

ASYMPTOTIC ESTIMATES  
FOR A TWO-DIMENSIONAL PROBLEM  
WITH POLYNOMIAL NONLINEARITY

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ABSTRACT. In this paper we give asymptotic estimates of the least energy solution  $u_p$  of the functional

$$J(u) = \int_{\Omega} |\nabla u|^2 \quad \text{constrained on the manifold} \quad \int_{\Omega} |u|^{p+1} = 1$$

as  $p$  goes to infinity. Here  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^2$ . Among other results we give a positive answer to a question raised by Chen, Ni, and Zhou (2000) by showing that  $\lim_{p \rightarrow \infty} \|u_p\|_{\infty} = \sqrt{e}$ .

1. INTRODUCTION

In this paper we consider the following elliptic minimization problem. Let us define a  $C^2$ -functional on  $H_0^1(\Omega)$ :

$$(1.1) \quad J(u) = \int_{\Omega} |\nabla u|^2 \quad \text{constrained on the manifold} \quad \int_{\Omega} |u|^{p+1} = 1$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^2$  and  $p$  is a real number greater than 1. Then we define

$$(1.2) \quad S_p = \inf_{u \in H_0^1(\Omega)} J(u).$$

By standard results it is easy to see that  $S_p$  is achieved at a function  $u_p \in H_0^1(\Omega)$  that satisfies

$$(1.3) \quad \begin{cases} -\Delta u_p = S_p u_p^p & \text{in } \Omega, \\ u_p > 0 & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1 we get  $S_p = O\left(\frac{1}{p}\right)$  for  $p$  large. Setting  $v_p = S_p^{\frac{1}{p-1}} u_p$  we are in the framework of [8], [9] and [6] where some asymptotic results about this problem were obtained.

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In particular, it was proved in [8] and [9] that the minimizer  $u_p$  looks like a sharp “spike”. More precisely it was shown that, for a suitable sequence  $p_n \rightarrow \infty$ ,

$$(1.4) \quad \|u_{p_n}\|_\infty \leq C,$$

and in the sense of distribution,

$$(1.5) \quad \frac{u_{p_n}^{p_n}}{\|u_{p_n}\|_\infty^{p_n}} \rightarrow \delta_{x_0}.$$

Moreover, the point  $x_{p_n}$  where the minimizer  $u_{p_n}$  achieves its maximum converges to a critical point of the Robin function.

In this paper we obtain estimates of a different nature, greatly improving some partial results obtained in [5], where uniqueness and qualitative properties of the least energy solution were proved.

Here we use the two-dimensional blow-up technique introduced in [1]. The blow-up function is obtained by linearizing the nonlinear term  $\text{plog } u_p$  around the point of maximum of  $u_p$ . More precisely let us define the function  $z_p(x) : \Omega_p = \frac{\Omega - x_p}{\varepsilon_p} \mapsto \mathbb{R}$ ,

$$(1.6) \quad z_p(x) = \frac{p}{u_p(x_p)} (u_p(\varepsilon_p x + x_p) - u_p(x_p))$$

where  $x_p$  is the point where  $u_p$  achieves its maximum and  $\varepsilon_p^2 = \frac{1}{pS_p u_p(x_p)^{p-1}}$ . Then we obtain the following.

**Theorem 1.1.** *For any sequence  $z_{p_n}$  with  $p_n \rightarrow \infty$ , there exists a subsequence of  $z_{p_n}$ , still denoted by  $z_{p_n}$ , such that  $z_{p_n} \rightarrow z$  in  $C_{loc}^2(\mathbb{R}^2)$ , where  $z(x) = \log \frac{1}{(1 + \frac{|x|^2}{8})^2}$ .*

The main result of the paper is a consequence of the above theorem.

**Theorem 1.2.** *Let  $u_p$  be a solution to (1.3). Then*

$$(1.7) \quad \lim_{p \rightarrow \infty} \|u_p\|_\infty = \sqrt{e}.$$

Note that the estimate  $\limsup_{p \rightarrow \infty} \|u_p\|_\infty \leq \sqrt{e}$  was proved in [9]. Here we give a positive answer to a question raised in [4], where some numerical computations suggested the validity of (1.7).

## 2. PROOF OF THEOREM 1.1

In this section we recall some results about the solution  $u_p$ , and then we give the proof of Theorem 1.1.

We start by recalling the following estimates on  $S_p$ , due to Ren and Wei ([8]).

**Lemma 2.1.** *Let  $S_p$  be defined as in (1.2). Then*

$$(2.1) \quad \lim_{p \rightarrow \infty} pS_p = 8\pi e.$$

*Proof.* Setting  $v_p = (S_p)^{\frac{1}{p-1}} u_p$  we have that  $v_p$  also achieves  $S_p$  and satisfies

$$(2.2) \quad \begin{cases} -\Delta v_p = v_p^p & \text{in } \Omega, \\ v_p > 0 & \text{in } \Omega, \\ v_p = 0 & \text{on } \partial\Omega. \end{cases}$$

From Corollary 2.3 of [8] we get  $\lim_{p \rightarrow \infty} p \int_\Omega v_p^{p+1} = 8\pi e$ . Hence, recalling that  $\int_\Omega u_p^{p+1} = 1$  we derive that  $\lim_{p \rightarrow \infty} p(S_p)^{\frac{p+1}{p-1}} = 8\pi e$ , which implies (2.1).  $\square$

Let us denote by  $x_p$  the point where  $u_p(x_p) = \|u_p\|_\infty$ . By Lemma 4.1 of [8] we know that  $x_p$  is far away from the boundary of  $\Omega$ . The next lemma provides additional information on the rate of  $u_p(x_p)$ .

**Lemma 2.2.** *We have that*

$$(2.3) \quad \lim_{p \rightarrow \infty} u_p(x_p)^{p-1} = +\infty.$$

*Proof.* Let us denote by  $\lambda_1(\Omega)$  the first eigenvalue of  $-\Delta$  with zero Dirichlet boundary condition. Then we have

$$(2.4) \quad 1 = \int_\Omega |u_p|^{p+1} \leq u_p(x_p)^{p-1} \int_\Omega |u_p|^2 \leq u_p(x_p)^{p-1} \lambda_1(\Omega)^{-1} \int_\Omega |\nabla u_p|^2.$$

Recalling that  $\int_\Omega |\nabla u_p|^2 = S_p$  and Lemma 2.1, we deduce that  $\int_\Omega |\nabla u_p|^2 \rightarrow 0$  as  $p$  goes to infinity. By (2.4), we obtain the claim.  $\square$

*Proof of Theorem 1.1.* For any sequence  $p_n \rightarrow +\infty$ , let us set  $z_n : \Omega_n = \frac{\Omega - x_{p_n}}{\varepsilon_n} \mapsto \mathbb{R}$ ,

$$(2.5) \quad z_n(x) = \frac{p_n}{u_{p_n}(x_{p_n})} (u_{p_n}(\varepsilon_n x + x_{p_n}) - u_{p_n}(x_{p_n}))$$

where  $\varepsilon_n^2 = \frac{1}{p_n S_{p_n} u_{p_n}(x_{p_n})^{p_n-1}}$ . From Lemma 2.1 and Lemma 2.2, we get  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and “ $\Omega_n \rightarrow \mathbb{R}^2$ ” as  $n \rightarrow \infty$ . Now let us write down the equation satisfied by  $z_n$ ,

$$(2.6) \quad \begin{cases} -\Delta z_n = \left(1 + \frac{z_n}{p_n}\right)^{p_n} & \text{in } \Omega_n, \\ 0 < 1 + \frac{z_n}{p_n} \leq 1 & \text{in } \Omega_n, \\ z_n = -p_n & \text{on } \partial\Omega_n. \end{cases}$$

We want to pass to the limit in (2.6). To do this we use some ideas in [2]. Let  $B(0, R)$  be the ball centered at the origin with radius  $R$ , and let  $w_n$  be the solution of

$$(2.7) \quad \begin{cases} -\Delta w_n = \left(1 + \frac{z_n}{p_n}\right)^{p_n} & \text{in } B(0, R), \\ w_n = 0 & \text{on } \partial B(0, R). \end{cases}$$

By the maximum principle and the standard regularity theory, we have that  $0 \leq w_n \leq C$  with  $C$  independent of  $n$ . For  $x \in B(0, R)$  set  $\psi_n(x) = z_n(x) - w_n(x)$ . Hence  $\psi_n$  is a sequence of harmonic functions which are uniformly bounded above. Hence by Harnack’s inequality [7] we have the alternative:

- either
- i) a subsequence of  $\psi_n$  is bounded in  $L^\infty_{loc}(B(0, R))$ ,
- or
- ii)  $\psi_n$  converges uniformly to  $-\infty$  on compact subsets of  $(B(0, R))$ .

Since  $\psi_n(0) = z_n(0) - w_n(0) = -w_n(0) \geq -C$ , case ii) cannot occur. Hence, up to a subsequence, which we denote again by  $\psi_n$ , we have that  $\psi_n$  is bounded in  $L^\infty(B(0, R))$  for any  $R > 0$  and the same holds for  $z_n$ . From (2.6), and the standard regularity theory, we derive that  $z_n$  is bounded in  $C^2_{loc}(\mathbb{R}^2)$ , and then it converges to a function  $z \in C^2(\mathbb{R}^2)$ . Passing to the limit in (2.6), we get that  $z$  satisfies

$$(2.8) \quad -\Delta z = e^z \quad \text{in } \mathbb{R}^2.$$

Let us prove that  $\int_{\mathbb{R}^2} e^z < +\infty$ . To do this we observe that, since  $z_n \rightarrow z$  in  $C^2_{loc}(\mathbb{R}^2)$ , then

$$(2.9) \quad p_n \left( \log \left( 1 + \frac{z_n}{p_n} \right) - \frac{z_n}{p_n} \right) \rightarrow 0 \quad \text{pointwise in } \mathbb{R}^2.$$

Hence

$$(2.10) \quad z_n + p_n \left( \log \left( 1 + \frac{z_n}{p_n} \right) - \frac{z_n}{p_n} \right) \rightarrow z \quad \text{pointwise in } \mathbb{R}^2.$$

By Fatou’s Lemma, we deduce

$$(2.11) \quad \begin{aligned} \int_{\mathbb{R}^2} e^z &\leq \liminf_{n \rightarrow \infty} \int_{\Omega_n} e^{z_n + p_n \left( \log \left( 1 + \frac{z_n}{p_n} \right) - \frac{z_n}{p_n} \right)} = \liminf_{n \rightarrow \infty} \int_{\Omega_n} \left( 1 + \frac{z_n}{p_n} \right)^{p_n} \\ &= \liminf_{n \rightarrow \infty} \frac{1}{\varepsilon_n^2 u_{p_n}^{p_n}(x_{p_n})} \int_{\Omega} u_{p_n}^{p_n} \leq \liminf_{n \rightarrow \infty} \frac{p_n S_{p_n}}{u_{p_n}(x_{p_n})} |\Omega|^{\frac{1}{p_n+1}} \leq C \end{aligned}$$

since  $u_{p_n}(x_{p_n}) \geq C$  in  $\Omega$  for  $n$  large (see [8], p. 755).

By a result of Chen and Li ([3]), the solutions of (2.8) satisfying  $\int_{\mathbb{R}^2} e^z < +\infty$  are given by

$$(2.12) \quad z(x) = \log \frac{\mu}{\left( 1 + \frac{\mu}{8} |x - x_0|^2 \right)^2} \quad \text{for } \mu > 0 \text{ and } x_0 \in \mathbb{R}^2.$$

Since  $z(x) \leq z(0) = 0$  for any  $x \in \mathbb{R}^2$ , we derive that  $\mu = 1$  and  $x_0 = 0$  in (2.12), and this gives the claim of Theorem 1.1. □

### 3. PROOF OF THEOREM 1.2

The next estimate plays a role in the proof of Theorem 1.2. This estimate was proved in [9] but we stress that it follows easily by Theorem 1.1.

**Lemma 3.1.** *We have that*

$$(3.1) \quad \limsup_{n \rightarrow \infty} \|u_{p_n}\|_{\infty} \leq \sqrt{e}.$$

*Proof.* It follows directly by Theorem 1.1.

Setting  $u_{p_n} = u_n$  and  $L = \limsup_{n \rightarrow \infty} \|u_n\|_{\infty}$ , by using Fatou’s Lemma, we obtain

$$(3.2) \quad \begin{aligned} 1 &= \int_{\Omega} u_n^{p_n+1} = u_n(x_n)^{p_n+1} \varepsilon_n^2 \int_{\Omega_n} \left( 1 + \frac{z_n}{p_n} \right)^{p_n+1} \\ &= \frac{u_n(x_n)^2}{p_n S_{p_n}} \int_{\Omega_n} \left( 1 + \frac{z_n}{p_n} \right)^{p_n+1} \geq \frac{L^2}{8\pi e} \int_{\mathbb{R}^2} e^z. \end{aligned}$$

Recalling that  $\int_{\mathbb{R}^2} e^z = 8\pi$ , we deduce the claim. □

Let us consider the linearized operator associated to (1.3), i.e.,  $L_p : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ ,

$$(3.3) \quad L_p = -\Delta - p S_p u_p^{p-1}(x) I, \quad x \in \Omega,$$

and let us denote by  $\lambda_1(L_p), \lambda_2(L_p)$  the first and the second eigenvalue of  $L_p$ . Now let us recall a property of  $\lambda_2(L_p)$ .

**Lemma 3.2.** *We have that*

$$(3.4) \quad \lambda_2(L_p) \geq 0.$$

*Proof.* The proof is standard since  $u_p$  is a minimizer of  $J$  on the manifold  $\int_{\Omega} |u|^{p+1} = 1$ . □

We consider, for  $D \subset \Omega_p$ ,  $L_{p,D} : H_0^1(D) \rightarrow H^{-1}(D)$ ,

$$(3.5) \quad L_{p,D} = -\Delta - \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{u_p^{p-1}(x_p)} I, \quad x \in D,$$

and let us denote by  $\lambda_1(L_{p,D})$ ,  $\lambda_2(L_{p,D})$  the first and the second eigenvalue of  $L_{p,D}$ .

**Lemma 3.3.** *We have that*

$$(3.6) \quad \lambda_2(L_{p,\Omega_p}) \geq 0.$$

*Proof.* Using the scaling  $x \rightarrow \varepsilon_p x + x_p$  we get  $\lambda_2(L_{p,\Omega_p}) = \varepsilon_p^2 \lambda_2(L_p)$  and (3.6) follows by Lemma 3.2. □

**Lemma 3.4.** *Let us denote by  $B_1 = B(0, 1)$ . Let  $p_n \rightarrow \infty$  such that  $z_{p_n} \rightarrow z$  in  $C_{loc}^1(\mathbb{R}^2)$ . Then for large  $p_n$ , we have*

$$(3.7) \quad \lambda_1(L_{p_n,B_1}) < 0.$$

*Proof.*

$$(3.8) \quad w_p = x \cdot \nabla z_p + \frac{2}{p-1} z_p + \frac{2p}{p-1}.$$

By direct computation we get that  $w_p$  satisfies

$$(3.9) \quad -\Delta w_p = \frac{u_p^{p-1}(\varepsilon_p x + x_p)}{u_p^{p-1}(x_p)} w_p.$$

Moreover,  $w_{p_n}(0) \rightarrow 2$  and for  $|x| = 1$ ,  $w_{p_n}(x) \rightarrow -\frac{4}{9}$  as  $p_n \rightarrow \infty$ .

Hence, if we denote by  $A_p = \{x \in B_1 : w_p > 0\}$  and

$$(3.10) \quad \tilde{w}_p = \begin{cases} w_p & \text{if } x \in A_p, \\ 0 & \text{if } x \in B_1 \setminus \bar{A}_p, \end{cases}$$

we derive that for  $p_n$  large,  $\tilde{w}_{p_n} \in H_0^1(B_1)$ . From (3.9) we get

$$(3.11) \quad \int_{B_1} |\nabla \tilde{w}_{p_n}|^2 - \int_{B_1} \frac{u_{p_n}^{p_n-1}(\varepsilon_{p_n} x + x_{p_n})}{u_{p_n}^{p_n-1}(x_{p_n})} \tilde{w}_{p_n}^2 = 0,$$

and this implies that  $\lambda_1(L_{p_n,B_1}) < 0$ . □

**Lemma 3.5.** *Let  $p_n$  be a sequence as in Lemma 3.4. Then for  $p_n$  large we have*

$$(3.12) \quad \lambda_1(L_{p_n,\Omega_{p_n} \setminus B_1}) > 0.$$

*Proof.* By contradiction let us suppose that  $\lambda_1(L_{p_n,\Omega_{p_n} \setminus B_1}) \leq 0$ . Then from Lemma 3.4, for large  $p_n$ ,  $\lambda_1(L_{p_n,B_1}) < 0$  and hence  $\lambda_2(L_{p_n,\Omega_{p_n}}) < 0$ . This gives a contradiction with Lemma 3.3. □

*Remark 3.6.* Lemma 3.4 implies that the operator  $L_{p_n,\Omega_{p_n} \setminus B_1}$  satisfies the maximum principle in  $\Omega_{p_n} \setminus B_1$ .

*Proof of Theorem 1.2.* By Lemma 3.1 we know that (up to a subsequence)  $\lim_{n \rightarrow \infty} \|u_{p_n}\|_\infty \leq \sqrt{e}$ . By contradiction let us suppose that there exists a subsequence of  $u_{p_n}$  (still denoted by  $u_{p_n}$ ) such that

$$(3.13) \quad \lim_{n \rightarrow \infty} \|u_{p_n}\|_\infty < \sqrt{e}.$$

Now, we will show that for large  $p_n$ , (3.13) implies the following estimate:

$$(3.14) \quad z_n(x) \leq C + \log \frac{1}{\left(1 + \frac{|x|^2}{8}\right)^2} \quad \forall x \in \Omega_{p_n}$$

where  $C$  is a constant independent of  $n$ .

By Theorem 1.1,  $z_n \rightarrow z$  in  $C^0(\overline{B_1})$  and hence (3.14) fails for  $x \in B_1$ . It is enough to prove 3.14 for  $x \in \Omega_n \setminus B_1$ . To prove this let us observe that the function  $z$  satisfies

$$(3.15) \quad -\Delta z = e^z \geq \left(1 + \frac{z}{p}\right)^p$$

for any  $p > 1$ . Furthermore, let us consider  $\psi_n = z_n - z$  in  $\Omega_{p_n}$ . By computing  $\psi_n$  on  $\partial(\Omega_{p_n} \setminus B_1)$  and by applying the maximum principle, if  $x \in \partial\Omega_{p_n}$ , we get

$$(3.16) \quad \begin{aligned} \psi_n(x) &= z_n(x) - z(x) = -p_n + 2 \log\left(1 + \frac{|x|^2}{8}\right) \leq -p_n + 2 \log \frac{1}{\varepsilon_{p_n}^2} + C \\ &\leq -p_n + 2 \log u_{p_n}(x_{p_n})^{p_n-1} + C \leq C \end{aligned}$$

where we had used  $u_{p_n}(x_{p_n}) < \sqrt{e}$ .

Now if  $x \in \partial B_1$ , by Theorem 1.1 we can derive again that  $\psi_n(x) \leq C$ .

Finally, we write down the equation satisfied by  $\psi_n$ . Using the convexity of  $F(s) = \left(1 + \frac{s}{p}\right)^p$  for  $p > 1$  we have

$$(3.17) \quad \begin{aligned} -\Delta \psi_n &= \left(1 + \frac{z_n}{p_n}\right)^{p_n} - \left(1 + \frac{z}{p_n}\right)^{p_n} \\ &\leq \left(1 + \frac{z_n}{p_n}\right)^{p_n-1} \psi_n = \frac{u_{p_n}^{p_n-1}(\varepsilon_{p_n} x + x_{p_n})}{u_{p_n}^{p_n-1}(x_{p_n})} \psi_n. \end{aligned}$$

Since the maximum principle holds in  $\Omega_{p_n} \setminus B_1$  for  $L_{p_n, \Omega_{p_n} \setminus B_1}$ , we now deduce that  $\psi_n \leq C$  in  $\Omega_{p_n} \setminus B_1$  and this gives (3.14).

From (3.14), a contradiction follows easily. Indeed, using Theorem 1.1 and Lebesgue's Theorem we derive

$$(3.18) \quad \begin{aligned} 1 &= \int_{\Omega} u_n^{p_n+1} = u_n(x_n)^{p_n+1} \varepsilon_n^2 \int_{\Omega_n} \left(1 + \frac{z_n}{p_n}\right)^{p_n+1} \\ &= \frac{u_n^2(x_n)}{8\pi e + o(1)} (8\pi + o(1)), \end{aligned}$$

which proves that  $\lim_{n \rightarrow \infty} \|u_n\|_\infty = \sqrt{e}$ , a contradiction with (3.13). □

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