varepsilon_n = 1\}$ , we put  $w(x) = \varepsilon_0 \cdots \varepsilon_m$ . If  $x = 0$ , we put  $w(x) = \emptyset$ .

For every  $w \in F$ ,  $|w|$  will denote the length of  $w$ . For every  $n < \omega$ , put  $W_n = \{w \in F : |w| = n\}$ . Each nonempty  $w \in F$  has a unique representation in the form  $w = w_1 w_2 \cdots w_k$ , where  $w_l = 0^{i_l} 1^{j_l}$ ,  $1 \leq l \leq k$ ,  $i_l, j_l \in \omega$ ,  $j_1, i_2, j_2, \dots, i_k \in \mathbb{N}$  (if  $k = 1$ , the requirement is  $i_1, j_1 \in \omega$  and  $i_1 + j_1 \in \mathbb{N}$ ). This representation will be called canonical. Words of the form  $0^i 1^j$ , where  $i, j \in \omega$  and  $i + j \in \mathbb{N}$ , will be called basic. If a word  $w$  is basic or  $w = \emptyset$ , we put  $w' = \emptyset$  and  $w^* = w$ . Otherwise, if  $w = w_1 w_2 \cdots w_k$  is the canonical representation, we put  $w' = w_1 w_2 \cdots w_{k-1}$  and  $w^* = 0^{|w_1| + \cdots + |w_{k-1}|} w_k$ .

Enumerate  $Y \setminus \{1_Y\}$  as  $\{y_n : 0 < n < \omega\}$ . We shall construct a clopen  $Y(w) \subseteq Y$  and  $y(w) \in Y(w)$  for every  $w \in F$  such that  $Y(\emptyset) = Y$ ,  $y(\emptyset) = 1_Y$  and the following conditions hold for all  $n \in \mathbb{N}$ :

- (1)  $Y(w0) \cup Y(w1) = Y(w)$  and  $Y(w0) \cap Y(w1) = \emptyset$  for all  $w \in W_{n-1}$ ;
- (2)  $y(w0) = y(w)$  for all  $w \in W_{n-1}$ ;
- (3)  $Y(w) = y(w')Y(w^*)$  and  $y(w) = y(w')y(w^*)$  for all  $w \in W_n$ ;
- (4)  $y_n \in \{y(w) : w \in W_n\}$ .

We take as  $Y(0)$  a clopen neighborhood of  $1_Y$  such that  $y_1 \notin Y(0)$ . Put  $Y(1) = Y \setminus Y(0)$ ,  $y(0) = 1_Y$  and  $y(1) = y_1$ .

Suppose that  $Y(w)$  and  $y(w)$  have already been constructed for all  $w \in W_n$  such that conditions (1)–(4) hold.

It is obvious that the subsets  $Y(w)$ , where  $w \in W_n$ , form a partition of  $Y$ . So, one of them, say  $Y(u)$ , contains  $y_{n+1}$ . For some  $z_{n+1} \in Y(u^*)$ ,  $y_{n+1} = y(u')z_{n+1}$ . If  $z_{n+1} = y(u^*)$ , we take as  $Y(0^{n+1})$  any clopen neighborhood of  $1_Y$  such that  $Y(w) \setminus y(w)Y(0^{n+1}) \neq \emptyset$  for all basic  $w \in W_n$ . Then for every basic  $w \in W_n$ , put  $y(w0) = y(w)$ ,  $Y(w0) = y(w)Y(0^{n+1})$  and  $Y(w1) = Y(w) \setminus Y(w0)$ , and take as  $y(w1)$  any element of  $Y(w1)$ . If  $z_{n+1} \neq y(u^*)$ , we take as  $Y(0^{n+1})$  in addition such that  $z_{n+1} \notin y(u^*)Y(0^{n+1})$  and put  $y(u^*1) = z_{n+1}$ . For all non-basic  $v \in W_{n+1}$ , we define  $Y(v)$  and  $y(v)$  by condition (3). Then  $y(v) = y(v')y(v^*) \in y(v')Y(v^*) = Y(v)$  and  $y_{n+1} = y(u')y(u^*1) = y((u1)')y((u1)^*) = y(u1)$ . To check conditions (2) and (1), let  $w \in W_n$ . Then

$$\begin{aligned} y(w0) &= y((w0)')y((w0)^*) = y(w)y(0^{n+1}) = y(w), \\ Y(w0) &= y(w)Y(0^{n+1}) = y(w')y(w^*)Y(0^{n+1}) = y(w')Y(w^*0), \\ Y(w1) &= y((w1)')Y((w1)^*) = y(w')Y(w^*1), \end{aligned}$$

and so

$$Y(w0) \cup Y(w1) = y(w')[Y(w^*0) \cup Y(w^*1)] = y(w')Y(w^*) = Y(w),$$

$$Y(w0) \cap Y(w1) = \emptyset.$$

After this construction we have obtained the mapping  $F' \ni w \mapsto y(w) \in Y$ . It follows from (4), (2) and (1) that it is a bijection. Define the bijection  $f : X \rightarrow Y$  by  $f(x) = y(w(x))$ . To verify that  $f$  is a local homomorphism, let  $x \in X$ ,  $w(x) = u$ . Take any  $z \in U_{|u|+1}$ , where  $U_n = \{(\varepsilon_i)_{i < \omega} \in X : \varepsilon_i = 0 \text{ for all } i < n\}$ . Then  $w(z) = 0^{|u|+1}v$  and  $w(x+z) = u0v$ . It is easy to show by induction on the length of the canonical representations using (3) that  $y(u0v) = y(u)y(0^{|u|+1}v)$ . Therefore,  $f(x+z) = y(u0v) = y(u)y(0^{|u|+1}v) = f(x)f(z)$ . To see that  $f$  is a local isomorphism, we note that  $f(U_{n+1}) = Y(0^n)$ ; so  $f$  is open.  $\square$

Now we can prove our main result.

**Theorem 3.** *Let  $X$  be a countably infinite homogeneous regular space, and let  $G$  be a countably infinite group. Then there is a group operation  $*$  on  $X$  with continuous left shifts such that  $(X, *)$  is isomorphic to  $G$ .*

*Proof.* We may suppose that  $X$  is not discrete. By Theorem 1, there is a Boolean group operation  $+$  on  $X$  with continuous shifts. We endow  $G$  with any non-discrete regular topology with continuous left shifts. By Theorem 2, there is a local isomorphism  $f : (X, +) \rightarrow G$  (it even suffices that  $f$  be a bijective local homomorphism). For any  $x, y \in X$ , we define  $x * y = f^{-1}(f(x)f(y))$ . Obviously,  $(X, *)$  is a group isomorphic to  $G$ . Next, given any  $x \in X$ , we can choose a neighborhood  $U$  of the identity such that  $f(x+z) = f(x)f(z)$  for all  $z \in U$ , and then  $x * z = f^{-1}(f(x)f(z)) = f^{-1}(f(x+z)) = x + z$ . It follows from this that the left shifts of  $(X, *)$  are continuous and open at the identity. Consequently, the left shifts of  $(X, *)$  are continuous.  $\square$

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